16.2: Line Integrals

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Overview

Now that we have some idea of what vector fields are, we would like to turn to the calculus of these objects. In particular, we will be interested in studying what's called the **line integral** of a vector field. This turns out to be a surprisingly rich concept. Indeed, after introducing and learning one method for evaluating such integrals in this section, we will spend the rest of this chapter developing alternative methods that can be used to evaluate these integrals — and these alternative methods will turn out to have surprisingly deep connections to the rest of the material we've encountered in this course.

Before we can do any of this, however, we will first need to define and learn to evaluate two different varieties of line integrals of real-valued two- and three-variable functions, as these will be indispensable instruments for evaluating line integrals of vector fields.

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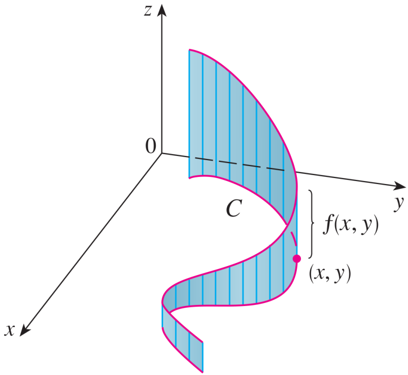
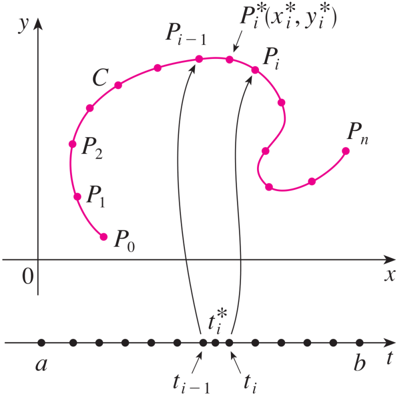
The Core Concept

Suppose we have a real-valued, two-variable function $f(x, y)$ and a smooth curve C in the xy-plane, parametrized by the vector function $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$. In the spirit of single-variable calculus, one might ask how much signed area is in the curtain between the curve C and the surface $z = f(x, y)$:



Approximating

As has hopefully become somewhat routine by this point, we start by estimating this area. We first break the curve C into n subarcs of length Δs, and from each subarc we choose a test point (x_i^*, y_i^*):



Approximation, cont.

Then, we form boxes between $f(x, y)$ and C on each subarc, with height $f(x_i^*, y_i^*)$. The area of each box is given by:

$$f(x_i^*, y_i^*)\Delta s$$

So that we estimate the total area A by adding up the area of each box:

$$A \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s$$

Letting n approach infinity yields the actual total area, which we call the **line integral of $f(x, y)$ along C (with respect to arc length)**:

$$A = \int_C f(x, y)\, \mathrm{d}s = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s$$

Calculation

So, we now know what the line integral represents, but how do we actually calculate it? It looks unlike any integral we've encountered before. Indeed, instead of bounds on the integral, we have just C. And we're told that the variable of integration is s, which has something to do with the arc length of C. How can we use all of this?

Recall C is parametrized by $\vec{r}(t) = \left\langle x(t), y(t) \right\rangle$. The arc length function for an arc parametrized this way is:

$$s(t) = \int_a^t \left| \vec{r}\,'(u) \right| \mathrm{d}u$$

Therefore, by the second fundamental theorem of calculus, we have:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left| \vec{r}\,'(t) \right|$$

i.e. $\mathrm{d}s = \left| \vec{r}\,'(t) \right| \mathrm{d}t$.

Calculation, cont.

If $\vec{r}(t) = \langle x(t), y(t) \rangle$, it is common to abuse notation and use the shorthand

$$f(\vec{r}(t)) := f(x(t), y(t))$$

Therefore, since C is parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$ with $a \leq t \leq b$, writing everything in terms of the variable t gives:

$$\begin{aligned} \int_C f(x, y)\, \mathrm{d}s &= \int_a^b f(x(t), y(t)) \left| \vec{r}\,'(t) \right| \mathrm{d}t \\ &= \boxed{\int_a^b f(\vec{r}(t)) \left| \vec{r}\,'(t) \right| \mathrm{d}t} \end{aligned}$$

Example

Evaluate $I_1 = \int_{C_1} (2 + x^2 y)\, \mathrm{d}s$, where C_1 is the upper half of the unit circle $x^2 + y^2 = 1$, traced counterclockwise.

We will proceed using the formula for the line integral of a real-valued function with respect to arc length given on the previous slide.

With this in mind, let $f_1(x, y) = 2 + x^2 y$ be the integrand of the integral.

In the formula on the previous slide, everything hinges on the parametrization of C_1. Therefore, our first goal is to come up with such a parametrization.

Example, cont.

Recall from your previous experience with parametric equations (see section 10.1 for a review of this material) that one way to parametrize the unit circle is with the following pair of parametric equations:

$$x_1(t) := \cos(t) \quad \text{and} \quad y_1(t) := \sin(t)$$

Therefore, we let $\vec{r_1}(t) = \left\langle \cos(t), \sin(t) \right\rangle$.

Of course, C_1 is only the upper half of the unit circle. Notice that on the interval $0 \leq t \leq \pi$, the graph of $\vec{r_1}(t)$ traces the the curve C_1.

Thus, we have a complete parametrization of C_1.

Example, cont.

With this parametrization out of the way, we can finally compute I_1:

$$\begin{aligned}
I_1 &= \int_{C_1} f_1(x, y)\,\mathrm{d}s \\
&= \int_0^{\pi} f_1(\vec{r_1}(t))\left|\vec{r_1}'(t)\right|\mathrm{d}t \\
&= \int_0^{\pi} (2 + \cos^2(t)\sin(t))\left|\left\langle -\sin(t), \cos(t)\right\rangle\right|\mathrm{d}t \\
&= \int_0^{\pi} (2 + \cos^2(t)\sin(t))\sqrt{(-\sin(t))^2 + (\cos(t))^2}\,\mathrm{d}t \\
&= \int_0^{\pi} (2 + \cos^2(t)\sin(t)) \cdot 1\,\mathrm{d}t \\
&= \boxed{2\pi + \frac{2}{3}}
\end{aligned}$$

where the latter integral is completed using the substitution $u := \cos(t)$.

Example

Calculate the signed area A_2 between the graph of the function $g(x, y) = 2x$ and the curve C_2, the vertical line segment extending from $(1, 1)$ to $(1, 2)$.

Note that, by definition, we have:

$$A_2 = \int_{C_2} g(x, y)\, \mathrm{d}s$$

and thus, we must once again compute the line integral of a two-variable, real-valued function. Just as in the previous example, the result of the problem hinges on our finding a parametrization of C_2.

Example, cont.

One of the reasons I chose this example was to present you with the following handy formula for parametrizing a line segment. Consider the following:

$$\vec{r_2}(t) := (1-t)\langle 1, 1\rangle + t\langle 1, 2\rangle = \langle 1, 1+t\rangle$$

You can verify (using our work in chapter 13) that the graph of this vector equation is a line segment that begins at the point $(1, 1)$ (when $t = 0$), and ends at the point $(1, 2)$ (when $t = 1$).

Thus, a parametrization of C_2 is $\vec{r_2}(t) = \langle 1, t+1\rangle$, with $0 \leq t \leq 1$.

Example, cont.

In general, if C is the line segment in $\mathbb{R}^2$ extending from the point (a, b) to the point (c, d), a parametrization of C is

$$\vec{r}(t) := (1 - t)\langle a, b\rangle + t\langle c, d\rangle$$

with $0 \leq t \leq 1$.

An analogous formula can be used to parametrize line segments in $\mathbb{R}^3$.

Of course, this is *not* the only way to parametrize such line segments! Indeed, the parametrization of lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ that we learned in chapter 12 also leads to a perfectly fine parametrization of line segments, as would plenty of other methods (indeed, there are infinitely-many ways to parametrize any curve). This is just one technique I find particularly memorable.

Example, cont.

In any case, we have:

$$\begin{aligned}
A_2 &= \int_{C_2} g(x,y)\,\mathrm{d}s \\
&= \int_0^1 g(\vec{r_2}(t))\left|\vec{r_2}'(t)\right|\mathrm{d}t \\
&= \int_0^1 2\left|\langle 0,1\rangle\right|\mathrm{d}t \\
&= \int_0^1 2\sqrt{0^2+1^2}\,\mathrm{d}t \\
&= \int_0^1 2\,\mathrm{d}t \\
&= 2t\Big|_0^1 \\
&= \boxed{2}
\end{aligned}$$

Piecewise-Smooth Curves

Our work up to this point relied on C being a smooth curve (loosely, one with no breaks or sharp corners along its length). This raises a follow-up question: how could we evaluate

$$\int_C f(x, y)\,\mathrm{d}s$$

if C is *not* smooth?

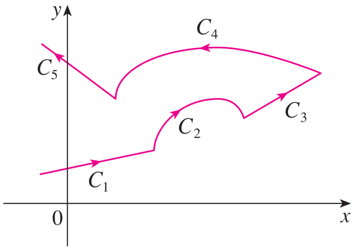
If C may be broken into a finite number of smooth pieces $C_1, C_2, \ldots, C_n$, then we may evaluate $\int_C f(x, y)\,\mathrm{d}s$ piecewise as

$$\int_C f(x, y)\,\mathrm{d}s = \int_{C_1} f(x, y)\,\mathrm{d}s + \int_{C_2} f(x, y)\,\mathrm{d}s + \cdots + \int_{C_n} f(x, y)\,\mathrm{d}s$$

Indeed, the total signed area between the graph of $f(x, y)$ and the curve C in the xy-plane is the sum of the signed areas between the graph of $f(x, y)$ and each of the curves C_1, C_2, $\ldots$, C_n.

Example

For example, if C were the following curve:



We could evaluate $\int_C f(x, y)\, \mathrm{d}s$ as

$$\int_C f(x, y)\, \mathrm{d}s = \sum_{i=1}^{5} \int_{C_i} f(x, y)\, \mathrm{d}s$$

Two Other Line Integrals

There are two more line integrals (with no nice graphical interpretation) that will be important for us: the **line integral of** $f(x, y)$ **along** C **with respect to** x **(resp.** y**)**. If C is parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$ with $a \leq t \leq b$, they are:

$$\int_C f(x, y)\,\mathrm{d}x = \int_a^b f(\vec{r}(t))x'(t)\,\mathrm{d}t$$

and

$$\int_C f(x, y)\,\mathrm{d}y = \int_a^b f(\vec{r}(t))y'(t)\,\mathrm{d}t$$

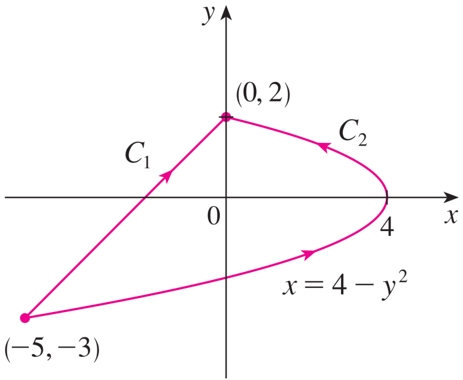
We will see some uses for these at the end of the section. We often complete such integrals in pairs, so there is a customary shorthand for this situation:

$$\int_C P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y := \int_C P(x, y)\,\mathrm{d}x + \int_C Q(x, y)\,\mathrm{d}y$$

Example

Evaluate $\int_C y^2\,\mathrm{d}x + x\,\mathrm{d}y$ in two cases: first, for $C = C_1$, the line segment extending from $(-5,-3)$ to $(0,2)$; and second, for $C = C_2$, the arc of the parabola $x = 4 - y^2$ extending from $(-5,-3)$ to $(0,2)$.

Let's draw our two curves, just to get a clear picture of things:



Example, cont.

Recall that, by definition, we have:

$$\int_C y^2\,\mathrm{d}x + x\,\mathrm{d}y = \int_C y^2\,\mathrm{d}x + \int_C x\,\mathrm{d}y$$

Therefore, for each case, we wish to evaluate a line integral along C with respect to x, and a second line integral along C with respect to y.

Example, cont.

Let's begin with C_1. Just as with the previous line integrals we've evaluated, how we proceed depends heavily on how we decide to parametrize this curve. C_1 is a line segment, so we can use the trick from the previous example to parametrize it:

$$\vec{r_3}(t) := (1-t)\langle -5, -3\rangle + t\langle 0, 2\rangle = \langle 5t-5, 5t-3\rangle$$

Thus, $\vec{r_3}(t) = \langle 5t-5, 5t-3\rangle$ with $0 \leq t \leq 1$ is a parametrization of C_1.

Of course, to utilise the method of evaluating these line integrals outlined in the slide above, it will help us to have names for the components of $\vec{r_3}(t)$. Therefore, let:

$$x_3(t) = 5t - 5 \quad \text{and} \quad y_3(t) = 5t - 3$$

Example, cont.

With all of this in mind, we have:

$$\begin{aligned}
\int_{C_1} y^2\,\mathrm{d}x + x\,\mathrm{d}y &= \int_{C_1} y^2\,\mathrm{d}x + \int_{C_1} x\,\mathrm{d}y \\
&= \int_0^1 (5t-3)^2 x_3'(t)\,\mathrm{d}t + \int_0^1 (5t-5) y_3'(t)\,\mathrm{d}t \\
&= \int_0^1 5(25t^2 - 30t + 9)\,\mathrm{d}t + \int_0^1 5(5t-5)\,\mathrm{d}t \\
&= \int_0^1 \left(125t^2 - 125t + 20\right)\mathrm{d}t \\
&= \boxed{\frac{-5}{6}}
\end{aligned}$$

Example, cont.

Now we turn to C_2. Again, to proceed we must begin by parametrizing this curve. How can we do so? The secret lies in the given relationship $x = 4 - y^2$.

Indeed, if we replace y in the equation above with t, then we obtain $x = 4 - t^2$. Therefore,

$$\vec{r_4}(t) := \left\langle 4 - t^2, t \right\rangle$$

is a parametrisation of the curve $x = 4 - y^2$. In particular, since C_2 extends from $(-5, -3)$, where y is equal to -3, to $(0, 2)$, where y is equal to 2; and since we set y equal to t, we see that $\vec{r_4}(t)$ with $-3 \leq t \leq 2$ is a parametrization of C_2 (You can also use the methods of chapters 10 and 13 to sketch the graph of $\vec{r_4}(t)$ between $t = -3$ and $t = 2$ to verify this).

Example, cont.

Of course, once again, to utilize the method of evaluating these particular line integrals outlined in the slide above, it will help us to have names for the components of $\vec{r_4}(t)$. Therefore, let:

$$x_4(t) = 4 - t^2 \qquad \text{and} \qquad y_4(t) = t$$

Putting everything together, we have:

$$\begin{aligned}
\int_{C_2} y^2 \, \mathrm{d}x + x \, \mathrm{d}y &= \int_{C_2} y^2 \, \mathrm{d}x + \int_{C_2} x \, \mathrm{d}y \\
&= \int_{-3}^{2} (t^2) x_4'(t) \, \mathrm{d}t + \int_{-3}^{2} (4 - t^2) y_4'(t) \, \mathrm{d}t \\
&= \int_{-3}^{2} \left[t^2(-2t) + (4 - t^2)(1) \right] \mathrm{d}t \\
&= \boxed{\frac{245}{6}}
\end{aligned}$$

Orientation

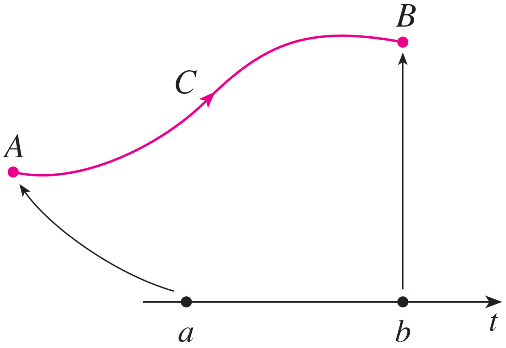
You may have noticed that in each of the examples above, the curves along which we integrated were drawn in a particular direction to connect two points. That is, these curves were given an **orientation**.

A natural question, to ask, then, is whether the orientation of a given curve makes any difference to the line integrals we might compute along said curve. The answer? It depends.

Let's take a closer look at this question.

Orientation, cont.

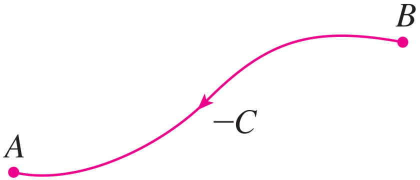
Suppose we have a curve C parametrized by $\vec{r_1}(t)$ as below:



The arrow here indicates the direction that C is traced in as t increases. This parametrization gives an **orientation** to C.

Orientation, cont.

The symbol $-C$ denotes the curve consisting of the same points as C, but with the opposite orientation, as in the following figure:



(Note: There's nothing inherent to the curve C itself that determines its orientation. Indeed, if we first had parametrized C so that it was traced from B to A as on this slide, then we would denote the orientation on the previous slide by $-C$. Opposite orientations have opposite signs; that's all. Nothing forces one to be the "positive" orientation and the other the "negative" orientation — at least, not for a generic curve. More on that later.)

Orientation, cont.

How does this affect line integrals? Well, if we integrate with respect to arc length, orientation makes no difference, as $\mathrm{d}s$ is always positive:

$$\int_{-C} f(x, y)\,\mathrm{d}s = \int_C f(x, y)\,\mathrm{d}s$$

However, if we integrate with respect to x and y, changing orientation flips the signs of $x'(t)$ and $y'(t)$, hence of the integral:

$$\int_{-C} f(x, y)\,\mathrm{d}x = -\int_C f(x, y)\,\mathrm{d}x$$

and

$$\int_{-C} f(x, y)\,\mathrm{d}y = -\int_C f(x, y)\,\mathrm{d}y$$

These latter relationships help to explain the notation $-C$ that we used to indicate the opposite orientation of C.

$\mathbb{R}^3$

It is worth noting that everything above can be repeated in $\mathbb{R}^3$. If we have a space curve C parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$, we have:

$$\int_C f(x, y, z)\,\mathrm{d}s = \int_a^b f(\vec{r}(t)) \left| \vec{r}\,'(t) \right| \mathrm{d}t$$

and the line integrals with respect to x, y, and z are also defined similar to the $\mathbb{R}^2$ case. For example:

$$\int_C f(x, y, z)\,\mathrm{d}z = \int_a^b f(\vec{r}(t)) z'(t)\,\mathrm{d}t$$

A Special Case

Note also that if $f(x, y) = 1$ or $f(x, y, z) = 1$, we have:

$$\int_C 1\,\mathrm{d}s = \int_a^b \left|\vec{r}\,'(t)\right|\mathrm{d}t = L(C)$$

where $L(C)$ is the length of the arc C.

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Framing

Up to this point in the section, we really haven't talked about anything we couldn't have covered in chapter 13. In fact, it almost looks like maybe we *should* have covered this material then, given that it draws on so many concepts from that chapter. So why did we wait until now?

We will find the answer as we investigate line integrals of vector fields.

But what is the line integral of a vector field? How is it defined? What does it represent? We will find one possible answer in physics: computing work.

Framing: Work

First, let's talk recall a basic work calculation: if a constant force $\vec{F}$ acts on an object as it moves in a straight line between two points, the work W done by that force on the object is

$$W = \vec{F} \cdot \vec{D}$$

where $\vec{D}$ is the displacement vector between the points.

Now let's consider a more general question: suppose that we wish to move an object between two points in $\mathbb{R}^3$ along some *curve* C, and the force $\vec{F}(x, y, z)$ varies depending on which point we're at along this curve. How much work W is done on the object by this variable force in this case?

In other words, if we move the object along C through a force field $\vec{F}(x, y, z)$ (i.e. a vector field which specifies a force at every point in $\mathbb{R}^3$), how much work W does the force field do on the object?

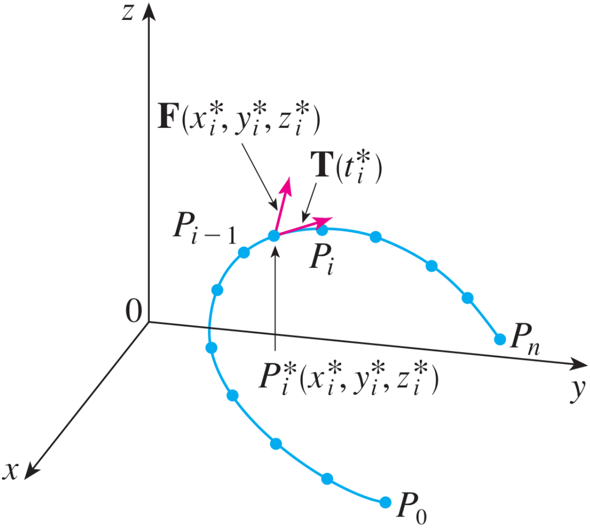
Framing: Work, cont.

To help us get started, let's assume that C is parametrized by the vector function $\vec{r}(t)$ for $a \leq t \leq b$.

As is probably starting to feel familiar by now, we won't compute the work W directly, not at first. Instead, we'll estimate it as follows:

1. We will break C up into n subarcs, each with length Δs, and choose a test point (x_i^*, y_i^*, z_i^*) on each subarc.
2. We will assume that the force acting on the object along the length of each subarc is a constant, equal to $\vec{F}(x_i^*, y_i^*, z_i^*)$, the force exerted on the object at the test point on each subarc.
3. Finally, we will assume that the object is moving in a straight line along each subarc, in the direction of the unit tangent vector to C at (x_i^*, y_i^*, z_i^*), which we will call $\vec{T}(x_i^*, y_i^*, z_i^*)$ (we will assume this direction for technical reasons, to make some calculations easier).

Estimation



Estimation, cont.

Why these assumptions? Well, with these simplifying assumptions, we can estimate the work W_i done by the force field on the object as it moves along the ith subarc using our first formula for work above, as follows:

$$W_i \approx \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \left[\Delta s \vec{T}(x_i^*, y_i^*, z_i^*)\right]$$

Therefore, the total work done by the force field on the object is approximately:

$$W \approx \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \left[\Delta s \vec{T}(x_i^*, y_i^*, z_i^*)\right]$$

Work Done by the Force Field

As we break C into more and more subarcs, our approximation continually improves. The actual work done by the force field on the particle is therefore:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \left[\Delta s \vec{T}(x_i^*, y_i^*, z_i^*)\right] \\ &= \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\,\mathrm{d}s \\ &= \int_C \vec{F} \cdot \vec{T}\,\mathrm{d}s \end{aligned}$$

Indeed, the second equation holds because the dot product of the vector fields $\vec{F}(x,y,z)$ and $\vec{T}(x,y,z)$ is a *scalar* function, so our Riemann sum is actually the line integral of a three-variable, real-valued function with respect to arc length.

Computation, In Practice

We can now calculate W precisely! We could stop here and use this formula, but our calculations would actually be somewhat inefficient in practice. Let's clean things up a bit.

Computation, In Practice cont.

Recall that C is parametrized by the vector function $\vec{r}(t)$, where $a \le t \le b$. Furthermore, recall from chapter 13 that the unit tangent vector $\vec{T}(t)$ to the graph of $\vec{r}(t)$ (i.e. to C) at a given value of t is:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\left|\vec{r}\,'(t)\right|}$$

Since the integral on the previous slide is a line integral of a scalar function with respect to arc length, our work from the beginning of the section applies. Writing everything in terms of our parametrization of C, we have:

$$\begin{aligned} W = \int_C \vec{F} \cdot \vec{T}\,\mathrm{d}s &= \int_a^b \left[\vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\left|\vec{r}\,'(t)\right|}\right] \left|\vec{r}\,'(t)\right| \mathrm{d}t \\ &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,\mathrm{d}t \end{aligned}$$

Definition

We can separate this entire discussion from the work example that we gave and make the following definition:

Definition: Let $\vec{F}(\vec{x})$ be a continuous vector field defined on a smooth curve C parametrized by the vector function $\vec{r}(t)$ for $a \leq t \leq b$. The **line integral of $\vec{F}$ along** C is:

$$\int_C \vec{F} \cdot \vec{T} \, \mathrm{d}s = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, \mathrm{d}t$$

We often abbreviate this integral as

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r}$$

This whole discussion also applies to vector fields on $\mathbb{R}^2$.

Example

Calculate the work W_4 done by the force field $\vec{H}(x, y) = \langle x^2, -xy \rangle$ on a particle as it moves from the point $(2, 0)$ to the point $(0, 2)$ along the quarter-circle C_4 of radius 2 centered at the origin in the first quadrant.

Just as in all of our previous examples in this section, the way our calculation proceeds depends thoroughly on giving a parametrization of C_4. Thus, we must first find such a parametrization.

Recall that, by your previous work in parametric equations (see chapter 10 if you need a refresher), a parametrization of C_4 is:

$$\vec{r}(t) := \langle 2\cos(t), 2\sin(t) \rangle$$

with $0 \leq t \leq \frac{\pi}{2}$.

Example, cont.

We thus have:

$$\begin{aligned}
W_4 &= \int_0^{\pi/2} \vec{H}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,\mathrm{d}t \\
&= \int_0^{\pi/2} \Big\langle 4\cos^2(t), -4\cos(t)\sin(t) \Big\rangle \cdot \big\langle -2\sin(t), 2\cos(t) \big\rangle \,\mathrm{d}t \\
&= \int_0^{\pi/2} \Big(-8\sin(t)\cos^2(t) - 8\sin(t)\cos^2(t) \Big)\,\mathrm{d}t \\
&= \int_0^{\pi/2} \Big(-16\sin(t)\cos^2(t) \Big)\,\mathrm{d}t \\
&= \boxed{\frac{-16}{3}}
\end{aligned}$$

where the latter integral is computed using the substitution $u = \cos(t)$.

Orientation

We know the effect that reversing the orientation of a curve has on line integrals with respect to arc length, x, y, and z, so it's only natural to ask: what happens in this case?

If we reverse the direction that C is traced out in, all of the tangent vectors along the length of C are also reversed. Therefore:

$$\int_{-C} \vec{F} \cdot \mathrm{d}\vec{r} = -\int_{C} \vec{F} \cdot \mathrm{d}\vec{r}$$

Another Connection to Real Line Integrals

Finally, suppose that we write $\vec{r}(t) = \left\langle x(t), y(t), z(t) \right\rangle$, so that $\vec{F}(\vec{r}(t)) = \left\langle P(x(t), y(t), z(t)), Q(x(t), y(t), z(t)), R(x(t), y(t), z(t)) \right\rangle$. Then we have:

$$\begin{aligned} \int_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_a^b \langle P, Q, R \rangle \cdot \left\langle x'(t), y'(t), z'(t) \right\rangle \mathrm{d}t \\ &= \int_a^b \left[Px'(t) + Qy'(t) + Rz'(t) \right] \mathrm{d}t \\ &= \boxed{\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z} \end{aligned}$$

This gives another way to compute such integrals, and also helps explain why we developed the concept of line integrals with respect to x, y, and z. An analogous result holds for vector fields on $\mathbb{R}^2$.

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Exercises

1. Evaluate the line integral

$$I_1 = \int_{C_1} \frac{x}{y}\, \mathrm{d}s$$

   where C_1 is the curve parametrized by $x = t^3$ and $y = t^4$, with $1 \leq t \leq 2$.

2. Evaluate the line integral

$$I_2 = \int_{C_2} y\, \mathrm{d}x + z\, \mathrm{d}y + x\, \mathrm{d}z$$

   where C_2 is the curve parametrized by $x = \sqrt{t}$, $y = t$, and $z = t^2$, with $1 \leq t \leq 4$.

Exercises, cont.

3. Evaluate the line integral

$$I_3 = \int_{C_3} y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z$$

where C_3 consists of two line segments: L_1, from $(2,0,0)$ to $(3,4,5)$, and L_2, from $(3,4,5)$ to $(3,4,0)$.

4. Evaluate

$$I_4 = \int_{C_4} \vec{F} \cdot \mathrm{d}\vec{r}$$

where $\vec{F}(x,y,z) = \langle xy, yz, xz \rangle$ and C_4 is the twisted cubic $x = t$, $y = t^2$ and $z = t^3$, with $0 \le t \le 1$.

Solutions

1. One possible solution:

$$I_1 = \int_1^2 \frac{1}{t}\sqrt{9t^4 + 16t^6}\,\mathrm{d}t = \int_1^2 t\sqrt{9 + 16t^2}\,\mathrm{d}t = \boxed{\frac{1}{48}\left(73^{3/2} - 125\right)}$$

2. One possible solution:

$$I_2 = \int_1^4 \left(\frac{1}{2}\sqrt{t} + t^2 + 2t^{3/2}\right)\mathrm{d}t = \boxed{\frac{722}{15}}$$

3. One possible solution:

$$I_3 = \int_0^1 (29t + 10)\,\mathrm{d}t + \int_0^1 (-15)\,\mathrm{d}t = \int_0^1 (29t - 5)\,\mathrm{d}t = \boxed{\frac{19}{2}}$$

4. One possible solution:

$$I_4 = \int_0^1 \left\langle t^3, t^5, t^4\right\rangle \cdot \left\langle 1, 2t, 3t^2\right\rangle \mathrm{d}t = \int_0^1 (t^3 + 5t^6)\,\mathrm{d}t = \boxed{\frac{27}{28}}$$