Putnam Problems involving Calculus

$\boxed{\text{B1, 1990}}$

FIND ALL REAL-VALUED CONTINUOUSLY DIFFERENTIABLE FUNCTIONS f ON THE REAL LINE SUCH THAT FOR ALL x,

$$(f(x))^2 = \int_0^x \left((f(t))^2 + (f'(t))^2 \right) dt + 1990.$$

$\boxed{\text{A2, 1989}}$

EVALUATE
$$\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} \, dy \, dx,$$

WHERE a AND b ARE POSITIVE.

$\boxed{\text{B2, 1985}}$

DEFINE POLYNOMIALS $f_n(x)$ FOR $n \geq 0$ BY $f_0(x) = 1$, $f_n(0) = 0$ FOR $n \geq 1$, AND

$$\frac{d}{dx}\left(f_{n+1}(x) \right) = (n+1) f_n(x+1)$$

FOR $n \geq 0$. FIND, WITH PROOF, $f_{100}(1)$.

$\boxed{\text{B4, 1985}}$

LET C BE THE UNIT CIRCLE $x^2 + y^2 = 1$. A POINT p IS CHOSEN RANDOMLY ON THE CIRCUMFERENCE OF C AND ANOTHER POINT q IS CHOSEN RANDOMLY FROM THE INTERIOR OF C (THESE POINTS ARE CHOSEN INDEPENDENTLY AND UNIFORMLY OVER THEIR DOMAINS). LET R BE THE RECTANGLE WITH SIDES PARALLEL TO THE x- AND y-AXES WITH DIAGONAL pq. WHAT IS THE PROBABILITY THAT NO POINT OF R LIES OUTSIDE OF C?