Pell Equations

Putnam Practice

November 17, 2004

A famous example of a Diophantine equation is Pell equation. It is an equation of the form

$$x^2 - Dy^2 = 1$$

with D a positive integer that is not a perfect square. To find all positive integer solutions of this equation, one first determines a minimal solution (i.e. the solution (x_0, y_0) for which $x_0+y_0\sqrt{D}$ is minimal). There is a general way to compute this minimal solution, however, in all problems below the minimal solution is easy to guess.

The other solutions are given by

$$x_n + y_n\sqrt{D} = (x_0 + y_0\sqrt{D})^n$$

It is easy to see that this formula yields solutions of the equation (multiply $x_n + y_n\sqrt{D}$ by its conjugate $x_n - y_n\sqrt{D} = (x_0 - y_0\sqrt{D})^n$). It is also easy to see that there are no other solutions.

For a generalized Pell equation

$$ax^2 - by^2 = c$$

where a, b are not divisible by any square the solutions might not exist. If $c = 1$, $a, b \neq 1$ and the minimal solution (x_0, y_0) exists, then the general solution is generated by

$$x_n\sqrt{a} + y_n\sqrt{b} = (x_0\sqrt{a} + y_0\sqrt{b})^{2n+1}$$

Example 1 *Prove that $n^2 + (n+1)^2$ is a perfect square for infinitely many natural numbers.*

Solution: Write $n^2 + (n+1)^2 = m^2$ as

$$(2n+1)^2 - 2m^2 = -1$$

Thus values of n for which $n^2 + (n+1)^2$ is a perfect square correspond to solutions of the Pell equation $x^2 - 2y^2 = -1$, where $x = 2n+1$. One solution is $x = y = 1$. Let

$$x_k + y_k\sqrt{2} = (1+\sqrt{2})^{2k+1}, k \geq 2$$

It is easy to check these are solutions of $x^2 - 2y^2 = -1$ as well. Thus $n_k = \frac{x_k - 1}{2}$ (note x_k is always odd). The first 3 values for n are 3, 20, 119.

1 Problems

1. Find all natural numbers of the form $m(m+1)/3$ that are perfect squares.

2. Solve the equation $(x+1)^3 - x^3 = y^2$ in positive integers.

3. Find all positive integers n for which both $2n+1$ and $3n+1$ are perfect squares.

4. Let $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$, and for positive integers n, define d_n as the greatest common divisor of the entries of $A^n - I$, where I is the identity matrix. Prove that $d_n \to \infty$ as $n \to \infty$.

5. Prove that there exist infinitely many positive integers n such that $n^2 + 1$ divides $n!$.