NORTHWESTERN UNIVERSITY Tuesday, October 6th, 2015

NAME: _____

FALL 2015 NU PUTNAM SELECTION TEST

Problem A1. Show that $\log(1+x) > x/(1+x)$ for all $x > 0$.

Problem A2. Define the sequence $a_0 = 0$, $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$ for $n \geq 0$. Find

$$S = \sum_{n=0}^{\infty} \arccos a_n \, .$$

(Note: $y = \arccos x \Leftrightarrow y \in [0, \pi]$ and $\cos y = x$.)

Problem A3. Let r be a real number in the interval $[0, 1)$. Find the sum

$$S = \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor 2^k r \rfloor}}{2^k} \, ,$$

where $\lfloor x \rfloor$ = integer part of x = greatest integer less that or equal to x.

Problem A4. One hundred passengers board a plane with exactly 100 seats. The first passenger takes a seat at random. The second passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. The third passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. This process continues until all the 100 passengers have boarded the plane. What is the probability that the last passenger takes his own seat?

Problem A5. Prove that the following divisibility criteria by 61 actually works. Divisibility by 61: Let n be a positive integer. Let d be the rightmost digit of n (in decimal notation), and let n' be the number obtained by removing from n its rightmost digit (if n has only one digit then $n' = 0$). Replace n with $n' - 6d$. Repeat those steps while the result is still a positive integer. If the process ends in zero then the original number is divisible by 61, otherwise it is not. Example for $n = 21045$: $2104 - 6 \cdot 5 = 2074$, $207 - 6 \cdot 4 = 183$, $18 - 6 \cdot 3 = 0$. Hence 21045 is divisible by 61.

Problem A6. Flip a fair coin until heads turns out twice consecutively. What is the expected number of flips?