13.3: Arc Length

Julia Jackson

Department of Mathematics
The University of Oklahoma

Fall 2021

Overview

Now that we have learned how to calculate derivatives and integrals of vector functions, we examine applications of these. We begin with measuring the length of an arc on a space curve, extending the idea from single-variable calculus.

Table of Contents

A Refresher and a Connection

Recall that if we are given a curve in the xy-plane with parametric equations $x = f(t)$ and $y = g(t)$ with f and g differentiable, the length L of the arc between $t = a$ and $t = b$ is given by the formula:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, \mathrm{d}t$$

To put it another way, if $\vec{r}(t) = \langle f(t), g(t) \rangle$ is a vector function in $\mathbb{R}^2$ with graph C, the length of the arc on C between $t = a$ and $t = b$ is given as above.

Generalizing

How might we generalize the formula on the previous slide to $\mathbb{R}^n$ for $n \geq 3$? Let's adjust it slightly for the answer. If $\vec{r}(t) = \langle f(t), g(t) \rangle$, then $\vec{r}\,'(t) = \langle f'(t), g'(t) \rangle$. The length L of the arc on the graph of $\vec{r}(t)$ between $t = a$ and $t = b$ is then:

$$\begin{aligned} L &= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, \mathrm{d}t \\ &= \int_a^b \left|\vec{r}\,'(t)\right| \mathrm{d}t \end{aligned}$$

It turns out that the latter formula holds in general. Thus, once and for all: if $\vec{r}(t)$ is a vector-valued function in $\mathbb{R}^n$ for *any* n, the length of the arc on the graph of $\vec{r}(t)$ between $t = a$ and $t = b$, where $a \leq b$, is:

$$\boxed{L = \int_a^b \left|\vec{r}\,'(t)\right| \mathrm{d}t}$$

Example

Find the length L of the arc lying on the graph of $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ between $(1, 0, 0)$ and $(1, 0, 2\pi)$.

From above, we have:

$$L = \int_a^b \left| \vec{r}\,'(t) \right| \mathrm{d}t$$

To use this formula, we must determine a and b. Well, note that the tip of $\vec{r}(t)$ is at $(1, 0, 0)$ when $t = 0$ by comparing the third component of $\vec{r}(t)$ to the third coordinate of $(1, 0, 0)$; and the tip of $\vec{r}(t)$ is at $(1, 0, 2\pi)$ when $t = 2\pi$ similarly. Thus, we have:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-\sin(t))^2 + \cos^2(t) + 1}\, \mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{2}\, \mathrm{d}t \\ &= \boxed{2\pi\sqrt{2}} \end{aligned}$$

Toward The Arc Length Function

At this point, we can quickly find the length of an arc between two points on the graph of a vector function. Here's an interesting related question, which will motivate our work through the rest of the section:

Find the point P on the graph of the vector function

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

which is four units away from the point $(1, 0, 0)$ in the direction of increasing t.

So, rather than find the distance between two given points, start at a given point, go a prescribed distance away, and find the second endpoint of the arc. We will solve this problem in two ways, one of which introduces a new concept, important in its own right: the **arc length function**.

First Solution

Find the point P on the graph of the vector function

$$\vec{r}(t) = \left\langle \cos(t), \sin(t), t \right\rangle$$

which is four units away from the point $(1, 0, 0)$ in the direction of increasing t.

Note first from the previous example that $(1, 0, 0)$ is the point on the graph of $\vec{r}(t)$ corresponding to $t = 0$. Beginning at this point, we sweep out an arc on the graph of $\vec{r}(t)$ of length four, and finish at some t-value, say $t = b$. Therefore, using the arc length formula we can set up the following equations:

$$\begin{aligned} 4 &= \int_0^b \sqrt{2}\,\mathrm{d}t \\ &= t\sqrt{2}\Big|_0^b = b\sqrt{2} \end{aligned}$$

First Solution, cont.

Thus, solving for b, we have

$$b = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

and hence P is at the tip of $\vec{r}(2\sqrt{2}) = \left\langle \cos(2\sqrt{2}), \sin(2\sqrt{2}), 2\sqrt{2} \right\rangle$:

$$\boxed{P = \left(\cos(2\sqrt{2}), \sin(2\sqrt{2}), 2\sqrt{2} \right)}$$

Follow-Up

Great! Our motivating problem is really just a twist on the first type of problem we learned to solve.

Now let's take a slightly different perspective, to introduce a new concept: the **arc length function**.

Second Solution

Find the point P on the graph of the vector function

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

which is four units away from the point $(1, 0, 0)$ in the direction of increasing t.

Again, the point $(1, 0, 0)$ on the graph of $\vec{r}(t)$ corresponds to $t = 0$. Now we proceed slightly differently: note that, using the arc length formula, we have that the length of any arc on the graph of $\vec{r}(t)$ starting at $t = 0$ and ending at some second, arbitrary t-value is given by the function:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{2}\,\mathrm{d}u = \left. u\sqrt{2}\right|_0^t \\ &= t\sqrt{2} \end{aligned}$$

This is called the **arc length function of $\vec{r}(t)$ starting at $t = 0$.**

Second Solution, cont.

The arc length function truly is a function: given any second t-value $t = b$ with $b \geq a$, we can evaluate $s(b)$ and obtain the length of the arc on the graph of $\vec{r}(t)$ between $t = 0$ and $t = b$.

Second Solution, cont.

How does this help us toward an answer? Here's the key insight:

From the equation $s(t) = t\sqrt{2}$, we can solve for t in terms of the variable s:

$$t = \frac{s}{\sqrt{2}}$$

Now we can swap out t in $\vec{r}(t)$ to obtain a new **parametrization** of the graph of $\vec{r}(t)$:

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$$

This is called the **reparametrization of $\vec{r}$ with respect to arc length**.

Second Solution, cont.

This parametrization of the graph of $\vec{r}$ is interesting in its own right. Rather than finding the points on the graph of $\vec{r}$ by relying on an outside parameter t, one can find them using only their distance s from $(1, 0, 0)$!

So, for example, to find what point is four units away from $(1, 0, 0)$ on the graph of $\vec{r}(s)$ in the direction of increasing t, we plug in $s = 4$:

$$\vec{r}(4) = \left\langle \cos\left(\frac{4}{\sqrt{2}}\right), \sin\left(\frac{4}{\sqrt{2}}\right), \frac{4}{\sqrt{2}} \right\rangle$$

and obtain the endpoint

$$\boxed{\left(\cos(2\sqrt{2}), \sin(2\sqrt{2}), 2\sqrt{2}\right)}$$

as desired.

The Arc Length Function

Generally, given a vector function $\vec{r}(t)$, the length of the arc on the graph of $\vec{r}(t)$ between $t = a$ and any second value of t is given by the function:

$$\boxed{s(t) = \int_a^t \left|\vec{r}\,'(u)\right| \mathrm{d}u}$$

This is called the **arc length function of $\vec{r}(t)$ starting at $t = a$**. Notice: u is a *dummy variable* of integration, and t is the actual variable of interest to the function $s(t)$.

Dummy Variables

What is a *dummy variable*? It's an extra variable that we introduce whose sole purpose is to help us evaluate the integral. The reason it's there is because when we evaluate, say, $s(b)$, we only want to plug b into the upper bound on the integral, not the function under the integral sign. Indeed, the arc length formula tells us that the length L of the arc on the graph of $\vec{r}(t)$ between $t = a$ and $t = b$ is

$$L = \int_a^b \left|\vec{r}\,'(t)\right| \mathrm{d}t = \int_a^b \left|\vec{r}\,'(u)\right| \mathrm{d}u$$

and NOT

$$\cancel{L = \int_a^b \left|\vec{r}\,'(b)\right| \mathrm{d}b}$$

Introducing a dummy variable assures that b will *only* be plugged into $s(t)$ where it's supposed to: in the upper bound of the integral.

Summary of the Second Method

Let's put this all together, one last time:

Given a vector function $\vec{r}(t)$, its graph C, and an initial point (u, v, w) on C, to find the point on C that is ℓ units away from (u, v, w) as t increases:

1. Find the value of t, say a, such that $\vec{r}(a) = \langle u, v, w \rangle$.
2. Calculate the arc length function $s(t)$ of $\vec{r}(t)$ starting at $t = a$.
3. Solve the arc length function for t, to write $t = f(s)$.
4. Plug $t = f(s)$ into $\vec{r}(t)$ to obtain $\vec{r}(s)$.
5. Plug ℓ in for s in $\vec{r}(s)$ to get your answer.

Table of Contents

Exercises

1. Find the length L_1 of the curve given by the graph of $\vec{r}(t) = \left\langle 2t, t^2, \frac{t^3}{3} \right\rangle$ with $0 \leq t \leq 1$.
2. Find the arc length function $s(t)$ for the curve given by the graph of $\vec{r}(t) = \left\langle e^t \sin(t), e^t \cos(t), \sqrt{2}e^t \right\rangle$ measured from $(0, 1, \sqrt{2})$ in the direction increasing t.
3. Reparametrize the curve in the previous problem with respect to arc length.
4. Find the point P four units from $(0, 1, \sqrt{2})$ in the direction of increasing t, along the curve from the previous two problems.

Solutions

1. $L_1 = \dfrac{7}{3}$
2. $s(t) = 2e^t - 2$
3. $\vec{r}(s) =$
   $\left\langle e^{\ln\left(\frac{s+2}{2}\right)} \sin\left(\ln\left(\frac{s+2}{2}\right)\right), e^{\ln\left(\frac{s+2}{2}\right)} \cos\left(\ln\left(\frac{s+2}{2}\right)\right), \sqrt{2} e^{\ln\left(\frac{s+2}{2}\right)} \right\rangle$
4. $P = \left(e^{\ln(3)} \sin\left(\ln(3)\right), e^{\ln(3)} \cos\left(\ln(3)\right), \sqrt{2} e^{\ln(3)}\right)$