15.2: Double Integrals Over General Regions

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Fall 2021

Overview

In the previous section we learned to evaluate the double integral of a two-variable function over a rectangular subset of its domain, i.e., to find the net signed volume of the solid bounded by a rectangle in the xy-plane and a surface. This raises a question: how could we find the signed volume of a solid that lies between a surface and some other shape in the xy-plane? In this and the next section, we will learn techniques for solving such problems.

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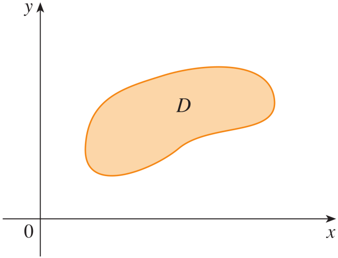
The Setup

Suppose that we wish to find the signed volume of the solid that lies between the graph of a continuous function $f(x, y)$ and a subset of its domain in the xy-plane such as the following:



Before we begin solving this problem, we introduce some notation: borrowing the language and notation of the previous section for this new problem, we refer to this signed volume as the double integral of $f(x, y)$ over D:

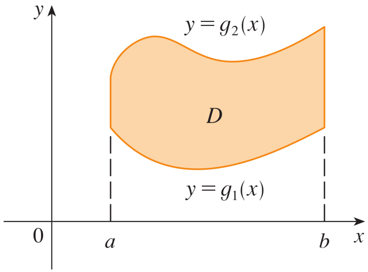
$$\iint_D f(x, y)\,\mathrm{d}A$$

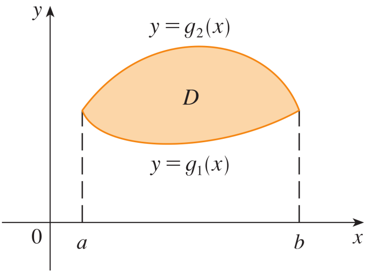
The Technique

To evaluate such integrals, we first need some definitions. A region D in the plane is said to be a **type I** region if it lies between two continuous functions of x, i.e.

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

For example:





The Technique, cont.

Analogous to our work over rectangles, we have the following:

If $f(x, y)$ is a continuous function on a type I region D given by

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

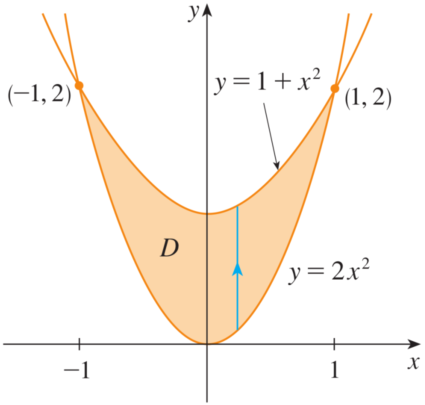
$$\iint_D f(x, y)\, \mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, \mathrm{d}y\, \mathrm{d}x$$

Note: The order of integration matters here! Fubini's theorem does not hold for such integrals.

Example

Evaluate $I = \iint_D (x + 2y)\,\mathrm{d}A$ where D is the region in the xy-plane bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Let's first sketch the region D:



Example, cont.

We note, then, that D may be described as a type I region as follows:

$$D = \left\{ (x, y) \;\middle|\; -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2 \right\}$$

Therefore, we have:

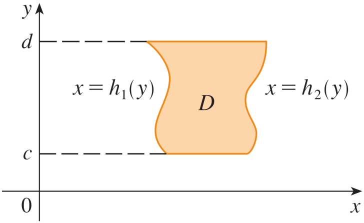
$$\begin{aligned}
I = \iint_D (x + 2y)\,\mathrm{d}A &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^{1} \left(xy + y^2\right)\Big|_{y=2x^2}^{y=1+x^2} \mathrm{d}x \\
&= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1\right)\mathrm{d}x \\
&= \left(\frac{-3}{5}x^5 - \frac{x^4}{4} + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x\right)\Bigg|_{-1}^{1} = \boxed{\frac{32}{15}}
\end{aligned}$$

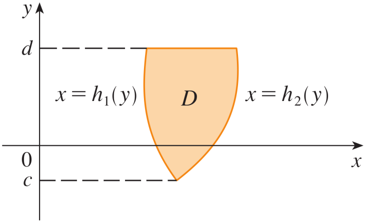
Type II Regions

Of course, our regions might not always be bounded by two functions of x; they could, for example, be bounded by two functions of y. More precisely, a region D in the plane is said to be a **type II** region if it lies between two continuous functions of y, i.e.

$$D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

For example:





Integrals

The method of evaluating a double integral over a type II region is analogous to evaluating a double integral over a type I region:

If $f(x, y)$ is a continuous function on a type II region D given by

$$D = \left\{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\right\}$$

then

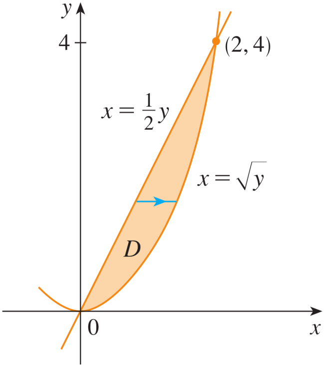
$$\iint_D f(x, y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Again, the order of integration matters; Fubini's theorem only holds when integrating over rectangles, not here.

Example

Find the signed volume V of the solid that lies between the paraboloid $z = x^2 + y^2$ and the region D in the xy-plane bounded by the line $x = \frac{y}{2}$ and the parabola $x = \sqrt{y}$.

Let's begin by sketching D:



Example, cont.

Now, note that we can think of D as a type II region:

$$D = \left\{ (x, y) \;\middle|\; 0 \le y \le 4, \frac{y}{2} \le x \le \sqrt{y} \right\}$$

Therefore, we have:

$$\begin{aligned} V = \iint_D (x^2 + y^2)\, dA &= \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy \\ &= \int_0^4 \left(\frac{x^3}{3} + xy^2 \right) \Bigg|_{x=y/2}^{x=\sqrt{y}} dy \\ &= \int_0^4 \left(\frac{1}{3} y^{3/2} + y^{5/2} - \frac{13}{24} y^3 \right) dy \\ &= \left(\frac{2}{15} y^{5/2} + \frac{2}{7} y^{7/2} - \frac{13}{96} y^4 \right) \Bigg|_0^4 = \boxed{\frac{216}{35}} \end{aligned}$$

Key Points

Note that the region D in the previous problem could also be conceived of as a type I region! Sometimes problems can be made much easier by conceiving of a region as type I instead of type II, or vice versa. See the exercises below for examples of this.

As has been mentioned a couple times already, Fubini's theorem *does not hold* for these integrals. If you wish to reverse the order of integration, you *must* change your conception of D from type I to type II or vice versa.

Properties

Many of the usual properties from single-variable calculus apply to double integrals. For example:

$$\iint_D [f(x,y) + g(x,y)]\,\mathrm{d}A = \iint_D f(x,y)\,\mathrm{d}A + \iint_D g(x,y)\,\mathrm{d}A$$

and

$$\iint_D cf(x,y)\,\mathrm{d}A = c\iint_D f(x,y)\,\mathrm{d}A$$

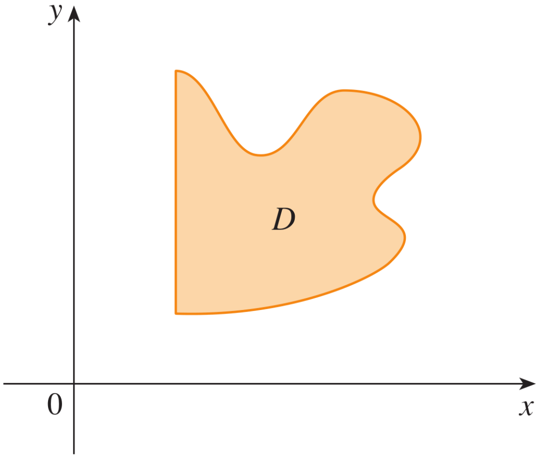
for a constant c.

If a region D may be written as $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap, except perhaps on their boundaries, then:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A$$

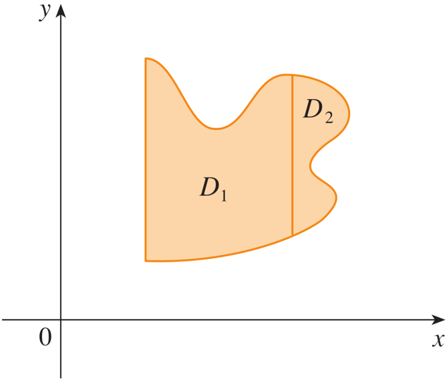
Properties, cont.

This latter property is useful, for example, in situations like the following. Consider the region D below, which is neither type I nor type II:



Properties, cont.

Note that we can split D into two regions as follows:



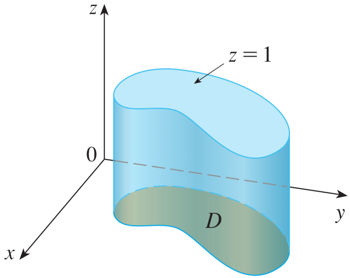
D_1 is type I and D_2 is type II. Therefore, if we wanted to evaluate $\iint_D f(x, y)\,\mathrm{d}A$, we could use the property above to do so.

Properties, cont.

One final property: Suppose we wanted to know the area of the base region D, denoted $A(D)$. It turns out that:

$$A(D) = \iint_D 1 \, \mathrm{d}A$$

Why? Well, $\iint_D 1 \, \mathrm{d}A$ is the volume of the cylinder of height 1 and base area D. The volume of such a cylinder is $A(D) \cdot 1$.



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Exercises

1. Find the signed volume V_1 of the solid that's bounded by the paraboloid $z = x^2 + y^2$ and the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.
2. Evaluate $\iint_R xy \, \mathrm{d}A$, where R is the region in the xy-plane bounded by the line $y = x - 1$ and the parabola $x = y^2 - 1$ [Hint: After sketching R, rewrite the equations of the boundaries of R so that it becomes a type II region].
3. Evaluate $\iint_D \sin(y^2) \, \mathrm{d}A$ where D is the region in the xy-plane bounded by the lines $y = 1$, $x = 0$, and $y = x$. [Hint: Think carefully about whether you should conceive of D as a type I or a type II region].
4. Find the signed volume V_4 of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$. [Hint: Conceive of this as a double integral after making two sketches: one of the tetrahedron, and one of the base region D in the xy-plane].

Solutions

1. $V_1 = \frac{216}{35}$
2. $\iint_R xy \, \mathrm{d}A = \frac{27}{8}$
3. $\iint_D \sin(y^2) \, \mathrm{d}A = \frac{-1}{2} \cos(1) + \frac{1}{2}$
4. $V_4 = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{3}$