NAME: ______________________________

FALL 2014 NU PUTNAM SELECTION TEST

Problem A1. Show that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(1+n)} \leq \pi .$$

Problem A2. Find the following infinite product:

$$P = \prod_{n=1}^{\infty} \left(1 + \left(\tfrac{1}{7}\right)^{2^n}\right)$$

Write the result as a fraction $P = \frac{a}{b}$ in least terms.

Problem A3. Let S be a set with even number of elements, and $f : S \to S$ a map of S into itself such that $f \circ f : S \to S$ is the identity map. Show that the set of the fixed points has even number of elements.

Problem A4. Let $f : \mathbb{R} \to \mathbb{R}$ a continuous function without fixed points, i.e., there is no $x \in \mathbb{R}$ such that $f(x) = x$. Let n be a positive integer. Prove that $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$ has no fixed points either.

Problem A5. The Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, \dots$ are defined as $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). The *digital root* of a non-negative integer is the (single digit) value obtained by an iterative process of summing digits, on each iteration using the result from the previous iteration to compute a digit sum. The process continues until a single-digit number is reached. For example, the digital root of 65,536 is 7, because $6 + 5 + 5 + 3 + 6 = 25$ and $2 + 5 = 7$. Prove that there are integers a, b, with $a > 0$ and $b \geq 0$, such that all Fibonacci numbers of the form F_{an+b}, $n = 0, 1, 2, 3, \dots$, have the same digital root.

Problem A6. Let a, b, c three positive real numbers prove:

$$\sqrt{a^2+1} + \sqrt{b^2+4} + \sqrt{c^2+9} \geq 2\sqrt{3}\sqrt{a+b+c} .$$