Putnam Seminar 2003

More problems from "Mathematical Miniatures".

Complete Sequences

A sequence $(a_m)_{m\geq 1}$ with positive integer terms is called *totally complete* if each positive integer can be expressed as a sum of distinct terms of this sequence.

1. Prove that every positive integer sequence $a_1, a_2, \ldots, a_n, \ldots$ satisfying the conditions $a_1 = 1$ and

$$a_{n+1} \leq 1 + a_1 + a_2 + \cdots + a_n, \qquad n = 1, 2, \ldots,$$

is totally complete.

2. (From the 1975 West German Olympiad) Two brothers inherited n pieces of gold with total weight $2n$. Each piece has integer weight and the heaviest of them is not heavier than the remaining ones combined. Prove that if n is even then the brothers can divide the inheritance into two parts with equal weights.

3. Show that every integer can be represented in infinitely many ways as

$$n = \pm 1^2 \pm 2^2 \pm \cdots \pm k^2$$

where k is a positive integer and each $\pm$ is replaced by either $+$ or $-$.