

13.1: Vector Functions and Space Curves

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Chapter Overview

In this course, we will discuss the calculus of various objects in \mathbb{R}^3 . To get us started, we begin with the simplest of these objects: curves. In this short chapter, we will discuss the calculus of so-called **space curves**; curves in \mathbb{R}^3 .

Section Overview

If we're going to create a calculus of curves in \mathbb{R}^3 , we need functions to be able to do this calculus on. In other words, we functions whose graphs are space curves.

In this section, we introduce exactly the functions we need: so-called **vector-valued functions**, or what are more commonly called just **vector functions**. We'll start by defining such functions, then learn techniques to graph them, before finally moving in the direction of calculus by learning to take their limits.

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Definition

A **vector-valued function** $\vec{r}(t)$, or simply **vector function**, is a function whose domain is a set of real numbers and whose range is a set of vectors.

Example

Let $\vec{r}(t) = \langle t, t^2, 2 \cos(t) \rangle$. Evaluate $\vec{r}(2)$ and $\vec{r}(\pi)$.

By plugging in, we have:

$$\vec{r}(2) = \langle 2, 4, 2 \cos(2) \rangle$$

and

$$\vec{r}(\pi) = \langle \pi, \pi^2, -2 \rangle$$

A Helpful Observation and Definition

Notice on the previous slide that each of the components of $\vec{r}(t)$ is, itself, a function of t . This is true of all vector functions. The functions that make up the components of the vector function $\vec{r}(t)$ are, helpfully, called its **component functions**. In this case:

$$x(t) := t$$

$$y(t) := t^2$$

$$z(t) := 2 \cos(t)$$

This isn't just an interesting observation: these component functions can come in especially handy when attempting to graph vector functions; more on that later.

Example

As another example, suppose we let $\vec{r}(t)$ be the line

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

Then $\vec{r}(t)$ is a vector function. Indeed, it accepts any real number as an argument, and returns a vector that, when its tail is placed at the origin, points to a point on our line.

We can see this even more explicitly if we break $\vec{r}(t)$ out into component form. Let $\vec{v} = \langle a, b, c \rangle$ and $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$. We have:

$$\begin{aligned}\vec{r}(t) &= \vec{r}_0 + t\vec{v} \\ &= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle\end{aligned}$$

The latter is certainly a vector.

Example, cont.

Notice that in this case, the component functions of $\vec{r}(t)$ are:

$$x(t) := x_0 + at$$

$$y(t) := y_0 + bt$$

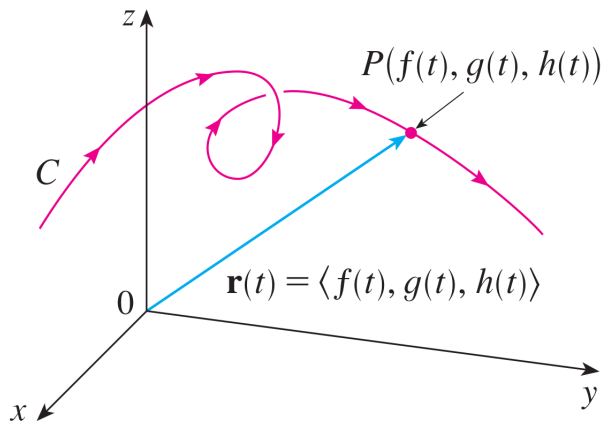
$$z(t) := z_0 + ct$$

In particular, the component functions of the line are its parametric equations! This hints at the reason component functions can be so useful for graphing: in general, the component functions of a vector function are the parametric equations of the graph of that vector function... but perhaps we are getting ahead of ourselves.

Let's step back for a moment. In general, how can we start with an arbitrary vector function and assign a graph to it? Well...

Space Curves

Suppose we have a vector function $\vec{r}(t)$. As t varies, the tip of the vector $\vec{r}(t)$, whose tail we will always place at the origin, traces out a **space curve**, i.e. a curve in \mathbb{R}^3 , C . Therefore, we think of the graph of an arbitrary vector function in \mathbb{R}^3 as a space curve.



Space Curves, cont.

Given the vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ and the corresponding space curve C , the equations $x = f(t)$, $y = g(t)$, and $z = h(t)$ giving the coordinates of each point on C are called the **parametric equations** of C .

In other words, the component functions of the vector function $\vec{r}(t)$ make up (the right side of) the parametric equations of the curve C .

This connection to parametric equations is fundamental in quickly creating the graph of a given vector function, as we shall see in the examples below.

Example

Describe the space curve C_1 which is the graph of the vector function $\vec{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle$.

Note that the component functions of $\vec{r}(t)$ are:

$$x(t) := 1 + t$$

$$y(t) := 2 + 5t$$

$$z(t) := -1 + 6t$$

These are the parametric equations of a line through $(1, 2, -1)$ parallel to the vector $\langle 1, 5, 6 \rangle$, and thus we have a complete description of C_1 .

Example

Sketch the curve C_2 which is the graph of the vector function $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Note once again that the component functions of $\vec{r}(t)$ are:

$$x(t) := \cos(t)$$

$$y(t) := \sin(t)$$

$$z(t) := t$$

Recall that in \mathbb{R}^2 the first two parametric equations plot the unit circle. Therefore, as t increases, x and y will trace out the unit circle in the xy -plane. Simultaneously, as t increases z will rise. The result is a helix:

Example, cont.

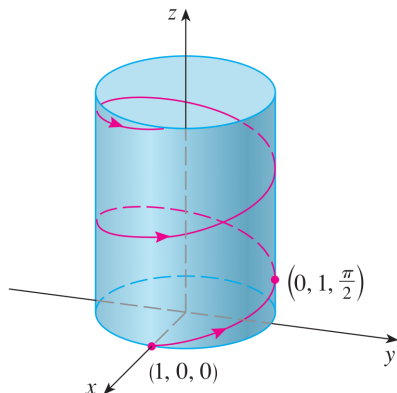


Figure: C_2

(Note that the curve does not include the cylinder, which is only included as scaffolding)

A Note On Computers

It is nontrivial to sketch an arbitrary space curve. You have probably noted that the methods we have used here are ad-hoc, and the sketching problems you will see in this course tend to be confined to using these very specialized techniques. Which is to say, if these problems feel a bit artificially chosen, they are intentionally so.

To sketch a randomly-chosen space curve, it is easiest to use a computer. Bear in mind that you may need to craft your input carefully. For example, to plot a space curve in Wolfram Alpha, you must specify it parametrically.

Limits

In the next section, we will discuss the calculus of vector functions. Toward that end, we first need to define the limit of such a function. We define it in the obvious way:

The **limit of a vector function** $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ **as** t **approaches** a is given by:

$$\lim_{t \rightarrow a} \vec{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

Example

Let $\vec{r}(t) = \left\langle 1 + t^3, te^{-t}, \frac{\sin(t)}{t} \right\rangle$. calculate $\lim_{t \rightarrow 0} \vec{r}(t)$.

By definition, we have:

$$\begin{aligned} \lim_{t \rightarrow 0} \vec{r}(t) &= \left\langle \lim_{t \rightarrow 0} (1 + t^3), \lim_{t \rightarrow 0} (te^{-t}), \lim_{t \rightarrow 0} \left(\frac{\sin(t)}{t} \right) \right\rangle \\ &= \boxed{\langle 1, 0, 1 \rangle} \end{aligned}$$

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1. Sketch the curve in \mathbb{R}^2 with vector equation $\vec{r}(t) = \langle t^2 - 1, t \rangle$. Use arrows to indicate the direction in which the curve flows as t increases.
2. Sketch the space curve with vector equation $\vec{r}(t) = \langle 2 \cos(t), 2 \sin(t), 1 \rangle$. Use arrows to indicate the direction in which the curve flows as t increases.
3. Evaluate $\lim_{t \rightarrow 1} \left\langle \frac{t^2 - t}{t - 1}, \sqrt{t + 8}, \frac{\sin(\pi t)}{\ln(t)} \right\rangle$.

Solutions

1. Use a graphing utility to check your answer.
2. Use a graphing utility to check your answer.
3. $\lim_{t \rightarrow 1} \left\langle \frac{t^2-t}{t-1}, \sqrt{t+8}, \frac{\sin(\pi t)}{\ln(t)} \right\rangle = \langle 1, 3, -\pi \rangle$; L'Hopital's Rule may help you!