

PUTNAM SEMINAR

1. Prove that there are infinitely many positive integers n with the property that if p is a prime divisor of $n^2 + 3$, then p is also a divisor of $k^2 + 3$ for some integer k with $k^2 < n$.

2. Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

3. For which real numbers c is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

4a. Prove that there exist integers a, b, c , not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$

4b. Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that

$$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}.$$

5. What is the units (i.e., rightmost) digit of

$$\left[\frac{10^{20000}}{10^{100} + 3} \right]?$$

6. Evaluate

$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$