14.8: Lagrange Multipliers

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Overview

In the previous section, in two distinct contexts we wanted to find the absolute maximum and minimum values of a two- or three-variable function subject to some constraint.

The first was when calculating the absolute maximum and minimum values of a two-variable function on a closed, bounded subset of its domain. In these problems, each boundary component provided a distinct constraint, and we sought the absolute maximum and minimum values of our function on each on our way to solving the overall problem.

Overview, cont.

The second was in solving optimization problems. In these, we wanted to calculate the absolute maximum or minimum value of a three- (or sometimes two-) variable function subject to a directly stated constraint.

In this section, we explore a second method for solving such problems. This new technique, called the method of Lagrange multipliers, is a handy alternative to the method we learned in the previous section. Each is useful in certain contexts, and mastery of both methods will maximize your flexibility in solving the problems of the previous section.

Table of Contents

The Method

To find the absolute maximum and minimum values of a function $f(x, y, z)$ (resp. $f(x, y)$) subject to the constraint $g(x, y, z) = k$ (resp. $g(x, y) = k$) for k a constant, perform the following:

1. Find all constants λ and ordered triples (x, y, z) (resp. ordered pairs (x, y)) such that
$$\nabla f = \lambda \nabla g$$
and
$$g = k$$
are simultaneously true.
2. Evaluate f at all the ordered triples (resp. ordered pairs) from the previous step. The largest value of f is the absolute maximum value of f subject to the constraint, and the smallest value of f is the absolute minimum value of f subject to the constraint.

Example

Find all the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Here we want to find all extreme values of the function $f(x, y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$. We already know a method for solving this problem, thanks to our work in the previous section, but let's see how the method of Lagrange multipliers handles it.

We begin by introducing a function name to the non-constant part of the constraint:

$$g(x, y) := x^2 + y^2$$

Example, cont.

This done, our next goal is to find all constants λ and ordered pairs (x, y) satisfying

$$\nabla f(x, y) = \langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle = \lambda \nabla g(x, y)$$

and

$$x^2 + y^2 = 1$$

By comparing the components of the vectors (more precisely, vector fields) in the first equation, we arrive at the more useful set of equations:

$$2x = \lambda 2x \tag{1}$$

$$4y = \lambda 2y \tag{2}$$

$$x^2 + y^2 = 1 \tag{3}$$

Thus, our goal, according to the method of Lagrange multipliers, is to find all constants λ and ordered pairs (x, y) satisfying all three of these equations at once.

Example, cont.

There are many ways to solve this system, but here's one that stands out to me: First, note that:

$$\begin{aligned} (1) \Rightarrow\ & 2x(1-\lambda) = 0 \\ \Rightarrow\ & x = 0 \text{ or } \lambda = 1 \end{aligned}$$

Therefore, for (1) to be true, we must have either $x = 0$ or $\lambda = 1$. We now investigate each case separately.

Example, cont.

First, suppose that $x = 0$. Note that:

$$\begin{aligned}(3) \text{ and } x = 0 &\Rightarrow y^2 = 1 \\ &\Rightarrow y = \pm 1\end{aligned}$$

Therefore, if $x = 0$, (1) and (3) can only be true at the same time if $y = \pm 1$. But then note that:

$$\begin{aligned}(2),\ x = 0, \text{ and } y = 1 &\Rightarrow 4 = 2\lambda \\ &\Rightarrow \lambda = 2\end{aligned}$$

and

$$\begin{aligned}(2),\ x = 0, \text{ and } y = -1 &\Rightarrow -4 = -2\lambda \\ &\Rightarrow \lambda = 2\end{aligned}$$

Example, cont.

Therefore, if $x = 0$, then all three equations are true at the same time:

at the point $(0, 1)$ with $\lambda = 2$

or

at the point $(0, -1)$ with $\lambda = 2$

We will hang onto the points $(0, 1)$ and $(0, -1)$ for later.

Example, cont.

Now we investigate the case where $\lambda = 1$. Note that:

$$\begin{aligned}
(2) \text{ and } \lambda = 1 &\Rightarrow 4y = 2y \\
&\Rightarrow 2y = 0 \\
&\Rightarrow y = 0
\end{aligned}$$

Therefore, if $\lambda = 1$, (1) and (2) can only be true at the same time if $y = 0$. But then note that:

$$\begin{aligned}
(3),\ \lambda = 1, \text{ and } y = 0 &\Rightarrow x^2 = 1 \\
&\Rightarrow x = \pm 1
\end{aligned}$$

Example, cont.

Therefore, if $\lambda = 1$, then all three equations are true at the same time:

at the point $(1, 0)$ with $\lambda = 1$

or

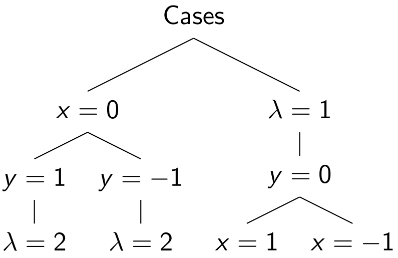
at the point $(-1, 0)$ with $\lambda = 1$

Example, cont.

At this point, we've found all possible solutions to this system of equations, as we've exhausted all the cases we discovered. Indeed, we found that for the first equation to be true, it must be that either $x = 0$ or $\lambda = 1$, and then we investigated which possible values of y and λ; and x and y, respectively, can satisfy the rest of the system in each case.

You may find it somewhat challenging to keep track of which case you're working on at a given time, especially if the number of cases grows large; and you may find it especially challenging to know when you've finished investigating all of the cases you found. To that end, you may find a tree like the one on the following slide helpful.

Example, cont.



This is a chart that you can construct as you go through solving the system of equations. For example, in this problem we would proceed as follows: the first equation tells us that for all equations in the system to be true simultaneously, we must have that either $x = 0$ or $\lambda = 1$. So, we add a node below “Cases” for each of these. Then, when we investigate the $x = 0$ case, we see that y must either be 1 or -1 when $x = 0$, so we add these nodes below $x = 0$, etc. When all nodes terminate with a value of x, y, and λ, you're finished!

Example, cont.

Let's finish up the problem. In solving the system of equations above, we obtained the following points:

$$(0,1),\ (0,-1),\ (1,0),\ \text{and}\ (-1,0)$$

We now plug all of these into $f(x,y)$. The method of Lagrange multipliers tells us that the largest value we get is the absolute maximum value of $f(x,y)$ on the circle $x^2+y^2=1$; and the smallest value we get is the absolute minimum value of $f(x,y)$ on the circle $x^2+y^2=1$.

Example, cont.

We have:

$$\begin{aligned} f(0,1) &= 0^2 + 2 \cdot 1^2 = 2 \\ f(0,-1) &= 2 \\ f(1,0) &= 1 \\ f(-1,0) &= 1 \end{aligned}$$

Therefore, the absolute maximum value of $f(x, y)$ on the unit circle is $f(0,1) = f(0,-1) = 2$, and the absolute minimum value of $f(x, y)$ on the unit circle is $f(1,0) = f(-1,0) = 1$.

Typical Challenges

As we got a peek at in the previous example, the most complicated portion of using the method of Lagrange multipliers is typically solving the system of (generally non-linear) equations which arises from the equations $\nabla f = \lambda \nabla g$ and $g = k$. In particular, the challenge often lies in keeping track of a number of cases.

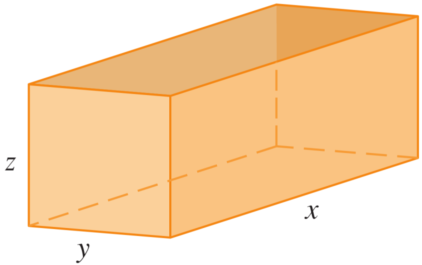
Below we examine another example, and give a handy technique that can be used to work one's way through a number of the typical systems of equations that arise in these problems.

Example

Find the maximum volume of a rectangular box with no lid constructed from 12m^2 of cardboard.

You may remember this problem from §14.7. Now we will use the method of Lagrange multipliers to obtain the same solution we did there.

We begin, again, by sketching and labelling the box.



Example

Once again, we wish to calculate the maximum volume of this box. Given our labels in the diagram, this means that we wish to find the maximum value of the three-variable function:

$$V(x, y, z) := xyz$$

subject to the constraint that the box must constructed from exactly 12m^2 of cardboard. That is, the surface area of the box (which, again, has no top) must be exactly 12m^2. Given our diagram labels on the previous slide, this means that x, y, and z must satisfy the constraint:

$$xy + 2xz + 2yz = 12$$

Ah! We see from this setup that the method of Lagrange multipliers could be used to solve this problem. So... let's give it a try!

Example, cont.

We begin by giving a name to the non-constant portion of the constraint equation. We'll stick with $g(x, y, z)$ for our name, since that's the one used in the statement of the method of Lagrange multipliers. That is, we declare:

$$g(x, y, z) := xy + 2xz + 2yz$$

With this squared away, the method of Lagrange multipliers says that to calculate the absolute maximum and minimum values of $V(x, y, z)$ subject to the constraint $g(x, y, z) = 12$, we must first find all ordered triples (x, y, z) and constants λ such that the equations:

$$\nabla V(x, y, z) = \langle yz, xz, xy \rangle = \lambda \langle y + 2z, x + 2z, 2x + 2y \rangle = \lambda \nabla g(x, y, z)$$

and

$$xy + 2xz + 2yz = 12$$

are true at the same time.

Example, cont.

By comparing the components of the vectors (more precisely, vector fields) of the first equation on the previous slide, we arrive at a more workable set of equations:

$$yz = \lambda(y + 2z) \tag{1}$$

$$xz = \lambda(x + 2z) \tag{2}$$

$$xy = \lambda(2x + 2y) \tag{3}$$

$$xy + 2xz + 2yz = 12 \tag{4}$$

Thus, our goal is to find all constants λ and all ordered triples (x, y, z) that satisfy all three of these equations at once.

Take a few minutes, and see if you can solve this system of equations on your own before moving on.

Example, cont.

There are many ways to solve any system of equations, but here's one technique that you may not have seen before, and that I find particularly useful in this case (and in many others).

Note that the left-hand sides of the first three equations are all very nearly the same, with one variable missing in each. So, we'll start by multiplying each of these by its "missing" variable to obtain the following modified set of equations:

$$xyz = \lambda x(y + 2z) \tag{1'}$$
$$xyz = \lambda y(x + 2z) \tag{2'}$$
$$xyz = \lambda z(2x + 2y) \tag{3'}$$
$$xy + 2xz + 2yz = 12 \tag{4}$$

The list of equations on the previous slide are all true at the same time precisely when this list of equations is. Therefore, we will work with this modified list.

Example, cont.

Now would be an excellent time to remind ourselves of the values that x, y, and z can take. All three variables represent edge lengths, and therefore can take only positive values in order to remain physically meaningful. There is, of course, another major restriction on all three (the constraint equation), but at the very least it's good to know that any cases we come across, or any ordered triples we find that involve negative values of x, y, or z may be discarded immediately.

Now let's proceed in solving the system on the previous slide.

Example, cont.

The left-hand side of the first three equations being the same provides a major advantage in how we may proceed, as this means that the right-hand sides of these equations must also be the same. In particular:

$$\begin{aligned}(1') \text{ and } (2') &\Rightarrow\ \lambda x(y+2z) = \lambda y(x+2z)\\ &\Rightarrow\ \lambda xy + 2\lambda xz = \lambda xy + 2\lambda yz\\ &\Rightarrow 2\lambda z(x-y) = 0\end{aligned}$$

Therefore, if $(1')$ and $(2')$ are true at the same time, we must have at least one of:

$$\lambda = 0, \qquad z = 0, \quad \text{or} \quad x = y$$

Example, cont.

Let's examine each case.

First, since z must be a positive number (as, again, it's the length of one of the edges of the box), we can immediately discard the case $z = 0$.

Similarly, consider the case $\lambda = 0$. Plugging $\lambda = 0$ into, say, $(1')$ gives:

$$xyz = 0$$

which in turns implies that at least one of x, y, and z is zero. So, again, we can discard this case, as all three of these numbers must be positive.

Thus, the only way that $(1')$ and $(2')$ can be true at the same time is if $x = y$.

Example, cont.

With this new understanding in mind, let's proceed in a similar way with equations $(1')$ and $(3')$:

$$\begin{aligned}
(1'),\ (3'),\ \text{and } x = y \Rightarrow\ & \lambda x(x + 2z) = \lambda z(4x) \\
\Rightarrow\ & \lambda x^2 + 2\lambda xz = 4\lambda xz \\
\Rightarrow\ & \lambda x^2 = 2\lambda xz \\
\Rightarrow\ & x\lambda(x - 2z) = 0
\end{aligned}$$

Therefore, if $(1')$, $(2')$, and $(3')$ are true at the same time, we must have that:

$$x = 0 \quad \text{or} \quad x = 2z$$

Example, cont.

Just as before, we can immediately discard the case $x = 0$, since x must be a positive number.

Therefore, combining all of our above work, the only way that $(1')$, $(2')$, and $(3')$ can be true at the same time is if:

$$x = y \quad \text{and} \quad x = 2z$$

We're now in a great position to find all of the ordered triples (x, y, z) and values of λ that satisfy all four equations at the same time.

Example, cont.

First, notice that:

$$\begin{aligned} x = y,\ x = 2z,\ \text{and } (4) \Rightarrow\ & x^2 + x^2 + x^2 = 12 \\ \Rightarrow\ & x^2 = 4 \\ \Rightarrow\ & x = \pm 2 \end{aligned}$$

Since x cannot be negative, this means that the only x-value that can satisfy all four equations at once is $x = 2$. For this value of x, we also have $y = 2$ and $z = 1$. Further:

$$\begin{aligned} x = 2,\ y = 2,\ z = 1,\ \text{and } (1') \Rightarrow\ & 4 = 8\lambda \\ \Rightarrow\ & \lambda = \frac{1}{2} \end{aligned}$$

Thus, the only way that all four equations in our system can be true at the same time is if $(x, y, z) = (2, 2, 1)$ and $\lambda = \frac{1}{2}$

Example, cont.

To be certain that we've covered all possible cases, we might make a tree like the following as we go:

- Cases
  - $\cancel{\lambda = 0}$
  - $\cancel{z = 0}$
  - $x = y$
    - $\cancel{x = 0}$
    - $x = 2z$
      - $x = 2$
        - $y = 2$
          - $z = 1$
            - $\lambda = \frac{1}{2}$

Example, cont.

Therefore, by the method of Lagrange multipliers, the absolute maximum volume of a rectangular box with no lid, constructed from 12m^2 of material is

$$V(2,2,1) = 4\text{m}^3$$

Comments

The previous examples demonstrate both the utility and the pitfalls of the method of Lagrange multipliers. Indeed, the first example was fairly efficient to solve using this method whereas some cleverness is required to solve it using the methods of the previous section (see the first exercise at the end of the section). On the other hand, the method of the previous section fairly efficiently solved our optimization problem, whereas the method of Lagrange multipliers required a bit of cleverness, as well as careful tracking of a number of cases, to solve a system of four equations.

This is a general feature of these two methods. The method of Lagrange multipliers trades the potential complexity of solving a slightly longer system of equations for the potential complexity of computing a list of partial derivatives and setting them equal to zero simultaneously.

Comments, cont.

All of this is to say that both methods have their utility, and you should bear both in mind when trying to find absolute extrema on a closed, bounded set; or when trying to solve optimization problems. Thoughtful practice and experience will help you to decide which will prove more useful for a given problem.

Table of Contents

Exercises

1. Use the methods from the previous section to solve the first example, above. Which do you find easier in this case: those methods, or the method of Lagrange multipliers?
2. Find the extreme values of the function $f(x, y) = 3x + y$ on the circle $x^2 + y^2 = 10$ two ways: first, by using Lagrange multipliers, and second, by using the methods of the previous section. Which do you prefer?
3. Find the extreme values of $h(x, y) = x^2 + 2y^2$ on the unit disk $x^2 + y^2 \leq 1$. At which step in the solution can the method of Lagrange multipliers be used?
4. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

Solutions

2. The absolute minimum value of $f(x, y)$ on the circle $x^2 + y^2 = 10$ is $f(-3, -1) = -10$; and the absolute maximum value of $f(x, y)$ on this same circle is $f(3, 1) = 10$.
3. The absolute minimum value of $h(x, y)$ on the unit disk is $h(0, 0) = 0$; and the absolute maximum value of $h(x, y)$ on the unit disk is $h(0, 1) = h(0, -1) = 2$. The method of Lagrange multipliers can only help us find the absolute maximum and minimum values of $h(x, y)$ on the boundary of the disk, i.e. on the unit circle.
4. The point on the sphere $x^2 + y^2 + z^2 = 4$ which is closest to the point $(3, 1, -1)$ is $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$; and the point on this same sphere farthest from the point $(3, 1, -1)$ is $\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.