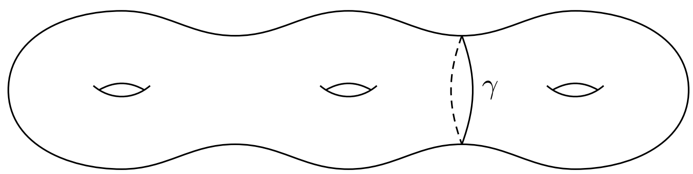
# Research Statement

Joel Louwsma

My research is in the areas of geometric group theory and low-dimensional topology. Broadly speaking, I am interested in groups that arise naturally in geometry and topology and in ways that geometry and topology can be used to answer algebraic questions about groups. Much of my work has involved stable commutator length and quasimorphisms, notions that will be defined shortly. These notions have topological, algebraic, and analytic interpretations, and in studying them I have used tools from these areas as well as from combinatorics, number theory, and optimization.

One type of problem I find interesting is that of finding the "simplest" surface with a prescribed boundary. As a simple example, consider the surface of genus 3, with the curve $\gamma$ as shown.




Here the curve $\gamma$ bounds a surface of genus two (the left-hand side of the figure) and a surface of genus one (the right-hand side of the figure). The surface on the right-hand side of $\gamma$ is "simpler" because it has smaller genus.

More generally, given a space $X$ and a closed curve $\gamma$ in $X$, one can ask which surfaces $S$ with one boundary component can be mapped to $X$ in such a way that the boundary of $S$ is mapped to $\gamma$ with degree 1. (Such maps can exist only if $\gamma$ represents the trivial element of $H_1(X)$.) This is usually a hard problem. However, it turns out to become more tractable if one also allows surfaces whose boundary maps to $\gamma$ with degree greater than 1 and normalizes the complexity of the surface accordingly. Since Euler characteristic is multiplicative under taking covers, whereas genus is not, it is most natural to keep track of the complexity of such surfaces in terms of Euler characteristic.

This perspective gives rise to a topological definition of stable commutator length. For fixed $X$ and $\gamma$, consider all maps $f\colon S \to X$ of a connected surface $S$ to $X$ such that the boundary of $S$ is mapped to $\gamma$ with positive degree. Let $\chi(S)$ denote the Euler characteristic of $S$, and let $n(S, f)$ denote the total degree of the map from the boundary of $S$ to $\gamma$. Then the *stable commutator length* of $\gamma$ is

$$\mathrm{scl}(\gamma) := \inf_{(S,f)} \frac{-\chi(S)}{2n(S,f)}. \tag{1}$$

If $G$ is the fundamental group of $X$ and $g$ is the element of $G$ represented by $\gamma$, we can also define $\mathrm{scl}(g) := \mathrm{scl}(\gamma)$. If $(S, f)$ achieves the infimum in (1), it is said to give an *extremal surface* for $g$ (or for $\gamma$).

While it is natural to study efficient surfaces with a prescribed boundary, it might seem quite mysterious from this topological perspective why this notion is called *stable commutator length*. This terminology comes from an equivalent algebraic definition. Given a group $G$, its commutator subgroup, denoted $[G, G]$, is generated by *commutators*, i.e. elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$. Thus every element $g \in [G, G]$ can be written as a product of commutators. Define the *commutator length* of $g$, denoted $\mathrm{cl}(g)$, to be the smallest number of commutators in a product of commutators that equals $g$. Then the stable commutator length of $g$ is

$$\mathrm{scl}(g) = \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}.$$

Details of the equivalence between the topological and algebraic definitions of stable commutator length are given in [9]. To summarize, a closed curve $\gamma$ bounds a surface of genus $k$ if and only if the corresponding element $g$ is a product of $k$ commutators. Therefore the original problem of finding the most efficient surface with a prescribed boundary is equivalent to computing commutator length. The relationship $\chi(S) = 2 - 2\,\mathrm{genus}(S)$ is used to convert between the problem in terms of genus and the problem in terms of Euler characteristic, and this is the reason for the 2 in the demoninator of (1).

Stable commutator length is also closely related to the theory of quasimorphisms; this connection was discovered by Bavard [2]. A *quasimorphism* on a group $G$ is a function $\phi\colon G \to \mathbb{R}$ satisfying the property that $|\phi(gh) - \phi(g) - \phi(h)| \le D$ for some constant $D$ that is independent of the choice of $g, h \in G$. Choose the smallest such $D$ and denote it by $D(\phi)$, referred to as the *defect* of $\phi$. A quasimorphism $\phi$ is called *homogeneous* if $\phi(g^n) = n\phi(g)$ for all $g \in G, n \in \mathbb{Z}$. Let $Q(G)$ denote the vector space of all homogeneous quasimorphisms on $G$. Bavard [2] shows that, if $g^k \in [G, G]$ for some $k \in \mathbb{N}$,

$$\mathrm{scl}(g) = \sup_{\substack{\phi\in Q(G)\\ D(\phi)\neq 0}} \frac{\phi(g)}{2D(\phi)}. \tag{2}$$

This gives a functional analytic interpretation of stable commutator length. A consequence is that every homogeneous quasimorphism with nonzero defect gives a lower bound on the stable commutator length of $g$. Moreover, it is known that the supremum in (2) is always achieved (see [9]). A homogeneous quasimorphism $\phi$ that achieves this supremum for some $g$, i.e. that satisfies $\mathrm{scl}(g) = \phi(g)/2D(\phi)$, is said to be *extremal* for that $g$.

One objective in the study of stable commutator length is to compute values it takes for specific elements. A major breakthrough in this direction was Calegari's algorithm [10] for computing stable commutator length in free groups. This algorithm can also be used to compute stable commutator length in certain classes of groups that are built from free groups in simple ways. However, there are few other instances in which stable commutator length can be computed explicitly, with the exception of certain elements and classes of groups for which it is known to vanish. Clay, Forester, and I [16] have shown how to compute stable commutator length for a certain class of elements of Baumslag–Solitar groups. Moreover, we have characterized exactly which elements in this class admit extremal surfaces. This work is discussed in more detail in Section 1.

One can also study the lower bounds on stable commutator length given by (2) for certain homogeneous quasimorphisms. Clay, Forester, and I [16] have done this for certain naturally defined quasimorphisms on groups acting on trees. We bound the defect of these quasimorphisms, thus obtaining lower bounds of $1/12$ on the stable commutator length of elements satisfying a certain condition. Moreover, calculations in Baumslag–Solitar groups show that our bounds are the best possible. This answers a question of Calegari and Fujiwara (Question 8.4 from [13]). Applying this work to Baumslag–Solitar groups, we show that there is a spectral gap: no element of a Baumslag–Solitar group has stable commutator length between 0 and $1/12$. This work is discussed in more detail in Section 2.

One can also ask when the lower bounds on stable commutator length given by a homogeneous quasimorphism are sharp, i.e. for which elements the quasimorphism is extremal. Although extremal quasimorphisms are known to exist, few examples of them have been found, due largely to the fact that the space of all homogeneous quasimorphisms is poorly understood for most groups. Finding quasimorphisms that can be certified to be extremal is of interest because it gives an indirect way to compute stable commutator length. Calegari and I [14] studied this question for an important group, the modular group $\mathrm{PSL}(2, \mathbb{Z})$, and a naturally occuring quasimorphism, the rotation quasimorphism. We showed that, for every element of the modular group, the product of this element with a sufficiently large power of a parabolic element is an element for which the rotation quasimorphism is extremal. This proves the natural analogue of Conjecture 3.16 from [10], with the free group $F_2$ replaced by the modular group $\mathrm{PSL}(2,\mathbb{Z})$. This work is discussed in more detail in Section 3.

# 1 Computing stable commutator length

It is generally difficult to compute values of stable commutator length. One class of groups for which stable commutator length can be computed explicitly is that of free groups. Calegari [10] showed how to use linear programming to construct an extremal surface corresponding to any element of the commutator subgroup of a free group. This gives an algorithm for computing stable commutator length in free groups and implies that it takes only rational values.

This algorithm can also be used to compute stable commutator length in certain classes of groups that are built from free groups in simple ways. For example, there is a general relationship between stable commutator length in a group and a finite-index subgroup (see [9]), and this can be used to compute stable commutator length in virtually free groups. In my thesis [25], I used the fact that the modular group $\mathrm{PSL}(2,\mathbb{Z})$ has an index-6 subgroup that is free to explain how to compute stable commutator length in $\mathrm{PSL}(2,\mathbb{Z})$.

Clay, Forester, and I have studied stable commutator length in Baumslag–Solitar groups. The Baumslag–Solitar group $BS(m,\ell)$ is the one-relator group definied by the presentation $\langle a,t \mid ta^m t^{-1}a^\ell \rangle$. It is the fundamental group of its presentation 2–complex $X$, where one thinks of $X$ as being constructed by attaching both ends of an annulus to a circle, by covering maps of degrees $m$ and $\ell$, respectively. Interpreting maps of surfaces to $X$ combinatorially, the problem of computing stable commutator length becomes a problem of minimizing a certain linear functional on an infinite dimensional vector space subject to certain constraints. When an element has what we call *alternating $t$–shape*, we are able to convert this optimization problem to one over a finite dimensional space and show that the output of this problem computes stable commutator length.

**Theorem 1** (Clay–Forester–L. [16])**.** *Suppose $g \in BS(m,\ell)$, $m \neq \ell$, has alternating $t$–shape. Then there is a finite dimensional, rational linear programming problem whose solution yields the stable commutator length of $g$. In particular,* $\mathrm{scl}(g)$ *is computable and is a rational number.*

In some cases the solution to this linear programming problem can be expressed in a closed form. We show that, if $m \nmid i$ and $\ell \nmid j$, we have

$$\mathrm{scl}(ta^i t^{-1} a^j) \;=\; \frac{1}{2}\left(1 - \frac{\gcd(i,m)}{|m|} - \frac{\gcd(j,\ell)}{|\ell|}\right). \tag{3}$$

This is interesting, because it is rare that a formula can be found for the stable commutator length of a class of elements.

We also characterize exactly which elements of alternating $t$–shape admit an extremal surface. It turns out many elements have extremal surfaces and many do not.

**Theorem 2** (Clay–Forester–L. [16])**.** *Let $g = \prod_{k=1}^{r} ta^{i_k}t^{-1}a^{j_k} \in BS(m,\ell)$, $m \neq \ell$. There is an extremal surface for $g$ if and only if*

$$\ell \sum_{k=1}^{r} i_k = -m \sum_{k=1}^{r} j_k.$$

This allows us to find many examples of elements with rational stable commutator length for which no extremal surface exists. Previous examples of this phenomenon were found in free products of abelian groups of higher rank (see [11]).

I hope to be able to extend this work to other elements and other classes of groups. In Baumslag–Solitar groups, the natural next question is to compute the stable commutator length of elements that are not of alternating $t$–shape. Our current methods give a lower bound on the stable commutator length of such elements, but we have examples that show these bounds are not always sharp. I would like to modify the linear programming problem to take into account the problems that arise in these examples, hopefully obtaining a linear programming problem that will compute the stable commutator length of all elements of Baumslag–Solitar groups.

More generally, it is natural to try to extend these techniques to so-called generalized Baumslag–Solitar groups, i.e. groups that act on trees with infinite cyclic edge and vertex stabilizers.

**Problem 1.** *Find an algorithm for computing stable commutator length in generalized Baumslag–Solitar groups.*

I also intend to study the problem of computing stable commutator length in other types of groups. In order for this problem to be tractable, one would like the group in question to be the fundamental group of a relatively simple space. Perhaps the natural next class of groups to study is surface groups, i.e. groups that are the fundamental group of a closed surface of genus at least 2.

**Problem 2.** *Find an algorithm for computing stable commutator length in surface groups.*

## 2 Quasimorphisms on groups acting on trees and spectral gaps

The relationship (2) implies that every homogeneous quasimorphism with nonzero defect gives a lower bound on stable commutator length. Specifically, if $f$ is a homogeneous quasimorphism on $G$ with $f(g) = 1$ and defect $D$, then $\mathrm{scl}(g) \geq 1/2D$. Therefore one can attempt to study stable commutator length by constructing homogeneous quasimorphisms and considering the lower bounds on stable commutator length that they give.

There are relatively few known ways to build quasimorphisms. Perhaps the simplest examples of quasimorphisms are the *counting quasimorphisms* on free groups [26, 7, 22]. Let $w$ be a reduced word in the generators of a free group $F_n$. Given an element $g \in F_n$, define $C_w(g)$ to be the number of occurrences of $w$ in the reduced representative of $g$. Then a quasimorphism $H_w$ on $F_n$ is defined by setting $H_w(g) = C_w(g) - C_{w^{-1}}(g)$. Variants of this construction have been used to construct quasimorphisms on word-hyperbolic groups [18], groups acting on Gromov hyperbolic spaces [19], amalgamated free products and HNN extensions [20], and mapping class groups [4, 5, 3].

Clay, Forester, and I [16] have studied homogeneous quasimorphisms of this type for groups acting on trees. We construct such quasimorphisms and show that their defect is at most 6. When an element of the group is what we call *well aligned*, we obtain a uniform lower bound on its stable commutator length. This notion of being well aligned agrees with the double coset condition in [13] in the case of an amalgamated free product of groups acting on its associated Bass–Serre tree.

**Theorem 3** (Clay–Forester–L. [16])**.** *Suppose $G$ acts on a simplicial tree $T$. If $g \in G$ is well aligned then* $\mathrm{scl}(g) \geq 1/12$.

This is an improvement on a result of Calegari–Fujiwara [13]. Moreover, this bound is optimal. Formula (3) shows that, in $BS(2,3)$, we have $\mathrm{scl}(tat^{-1}a) = 1/12$, and this element is well aligned with respect to the action on the associated Bass–Serre tree. This bound is also optimal if we restrict to amalgamated free products of groups because, in the group $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, the element $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ has stable commutator length 1/12 (see [25]), and this element is well aligned with respect to the action on the associated Bass–Serre tree. This answers Question 8.4 from [13].

Not all hyperbolic elements are well aligned. For example, there are 3–manifold groups that split as amalgamated free products and contain hyperbolic elements with very small stable commutator length; see [13]. However, by restricting to trees that are *acylindrical*, we can obtain lower bounds on stable commutator length that apply to all hyperbolic elements. These bounds are almost universal, depending only on the acylindricity constant. Alternatively, one can obtain a genuine uniform lower bound by considering only elements with translation length greater than or equal to the acylindricity constant.

**Theorem 4** (Clay–Forester–L. [16]). *Suppose $G$ acts $K$–acylindrically on a tree $T$ and let $N$ be the smallest integer greater than or equal to $\frac{K}{2}+1$.*

1. *If $g \in G$ is hyperbolic then either $\mathrm{scl}(g) = 0$ or $\mathrm{scl}(g) \geq 1/12N$.*
2. *If $g \in G$ is hyperbolic and $|g| \geq K$ then either $\mathrm{scl}(g) = 0$ or $\mathrm{scl}(g) \geq 1/24$.*

*In both cases, $\mathrm{scl}(g) = 0$ if and only if $g$ is conjugate to $g^{-1}$.*

**Spectral gaps.** We use Theorem 3 to show that there is a *gap* in the stable commutator length spectrum for Baumslag–Solitar groups. Results of a similar type have previously been established for several other classes of groups. Perhaps the first result in this direction is a result of Duncan–Howie [17] that implies that stable commutator length is always at least 1/2 in free groups. A result of Calegari–Fujiwara [13] shows that there is a gap above 0 in the stable commutator length spectrum for word-hyperbolic groups. A recent result of Bromberg–Bestvina–Fujiwara [3] shows that there is also a gap above 0 in the stable commutator length spectrum for mapping class groups.

We show that, if an element $g$ of a Baumslag–Solitar group is not well aligned with respect to the action on the associated Bass–Serre tree, then its action on the Bass–Serre tree must be such that we can conclude that $\mathrm{scl}(g) = 0$. Combining this with Theorem 3, we obtain the following gap theorem for Baumslag–Solitar groups.

**Theorem 5** (Clay–Forester–L [16]). *For every element $g \in BS(m,\ell)$, either $\mathrm{scl}(g) = 0$ or $\mathrm{scl}(g) \geq 1/12$.*

A natural question is how much more generally it is true that elements that are not well aligned must have stable commutator length 0.

**Problem 3.** *Determine whether the proof of Theorem 5 can be generalized to groups other than Baumslag–Solitar groups.*

Another interesting problem is to determine the precise size of the spectral gap for mapping class groups. This would involve understanding the defects of the quasimorphisms used in [3].

**Problem 4.** *Determine the exact size of the spectral gap for mapping class groups. Which mapping classes have minimal positive stable commutator length?*

## 3 Extremal quasimorphisms

For a fixed group element, it is known that the supremum in (2) is always achieved (see [9]). It is interesting to study which quasimorphisms achieve this supremum, i.e. when the lower bounds on stable commutator length discussed in Section 2 are sharp. Certifying that a quasimorphism is extremal for certain elements would give an indirect way to compute the stable commutator length of these elements. Two natural questions can be asked about when homogeneous quasimorphisms are extremal.

**Question 1.** *Given an element, which homogeneous quasimorphisms are extremal for it?*

**Question 2.** *Given a homogeneous quasimorphism, for which elements is it extremal?*

I have studied Question 2 for a well-known group, the modular group $\mathrm{PSL}(2,\mathbb{Z})$, and a naturally occuring quasimorphism, the rotation quasimorphism. The rotation quasimorphism on $\mathrm{PSL}(2,\mathbb{Z})$ is equal to the homogenization of the classical Rademacher function, and may therefore be described in a number of equivalent ways (see [1]). Perhaps the most explicit description is as follows (see [24]). Define $L = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Suppose $A$ is a hyperbolic element of $\mathrm{PSL}(2,\mathbb{Z})$. Then $A$ is conjugate to an element of the form $R^{a_1}L^{b_1}\dots R^{a_n}L^{b_n}$, $a_i, b_i > 0$, and this is unique up to cyclic permutation. In this case,

$$\mathrm{rot}(A) = \frac{\sum_{i=1}^n a_i - \sum_{i=1}^n b_i}{6}.$$

If $A$ is non-hyperbolic, then it either finite order, in which case the rotation quasimorphism takes the value 0, or parabolic, in which case it is conjugate to $R^a$ or $L^b$. Define $\mathrm{rot}(R^a) = a/6$ and $\mathrm{rot}(L^b) = -b/6$.

Progress in understanding the extremality of the rotation quasimorphism on the modular group has come from a geometric approach. Say that a curve $\gamma$ in a topological space $Y$ *virtually bounds an immersed surface* if there is an immersion $S \to Y$ mapping $\partial S$ to a cover of $\gamma$ in an orientation-preserving way. The following stability theorem shows that certain families of closed geodesics virtually bound immersed surfaces.

**Theorem 6** (Calegari–L. [14]). *For every hyperbolic element of the modular group, the product of this element with a sufficiently large power of a parabolic element is represented by a geodesic in the modular surface $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$ that virtually bounds an immersed surface.*

The rotation quasimorphism is easily seen to be extremal for finite order and parabolic elements, and it follows from work of Calegari [8] that it is extremal for a hyperbolic element of the modular group if and only if the corresponding geodesic on the modular surface $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$ virtually bounds an immersed surface. Therefore this theorem implies that the rotation quasimorphism is extremal for the corresponding elements of $\mathrm{PSL}(2,\mathbb{Z})$.

**Corollary** (Calegari–L. [14]). *For every element of the modular group $\mathrm{PSL}(2,\mathbb{Z})$, the product of this element with a sufficiently large power of a parabolic element is an element for which the rotation quasimorphism is extremal.*

It is worth mentioning that this result is surprising, as one expects that generically $\mathrm{scl}(A) \neq \mathrm{rot}(A)/2$. This is because Calegari–Maher [15] show that, in any hyperbolic group, the stable commutator length of a random word of length $k$ has order $k/\log k$. On the other hand, the rotation quasimorphism satisfies a central limit theorem, and hence its value on a random word of length $k$ has order $\sqrt{k}$ (see [12], [6]).

Although we have shown that the rotation quasimorphism is extremal for certain families of elements of the modular group, the exact condition controlling the extremality of the rotation quasimorphism is unclear. This is a natural question for further study.

**Problem 5.** *Characterize the elements of the modular group for which the rotation quasimorphism is extremal.*

Specifically, I hope to characterize these elements in terms of the arithmetic codings of geodesics on the modular surface given by Katok–Ugarcovici [23].

Since extremal quasimorphisms always exist, when the rotation quasimorphism is not extremal, there must be some other quasimorphism that is. A number of quasimorphisms on $\mathrm{PSL}(2,\mathbb{Z})$ can be constructed from the counting quasimorphisms $H_w$ on $F_2$ described in Section 2. Given an element of $\mathrm{PSL}(2,\mathbb{Z})$, find the corresponding formal sum of elements of $F_2$, as explained in [25]. Define a quasimorphism on $\mathrm{PSL}(2,\mathbb{Z})$ by setting its value on a particular element equal to the sum of the values of $H_w$ on the corresponding elements of $F_2$. I have an algorithm for computing the elements of $F_2$ corresponding to an element of $\mathrm{PSL}(2,\mathbb{Z})$ and have written a program that implements this algorithm. I will use this to calculate values of various such quasimorphisms on $\mathrm{PSL}(2,\mathbb{Z})$ and will compare these values with values of stable commutator length on $\mathrm{PSL}(2,\mathbb{Z})$ in an attempt to understand when these quasimorphisms are extremal.

Since the three-strand braid group is a central extension of the modular group, our procedure for computing stable commutator length in the modular group extends to the three-strand braid group. The rotation quasimorphism lifts from $\mathrm{PSL}(2,\mathbb{Z})$ to $B_3$, and our result gives information about when this lifted quasimorphism is extremal. The procedure for computing stable commutator length does not generalize to higher-strand braid groups, however, and therefore I would like to understand *why* certain quasimorphisms are extremal for three-strand braids, with the goal of carrying these reasons over to higher-strand braid groups. This would give an indirect method for computing the stable commutator length of higher-strand braids.

Another interesting family of quasimorphisms on braid groups are the signature quasimorphisms defined by Gambaudo–Ghys [21]. One can compute values of these quasimorphisms in certain special cases, but it seems difficult to do this in general. I have been working with a homological description of these quasimorphisms in which the computation may be more tractable, and hope to be able to determine when these quasimorphisms are extremal.

**Problem 6.** *Characterize the braids for which the signature quasimorphisms defined by Gambaudo–Ghys are extremal.*

# References


[1] Michael Atiyah, *The logarithm of the Dedekind $\eta$-function*, Math. Ann. **278** (1987), no. 1-4, 335–380. MR 909232 (89h:58177)

[2] Christophe Bavard, *Longueur stable des commutateurs*, Enseign. Math. (2) **37** (1991), no. 1-2, 109–150. MR 1115747 (92g:20051)

[3] Mladen Bestvina, Ken Bromberg, and Koji Fujiwara, *Stable commutator length on mapping class groups*, arXiv:1306.2394.

[4] Mladen Bestvina and Koji Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. **6** (2002), 69–89 (electronic). MR 1914565 (2003f:57003)

[5] ———, *Quasi-homomorphisms on mapping class groups*, Glas. Mat. Ser. III **42(62)** (2007), no. 1, 213–236. MR 2332668 (2008k:57002)

[6] Michael Björklund and Tobias Hartnick, *Biharmonic functions on groups and limit theorems for quasimorphisms along random walks*, Geom. Topol. **15** (2011), no. 1, 123–143. MR 2764114 (2011m:60061)

[7] Robert Brooks, *Some remarks on bounded cohomology*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 53–63. MR 624804 (83a:57038)

[8] Danny Calegari, *Faces of the scl norm ball*, Geom. Topol. **13** (2009), no. 3, 1313–1336. MR 2496047

[9] ———, *scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009. MR 2527432

[10] ———, *Stable commutator length is rational in free groups*, J. Amer. Math. Soc. **22** (2009), no. 4, 941–961. MR 2525776

[11] ———, *scl, sails, and surgery*, J. Topol. **4** (2011), no. 2, 305–326. MR 2805993 (2012g:57004)

[12] Danny Calegari and Koji Fujiwara, *Combable functions, quasimorphisms, and the central limit theorem*, Ergodic Theory Dynam. Systems **30** (2010), no. 5, 1343–1369. MR 2718897 (2011k:20088)

[13] ———, *Stable commutator length in word-hyperbolic groups*, Groups Geom. Dyn. **4** (2010), no. 1, 59–90. MR 2566301 (2011a:20109)

[14] Danny Calegari and Joel Louwsma, *Immersed surfaces in the modular orbifold*, Proc. Amer. Math. Soc. **139** (2011), no. 7, 2295–2308. MR 2784794 (2012e:20090)

[15] Danny Calegari and Joseph Maher, *Statistics and compression of scl*, arXiv:1008.4952.

[16] Matt Clay, Max Forester, and Joel Louwsma, *Stable commutator length in Baumslag–Solitar groups and quasimorphisms for tree actions*, arXiv:1310.3861.

[17] Andrew J. Duncan and James Howie, *The genus problem for one-relator products of locally indicable groups*, Math. Z. **208** (1991), no. 2, 225–237. MR 1128707 (92i:20040)

[18] David B. A. Epstein and Koji Fujiwara, *The second bounded cohomology of word-hyperbolic groups*, Topology **36** (1997), no. 6, 1275–1289. MR 1452851 (98k:20088)

[19] Koji Fujiwara, *The second bounded cohomology of a group acting on a Gromov-hyperbolic space*, Proc. London Math. Soc. (3) **76** (1998), no. 1, 70–94. MR 1476898 (99c:20072)

[20] ———, *The second bounded cohomology of an amalgamated free product of groups*, Trans. Amer. Math. Soc. **352** (2000), no. 3, 1113–1129. MR 1491864 (2000j:20098)

[21] Jean-Marc Gambaudo and Étienne Ghys, *Braids and signatures*, Bull. Soc. Math. France **133** (2005), no. 4, 541–579. MR 2233695 (2007d:57014)

[22] R. I. Grigorchuk, *Some results on bounded cohomology*, Combinatorial and geometric group theory (Edinburgh, 1993), London Math. Soc. Lecture Note Ser., vol. 204, Cambridge Univ. Press, Cambridge, 1995, pp. 111–163. MR 1320279 (96j:20073)

[23] Svetlana Katok and Ilie Ugarcovici, *Arithmetic coding of geodesics on the modular surface via continued fractions*, European women in mathematics—Marseille 2003, CWI Tract, vol. 135, Centrum Wisk. Inform., Amsterdam, 2005, pp. 59–77. MR 2223106 (2006k:37075)

[24] Robion Kirby and Paul Melvin, *Dedekind sums, $\mu$-invariants and the signature cocycle*, Math. Ann. **299** (1994), no. 2, 231–267. MR 1275766 (95h:11042)

[25] Joel Louwsma, *Extremality of the rotation quasimorphism on the modular group*, Ph.D. thesis, California Institute of Technology, CaltechTHESIS:05132010-155930760, 2011.

[26] A. H. Rhemtulla, *A problem of bounded expressibility in free products*, Proc. Cambridge Philos. Soc. **64** (1968), 573–584. MR 0225889 (37 #1480)