Putnam Seminar — Week 2 — Sept. 23, 2015

(2014, B1) A *base* 10 *over-expansion* of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \ldots, 10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

(1986, A1) Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

(1986, A2) What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor ?$$

(Here the notation $\lfloor x \rfloor$, sometimes called "the floor of x", is used to stand for the greatest integer less than or equal to x.)

(1986, A3) Evaluate

$$\sum_{n=0}^{\infty} \text{Arccot } (n^2 + n + 1),$$

where Arccot(t) for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot(\theta) = t$.