36

The Game of Life Problem

Apart from his numerous "serious" mathematical achievements, John Conway became famous by inventing his *Game of Life* in the early seventies. The game was brought to the public's attention by Martin Gardner through his column in *Scientific American*.

The game is played on an infinite chessboard. Some initial arrangement of "live cells" is given, and each new generation is determined by several rules, or the "genetic laws." They are quite simple and were chosen carefully after lots of experiments. The only detail we would like to emphasize is that the evolution is "discrete"; that is: Following the rules, one determines whether there should exist life in each particular cell of the chessboard throughout the next generation. Then the old generation vanishes and the next one comes to life. This is exactly the case with the problem we are going to consider.

It is not our intention to discuss the *Game of Life* here. If you want to learn more, read the corresponding three chapters of Martin Gardner's book *Wheels, Life and Other Mathematical Amusements*.

The *Game of Life* was cheered all over the world. Our story is about a problem of the 1973 All-Union Olympiad, obviously inspired by Conway's invention.

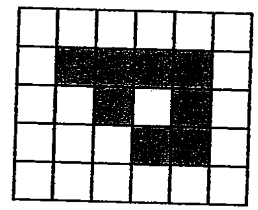
An infinite chessboard is given whose cells are initially all white. Then a set G_0 of n cells is chosen, and they are all colored black. At each moment $t = 1, 2, \ldots$ of time, a simultaneous change of color takes place for each cell on the chessboard according to the following rule: Every cell C gets the color that prevails in the three-cell configuration formed by C itself and its upper and right neighbors. That is, if two or three of these cells were white, then C becomes white; if two or three of them were black, then

C becomes black. In this way the consecutive steps give rise to new sets $G_1, G_2, \ldots, G_k, \ldots$ of black cells.

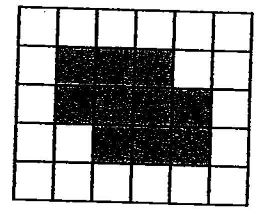
(a) *Prove that $G_k = \varnothing$ for some positive integer k (that is, there will be no black cells on the chessboard after a while).*

(b) *Prove that $G_n = \varnothing$ (that is, all black cells will disappear no later than the moment $t = n$).*

The majority of people who try to solve this problem expect that the number of black cells in each next generation G_{k+1} is *strictly less* than the one in the previous generation G_k. This is not true. Moreover, it is possible for G_{k+1} to contain *more* black cells than G_k, as the example below shows.



G_0



G_1

This observation is another piece of evidence of how unpredictable the behavior of the generations might be, even with simple rules of the Conway type. Still, another type of monotonicity yields an easy proof of (a).

(solution omitted!)

Part (b) is much harder. That one cannot expect it to be easy may be inferred after finding a large variety of generations G_0 that die out in exactly n steps. (It is not a bad idea to write a program and play this game on a computer.)

The next proof is an illustrative example of how simple and efficient the method of induction can be.

(solution omitted again!)

This is essentially the solution found by **A. Gomilko**, a Russian tenth-grader, during competition time. There is no wonder it won a special prize.