12.5: Equations of Lines and Planes

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Overview

You have spent a lot of time learning how to create vectors from raw data and manipulate them, but it is perhaps still unclear why we are so interested in them. The short, abstract answer is that vectors provide a convenient language to specify higher-dimensional objects. Today we will see a bit of this as we learn to use vectors to describe lines and planes in $\mathbb{R}^3$. This is not only a convenient exercise to ease us into thinking more concretely about the utility of vectors, but also practical, as these will be fundamental objects of study for us moving forward.

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Two Dimensions

In $\mathbb{R}^2$, we can give an equation for a line if we know two pieces of information: a point P on the line, and the slope m of the line. Moving to $\mathbb{R}^3$, it's not immediately obvious how we might generalize the slope concept.

There's another way we can think of the slope of a line in $\mathbb{R}^2$ that lends more insight: the slope of the line is the direction in which the line extends from the point P.

In other words, a line can be completely described by a point it contains and the direction it points in, i.e. an **initial point** and a **direction vector**. Let's see how this works in practice.

The Vector Equation

Think of the initial point P on the line as the tip of a vector $\vec{r_0}$. To get another point on the line, we could add the direction vector $\vec{v}$ to $\vec{r_0}$. To get the rest of the line, we can simply stretch or shrink this direction vector forward or backward as far as we like. This stretching or shrinking corresponds to multiplying $\vec{v}$ by a real scalar t.

In other words, the line is made up of every possible sum of $\vec{r_0}$ and a scalar multiple of $\vec{v}$.

Thus, we can specify a line in $\mathbb{R}^2$ or $\mathbb{R}^3$ symbolically with the following **vector equation**:

$$\boxed{\vec{r} = \vec{r_0} + t\,\vec{v}}$$

for any scalar value t, i.e., t is the variable (or **parameter**) here.

The Parametric Equations

The equation above is descriptive, but to be able to do many calculations we will want to know what it means in terms of the variables x and y (and z, if we are working in $\mathbb{R}^3$).

Let $\vec{r} = \langle x, y, z \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\vec{v} = \langle a, b, c \rangle$. The equation above then becomes:

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$$

We can write this as a set of three **parametric equations**:

$$\boxed{\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}}$$

Note that if we are given a line in this form, we can immediately read off the point on the line $\langle x_0, y_0, z_0 \rangle$ and a direction vector $\langle a, b, c \rangle$.

The Symmetric Equations

There's one more way that lines in $\mathbb{R}^3$ are often described. If we solve each of the equations above for the parameter t, we have:

$$t = \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

These are called the **symmetic equations** of the line.

Note again that if we are given a line in this form, we can immediately read off the point $\langle x_0, y_0, z_0 \rangle$ on the line and a direction vector $\langle a, b, c \rangle$.

Example

Find vector, parametric, and symmetric equations of the line L which passes through the points $A = (2, 4, -3)$ and $B = (3, 4, 1)$.

Remember: to give an equation of a line, we need a point on the line, and the direction it extends in.

We know a point on L immediately: A.

Since L extends between A and B, a direction vector for L is:

$$\vec{v} := \langle 3-2, 4-4, 1-(-3) \rangle = \langle 1, 0, 4 \rangle$$

Thus, a vector equation for L is:

$$\boxed{\langle x, y, z \rangle = \langle 2, 4, -3 \rangle + t\langle 1, 0, 4 \rangle}$$

Example, cont.

From the previous slide, a set of parametric equations for L is:

$$\boxed{\begin{aligned} &x = 2 + t \\ &y = 4 \\ &z = -3 + 4t \end{aligned}}$$

and a set of symmetric equations for L is:

$$\boxed{y = 4, \quad x - 2 = \frac{z + 3}{4}}$$

Example

Let L_1 and L_2 be lines in $\mathbb{R}^3$ described by the parametric equations:

$$
\begin{array}{llll}
L_1: & x = 1 + t & y = -2 + 3t & z = 4 - t \\
L_2: & x = 2s & y = 3 + s & z = -3 + 4s
\end{array}
$$

Are these lines parallel, intersecting, or **skew** (nonintersecting and nonparallel)?

By definition, parallel lines extend in the same direction. In other words, if L_1 has direction vector $\vec{v_1}$ and L_2 has direction vector $\vec{v_2}$, then L_1 and L_2 are parallel if and only if $\vec{v_1}$ and $\vec{v_2}$ point in the same (or exact opposite) direction, i.e., if $\vec{v_1}$ is a scalar multiple of $\vec{v_2}$.

Now, note that L_1 points in the direction of $\vec{v_1} := \langle 1, 3, -1 \rangle$ and L_2 points in the direction of $\vec{v_2} := \langle 2, 1, 4 \rangle$. Thus, L_1 and L_2 are not parallel, as $\vec{v_2}$ is not a scalar multiple of $\vec{v_1}$.

Example, cont.

If the lines intersect, there must be values of s and t such that the x, y, and z values of each are the same. This gives us the three equations:

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s \\ 4 - t &= -3 + 4s \end{aligned}$$

Solving the first gives $t = 2s - 1$. Plugging this into the second gives $s = \frac{8}{5}$, so that $t = \frac{11}{5}$. However, these values *don't* satisfy the third equation, so L_1 and L_2 do not intersect, i.e. they are $\boxed{\textit{skew}}$.

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Surfaces

As in $\mathbb{R}^2$, lines are just the first of many curves we that we will draw and describe in $\mathbb{R}^3$. For now, let's change gears and begin to discuss a new class of objects in $\mathbb{R}^3$ that don't exist in $\mathbb{R}^2$: surfaces. In particular, we will discuss planes.

A plane is an (important) example of a surface. Intuitively, a plane is the collection of all points lying in a flat sheet sitting in space.

This is in some way fundamentally different than a line or any other curve, as a plane has two (topological) dimensions, whereas a curve has only one. Put another way, any curve in $\mathbb{R}^3$ is like a distorted string, whereas a surface is like a distorted sheet of paper.

Let's now turn our full attention to planes.

Equation of a Plane

We have a qualitative description of what a plane is, but how can we specify one mathematically? Well, the first thing to note is that a plane can be specified uniquely if we know two pieces of information: a point $P = (x_0, y_0, z_0)$ in the plane, and a vector $\vec{n} = \langle a, b, c \rangle$ orthogonal to it (called a **normal** vector). (Take a moment to think about this).

How does this information help us? Think of the point P in the plane as the tip of a vector $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$. Suppose that (x, y, z) is any other point in the plane, which we can think of as the tip of the vector $\vec{r} = \langle x, y, z \rangle$ Form the vector $\vec{r} - \vec{r_0}$, which lies in the plane. Note that since $\vec{n}$ is orthogonal to the plane, it must also be orthogonal to the vector $\vec{r} - \vec{r_0}$. So, the plane is the collection of points (x, y, z) such that $\vec{r} - \vec{r_0}$ is orthogonal to $\vec{n}$. How can we express this mathematically?

Equation, cont.

Recall that two vectors are orthogonal precisely when their dot product is zero. Therefore, we can express the observation above as an equation:

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0}$$

This is called a **vector equation of the plane**. If we write the vectors into component form and expand the dot product, we obtain a **scalar equation of the plane**:

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$
$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Finally, if we group the constants $-ax_0, -by_0$, and $-cz_0$ together as one constant d, we obtain a **linear equation** of the plane:

$$\boxed{ax + by + cz + d = 0}$$

The plane is the set of all points (x, y, z) satisfying these equations.

Equation Note

Note that the latter equation is particularly useful because we can immediately read off the normal vector $\vec{n}$ from the coefficients of x, y, and z.

Example

Find vector, scalar, and linear equations of the plane T passing through the points $P = (1, 3, 2)$, $Q = (3, -1, 6)$, and $R = (5, 2, 0)$.

Remember: a plane is determined by a point it contains and a normal vector. We have a point in the plane: P. What about a normal vector?

Note that T contains the vectors $\overrightarrow{PQ} = \langle 2, -4, 4 \rangle$ and $\overrightarrow{PR} = \langle 4, -1, -2 \rangle$. The vector $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 12, 20, 14 \rangle$ is orthogonal to both of these, and hence T itself!

Thus, a vector equation of T is:

$$\boxed{\langle 12, 20, 14 \rangle \cdot \langle x - 1, y - 3, z - 2 \rangle = 0}$$

Example, cont.

From this, we immediately obtain a scalar equation of T:

$$\boxed{12(x-1)+20(y-3)+14(z-2)=0}$$

and a linear equation of T:

$$\boxed{12x+20y+14z-100=0}$$

Example

The angle between two planes is the angle between their normal vectors. Calculate the angle between the planes T_1 and T_2, given by the equations $x + y + z = 1$ and $x - 2y + 3z = 1$, respectively.

A vector orthogonal to T_1 is $\vec{n}_1 := \langle 1, 1, 1 \rangle$ and a vector orthogonal to T_2 is $\vec{n}_2 := \langle 1, -2, 3 \rangle$.

Let θ be the angle between the vectors $\vec{n}_1$ and $\vec{n}_2$ (which is the angle between T_1 and T_2). Recall that $\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos(\theta)$. Thus:

$$\cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{2}{\sqrt{42}}$$

giving

$$\boxed{\theta = \arccos \frac{2}{\sqrt{42}} \approx 72^\circ}$$

Example

Find the point P at which the line L with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane T with linear equation $4x + 5y - 2z = 18$.

For L and T to intersect, there must be a value of t such that the x, y, and z coordinates of the corresponding point on L satisfy the equation of T.

That is, we make a substitution:

$$\begin{aligned} & 4x + 5y - 2z = 18 \\ \Rightarrow \quad & 4(2 + 3t) + 5(-4t) - 2(5 + t) = 18 \\ \Rightarrow \quad & t = -2 \end{aligned}$$

For this t, we have $x = -4$, $y = 8$, and $z = 3$. Thus, these objects intersect at $\boxed{P = (-4, 8, 3)}$.

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Exercises

1. Find a set of vector, parametric, and symmetric equations of the line through the origin and the point $(4, 3, -1)$.
2. Find a set of vector, parametric, and symmetric equations of the line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$.
3. Is the line through $(-4, -6, 1)$ and $(-2, 0, -3)$ parallel to, intersecting with, or skew with the line through $(10, 18, 4)$ and $(5, 3, 14)$?
4. Find a set of vector, scalar, and linear equations of the plane through $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$.
5. Calculate the angle between the planes $9x - 3y + 6z = 2$ and $2y = 6x + 4z$.
6. Find the point at which the line $x = t - 1$, $y = 1 + 2t$, $z = 3 - t$ intersects the plane $3x - y + 2z = 5$.

Solutions

1. One solution:
   Vector: $\langle x, y, z\rangle = \langle 0, 0, 0\rangle + t\,\langle 4, 3, -1\rangle$
   Parametric:

$$x = 4t$$
$$y = 3t$$
$$z = -t$$

   Symmetric: $\frac{x}{4} = \frac{y}{3} = -z$. Bear in mind that there are many possible correct solutions that may look quite different from this one.

Solutions, cont.

2. One solution:
   Vector: $\langle x, y, z\rangle = \langle 0, 14, -10\rangle + t\,\langle 2, -3, 9\rangle$
   Parametric:

$$
\begin{aligned}
x &= 2t \\
y &= 14 - 3t \\
z &= -10 + 9t
\end{aligned}
$$

   Symmetric: $\frac{x}{2} = \frac{y-14}{-3} = \frac{z+10}{9}$

Solutions, cont.

3. The lines are parallel.
4. One solution:
   Vector: $\langle 5, -1, -1\rangle \cdot \langle x-1, y+1, z+1\rangle = 0$
   Scalar: $5(x-1) + (-y-1) + (-z-1) = 0$
   Linear: $5x - y - z - 7 = 0$
5. $\cos(\theta) = 1$, so $\theta = 0$, i.e. the planes are parallel. A way to see this without using the dot product: their normal vectors are parallel.
6. $(-4, -5, 6)$