14.6: Directional Derivatives and the Gradient Vector

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Overview

In section 14.3 we noted that the instantaneous rate of change, i.e. the derivative, of a multivariable function f at a point P is not unique; it varies depending on which direction one wishes to measure the change in: thinking of a contour map of a two-variable function like a topographic map, the steepness of one's path away from a given point on a hillside changes depending on the direction one walks in.

We then went on to measure the instantaneous rate of change of a two-variable function in exactly two directions: parallel to the x-axis in the direction of increasing x, and parallel to the y-axis in the direction of increasing y. But what about all the other possible directions? That will be the subject of this section: the so-called **directional derivatives**. Along the way we will a more generally useful concept that can also be used to simplify our work here: the **gradient vector**.

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What We Have So Far

In §14.3, we learned that the instantaneous rate of change of a two-variable function $f(x, y)$ parallel to the x-axis in the direction of increasing x is given by the partial derivative $f_x(x, y)$, and similarly the instantaneous rate of change of $f(x, y)$ parallel to the y-axis in the direction of increasing y is given by the partial derivative $f_y(x, y)$.

A pithier way of saying this is that $f_x(x, y)$ represents the instantaneous rate of change of $f(x, y)$ in the direction of the unit vector $\vec{i} = \langle 1, 0\rangle$, and $f_y(x, y)$ represents the instantaneous rate of change of $f(x, y)$ in the direction of the unit vector $\vec{j} = \langle 0, 1\rangle$.

The Setup

Now suppose we wanted to know the instantaneous rate of change of a $f(x,y)$ in a different direction. For example, suppose we have $f(x,y) = \sin(x) + e^{xy}$. What is its derivative at $(0,1)$ in the direction of the unit vector $\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$?

This looks like it might be quite complicated to answer; do we need some new concept of a partial derivative in any direction? No! It turns out that the two directions we already learned are sufficient.

The Result

Theorem: If f is a differentiable function of x and y, then its **directional derivative** in the direction of the unit vector $\vec{u} = \langle p, q \rangle$ is:

$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)p + f_y(x,y)q}$$

Note that $D_{\vec{i}} f(x,y) = f_x(x,y)$ or $D_{\vec{j}} f(x,y) = f_y(x,y)$, respectively. In other words, this theorem is consistent with what we've already learned about derivatives of multivariable functions, a very good thing, indeed.

Example

Calculate the derivative of $g(x,y) = \sin(x) + e^{xy}$ at the point $(0,1)$ in the direction of the vector $\vec{w} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

First, notice that the vector $\vec{w}$ is a unit vector, as:

$$|\vec{w}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9+16}{25}} = \sqrt{1} = 1$$

Therefore, we can apply the formula for the directional derivative from the previous slide. To do so, we will need the following information:

$$\begin{aligned} g_x(x,y) &= \cos(x) + ye^{xy} && \Longrightarrow && g_x(0,1) = 2 \\ g_y(x,y) &= xe^{xy} && \Longrightarrow && g_y(0,1) = 0 \end{aligned}$$

Example, cont.

Therefore, the derivative of $g(x,y)$ in the direction of $\vec{w}$ at the point $(0,1)$ is:

$$\begin{aligned} D_{\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle} g(0,1) &= \frac{3}{5} g_x(0,1) + \frac{4}{5} g_y(0,1) \\ &= 2\left(\frac{3}{5}\right) + 0 \\ &= \boxed{\frac{6}{5}} \end{aligned}$$

Example

Calculate the derivative of the function $h(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $\vec{w} = \langle 2,5\rangle$.

There's one major difference between this example and the previous one: the vector $\vec{w} = \langle 2,5\rangle$ is *not* a unit vector! The only tool we have at our disposal to compute directional derivatives is the theorem from a few slides back, but to use the theorem our direction vector must be a unit vector. So... what can we do?

Example

Let's start by introducing the following vector:

$$\vec{u} := \frac{\vec{w}}{|\vec{w}|} = \frac{\langle 2,5\rangle}{\sqrt{29}} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

Why this vector? Well, first, notice that $\vec{u}$ points in the same direction as $\vec{w}$ (as the two are scalar multiples of one another). Therefore:

$$D_{\vec{w}}h(2,-1) = D_{\vec{u}}h(2,-1)$$

Second, and just as important, by construction $\vec{u}$ is a unit vector (although you can verify this directly if you wish), which means that we can use the directional derivative formula to compute the right-hand side of this equation.

Example, cont.

To use that formula, we first need the following information:

$$\begin{aligned} h_x(x,y) &= 2xy^3 & &\Longrightarrow & h_x(2,-1) &= -4 \\ h_y(x,y) &= 3x^2y^2 - 4 & &\Longrightarrow & h_y(2,-1) &= 8 \end{aligned}$$

Therefore, putting everything together, we have:

$$\begin{aligned} D_{\vec{w}}h(2,-1) &= D_{\vec{u}}h(2,-1) \\ &= \frac{2}{\sqrt{29}}h_x(2,-1) + \frac{5}{\sqrt{29}}h_y(2,-1) \\ &= \frac{-8}{\sqrt{29}} + \frac{40}{\sqrt{29}} = \boxed{\frac{32}{\sqrt{29}}} \end{aligned}$$

Key Concept: The Gradient Vector

To facilitate calculating directional derivatives in higher dimensions and obtaining several results below, we introduce the following surprisingly useful concept:

For f a function of the variables $x_1, x_2, \dots, x_n$, we define the **gradient vector** of f:

$$\nabla f(x_1, x_2, \dots, x_n) = \left\langle f_{x_1}(x_1,\dots,x_n), f_{x_2}(x_1,\dots,x_n), \dots, f_{x_n}(x_1,\dots,x_n) \right\rangle$$

In particular, if $f(x,y)$ is a function of two variables, then:

$$\nabla f(x,y) = \left\langle f_x(x,y), f_y(x,y) \right\rangle$$

With this definition, note that we may write:

$$D_{\vec{u}}f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

Key Concept: The Gradient Vector, cont.

Why the latter? Well, if we write $\vec{u} = \langle p, q \rangle$ as we did in the formula for the directional derivative, then:

$$\begin{aligned} \nabla f(x,y) \cdot \vec{u} &= \big\langle f_x(x,y), f_y(x,y) \big\rangle \cdot \langle p, q \rangle \\ &= f_x(x,y)p + f_y(x,y)q \\ &= D_{\vec{u}} f(x,y) \end{aligned}$$

as claimed.

Functions of More Variables

To extend the directional derivative to higher dimensions, let $f(x_1, \ldots, x_n)$ be a function differentiable in each x_i, and let $\vec{u} = \langle u_1, \ldots, u_n \rangle$ be a unit vector. We then define the derivative of f at the point $(a_1, \ldots, a_n)$ in the direction of $\vec{u}$ to be:

$$D_{\vec{u}} f(a_1, \ldots, a_n) = \nabla f(a_1, \ldots, a_n) \cdot \vec{u}$$

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Maximizing the Directional Derivative

Suppose that f is a differentiable function of two or three variables. We now know how to take the derivative of f at a point in any direction. A natural question to ask at this point is in what direction f has the greatest instantaneous rate of change (after all, this is calculus, where maximizing and minimizing functions is our bread and butter). This has a surprisingly simple answer!

Theorem: Suppose that f is a differentiable function of two or three variables. The maximum value of $D_{\vec{u}} f$ at a point is $|\nabla f|$, and it occurs when the unit vector $\vec{u}$ points in the same direction as ∇f.

Maximizng the Directional Derivative, cont.

Why would this be? Well, assume for now that f is a two-variable function, $f(x, y)$. Recall that:

$$D_{\vec{u}} f(a, b) = \nabla f(a, b) \cdot \vec{u} = \left|\nabla f(a, b)\right| \left|\vec{u}\right| \cos(\theta) = \left|\nabla f(a, b)\right| \cos(\theta)$$

Where θ is the angle between $\nabla f(a, b)$ and $\vec{u}$. This is at a maximum when $\cos(\theta) = 1$, i.e. when $\nabla f(a, b)$ and $\vec{u}$ point in the same direction, as the theorem on the previous slide claims. Furthermore, when $\cos(\theta) = 1$, then $D_{\vec{u}} f(a, b) = \left|\nabla f(a, b)\right|$, again, as the theorem claims.

The argument when f is a three-variable function is identical.

Example

Let $g(x, y) = \sin(x) + e^{xy}$. Find the direction in which $g(x, y)$ has the maximum instantaneous rate of change at the point $(0, 1)$. What is that rate?

The theorem above tells us that the maximum instantaneous rate of change of $g(x, y)$ at $(0, 1)$ is $\left|\nabla g(0, 1)\right|$ and this rate occurs when we move from $(0, 1)$ in the direction of $\nabla g(0, 1)$ (or, given as a unit vector, when we move in the direction of $\frac{\nabla g(0,1)}{\left|\nabla g(0,1)\right|}$).

In a previous example, we calculated

$$\nabla g(0, 1) = \left\langle g_x(0, 1), g_y(0, 1)\right\rangle = \langle 2, 0\rangle$$

Therefore, the maximum instantaneous rate of change of $g(x, y)$ at $(0, 1)$ is $\left|\langle 2, 0\rangle\right| = 2$ and this rate occurs when we head out from $(0, 1)$ in the direction $\frac{\nabla g(0,1)}{\left|\nabla g(0,1)\right|} = \langle 1, 0\rangle$

Tangent Planes to Surfaces

Now we'll change gears slightly and examine how the gradient vector can be applied to our previous work on tangent planes. Recall from §14.4 that an equation of the plane that lies tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b)) = (a, b, c)$ is:

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + c$$

Suppose now that we wish to find an equation of a plane that lies tangent to a surface that isn't written explicitly as $z = f(x, y)$, but instead gives an implicit relationship between x, y, and z, such as: $x^2 + y^2 + z^2 = 1$, or $x^2 + y^2 + z^2 + z^4 = 1$. How could we proceed? To get the answer, let's twist our work on tangent planes of explicitly-defined surfaces slightly.

Tangent Planes to Surfaces, cont.

Let's begin again with the equation $z = f(x, y)$. Gather everything on one side of the equation to obtain $f(x, y) - z = 0$. Now z is no longer given as an explicit function of x and y, which is exactly the case we are interested in. On the one hand, an equation of the plane that lies tangent to this surface at (a, b, c) is still:

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) + c - z = 0$$

On the other hand, let's write $F(x, y, z) = f(x, y) - z$. Then note that:

$$F_x(x, y, z) = f_x(x, y), \quad F_y(x, y, z) = f_y(x, y), \quad F_z(x, y, z) = -1$$

With this in mind, notice that we can rewrite the equation of our tangent plane as:

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

i.e.:

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c\rangle = 0$$

Result

It turns out that the latter is the formula we were after. To summarize: Suppose we are given a surface $F(x, y, z) = 0$. An equation of the plane tangent to this surface at the point (a, b, c) is:

$$\boxed{\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0}$$

In particular, $\nabla F(a, b, c)$ is a normal vector for the tangent plane.

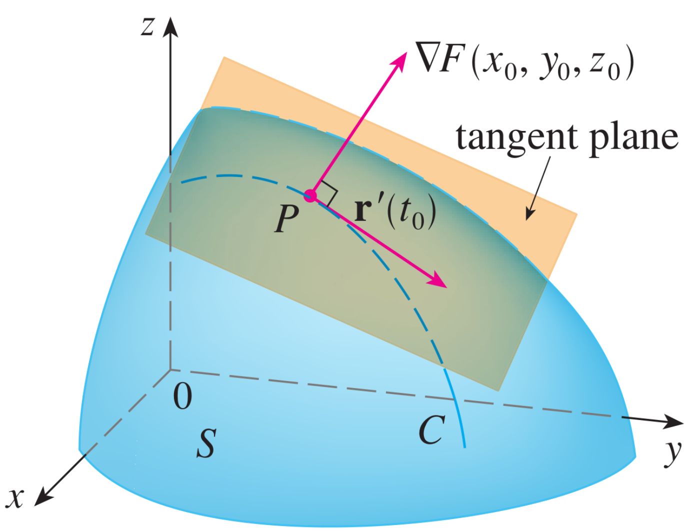
This formula for the tangent plane to a surface is more general than the one we previously learned. Indeed, we showed above that it holds for surfaces of the form $z = f(x, y)$ if we rewrite them as $f(x, y) - z = 0$, and the result here states that it also works for surfaces that aren't given explicitly as $z = f(x, y)$.

Result, cont.

One additional note: the line passing through (a, b, c) whose direction vector is $\nabla F(a, b, c)$ is called the **normal line** to the surface at (a, b, c).

The Picture

Here's a picture of the situation, with $P = (x_0, y_0, z_0)$:



Example

Find equations of the tangent plane T and normal line L to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

at the point $(-2, 1, -3)$.

We begin with T. The result on a previous slide gives us a formula for an equation of the tangent plane to any surface at any point on that surface, so long as that surface is of the form $F(x, y, z) = 0$. Therefore, we begin by putting our ellipsoid in the necessary form:

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3 = 0$$

and then introduce a name for the right-hand side:

$$F(x, y, z) := \frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3$$

Example, cont.

This gives:

$$\nabla F(x,y,z) = \left\langle \frac{x}{2}, 2y, \frac{2}{9}z \right\rangle$$

so that:

$$\nabla F(-2,1,-3) = \left\langle -1, 2, \frac{-2}{3} \right\rangle$$

Therefore, by the formula on a previous slide, an equation of T is:

$$\begin{aligned} &\nabla F(-2,1,3) \cdot \langle x+2, y-1, z+3 \rangle = 0 \\ \Rightarrow &\left\langle -1, 2, \frac{-2}{3} \right\rangle \cdot \langle x+2, y-1, z+3 \rangle = 0 \\ \Rightarrow &-(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0 \\ \Rightarrow &\boxed{-x + 2y - \frac{2}{3}z = 6} \end{aligned}$$

Example, cont.

Now we turn to L. By definition, the normal line L is the line through $(-2,1,-3)$ whose direction vector is the normal vector to the tangent plane at $(-2,1,-3)$. From the previous slide, that normal vector is $\nabla F(-2,1,-3) = \left\langle -1, 2, \frac{-2}{3} \right\rangle$. Thus, by our work in section 12.5, a set of parametric equations of L is:

$$\boxed{\begin{aligned} x &= -2 - t \\ y &= 1 + 2t \\ z &= -3 - \frac{2}{3}t \end{aligned}}$$

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Exercises

1. Let $f(x,y) = x/y$. Find the instantaneous rate of change of $f(x,y)$ at the point $(2,1)$ in the direction of $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.
2. Let $g(x,y,z) = x\sin(yz)$. Calculate the directional derivative of $g(x,y,z)$ at the point $(1,3,0)$ in the direction of $\langle 1,2,-1 \rangle$.
3. Let $h(x,y) = xe^y$. Calculate the instantaneous rate of change of $h(x,y)$ at $(2,0)$ as we head toward the point $(\frac{1}{2}, 2)$.
4. Let $s(x,y) = 4y\sqrt{x}$. Find the maximum instantaneous rate of change of $s(x,y)$ at $(4,1)$ and give the direction in which it occurs.
5. Give equations of the tangent plane S and the normal line L_0 to the surface $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ at the point $(3,3,5)$.

Solutions

1. The instantaneous rate of change of $f(x, y)$ at the point $(2, 1)$ in the direction of $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$ is $D_{\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle} f(2, 1) = -1$
2. $D_{\langle 1,2,-1 \rangle} g(1, 3, 0) = D_{\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle} g(1, 3, 0) = \frac{-3}{\sqrt{6}} = \frac{-\sqrt{6}}{2}$
3. The instantaneous rate of change of $h(x, y)$ at the point $(2, 0)$ as we head toward the point $\left(\frac{1}{2}, 2\right)$ is:

$$D_{\left\langle \frac{-3}{2}, 2 \right\rangle} h(2, 0) = D_{\left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle} h(2, 0) = 1$$

Solutions, cont.

4. The maximum instantaneous rate of change of $s(x, y)$ at $(4, 1)$ is $\sqrt{65}$, and this maximum rate occurs in the direction of $\vec{u} = \left\langle \frac{1}{\sqrt{65}}, \frac{8}{\sqrt{65}} \right\rangle$.
5. An equation of S is $4x + 4y + 4z = 44$, and a set of parametric equations of L_0 is:

$$\begin{aligned} x &= 4t + 3 \\ y &= 4t + 3 \\ z &= 4t + 5 \end{aligned}$$

(There are, of course, many other acceptable answers here!)