Self-Similar Surfaces: Involutions and Perfection

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Abstract

We investigate the problem of when big mapping class groups are generated by involutions. Restricting our attention to the class of *self-similar* surfaces, which are surfaces with self-similar ends space, as defined by Mann and Rafi, and with 0 or infinite genus, we show that, when the set of maximal ends is infinite, then the mapping class groups of these surfaces are generated by involutions, normally generated by a single involution, and uniformly perfect. In fact, we derive this statement as a corollary of the corresponding statement for the homeomorphism groups of these surfaces. On the other hand, among self-similar surfaces with one maximal end, we produce infinitely many examples in which their big mapping class groups are neither perfect nor generated by torsion elements. These groups also do not have the automatic continuity property.

1 Introduction

Consider a connected and oriented surface Σ. We distinguish two types of surfaces, those of finite type, i.e. a closed surface minus finitely many points, or of infinite type otherwise. Let $G(\Sigma)$ be either the group $\mathrm{Homeo}^+(\Sigma)$ of orientation preserving self-homeomorphisms of Σ or the mapping class group $\mathrm{MCG}(\Sigma)$ of Σ. We are interested in the algebraic structure of $G(\Sigma)$, especially when Σ has infinite type.

As a topological group, equipped with the compact open topology, $\mathrm{Homeo}^+(\Sigma)$ is a non-locally-compact Polish group. $\mathrm{MCG}(\Sigma)$, being a quotient of $\mathrm{Homeo}^+(\Sigma)$, inherits a topology. When Σ has finite type, then this topology is discrete and $\mathrm{MCG}(\Sigma)$ is finitely presented. But when Σ has infinite type, then $\mathrm{MCG}(\Sigma)$ is also a non-locally-compact Polish group, similar to the homeomorphism group. In particular, $\mathrm{MCG}(\Sigma)$ is not countably generated, justifying the nomenclature of *big mapping class group* in the literature.

An obvious group-theoretic problem is to identify canonical generating sets for $G(\Sigma)$. For any group, a natural choice is its set of involutions, or more broadly, its set of torsion elements. This leads us to ask if $G(\Sigma)$ is generated by involutions (or torsion elements). (The set of Dehn twists, being countable, can never generate a big mapping class group; and often, they do not even topologically generate [2].) For finite type surfaces, this question is well studied for their mapping class groups; see [16, 13, 4, 10, 11, 12, 19] and the references within for the story on generating by involutions. The goal of this paper is to explore this question for surfaces of infinite type.

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To answer this question for all surfaces of infinite type should be challenging, as $G(\Sigma)$ is as complicated as the homeomorphism group of the *ends space* of Σ. In trying to tame the world of surfaces of infinite type, Mann and Rafi [15] introduced a preorder on an ends space, and showed that the induced partial order always has maximal elements. They also introduced the notion of self-similar ends spaces. We call a surface self-similar if it has a self-similar ends space and 0 or infinite genus. Among these, we identity a subclass, called *uniformly* self-similar, which are self-similar with infinitely many maximal ends. This subclass exhibits additional symmetry, to which the sphere minus a Cantor set belongs. It was observed by Calegari [5] that the mapping class group of the sphere minus a Cantor set is uniformly perfect. We extend this result to all uniformly self-similar surfaces, along with answering the generation problem by involutions for these surfaces. Our main theorem is the following.

Theorem A. *Let Σ be a uniformly self-similar surface and $G(\Sigma)$ be either* $\mathrm{Homeo}^+(\Sigma)$ *or* $\mathrm{MCG}(\Sigma)$. *Then $G(\Sigma)$ is generated by involutions, normally generated by a single involution, and uniformly perfect. Moreover, each element of $G(\Sigma)$ is a product of at most 3 commutators and 12 involutions.*

Note that the case of $\mathrm{MCG}(\Sigma)$ follows immediately from that of $\mathrm{Homeo}^+(\Sigma)$. For the latter case, we use a method akin to *fragmentation*, a well known tool in the study of homeomorphism groups. In more detail, we first observe that a uniformly self-similar ends space E behaves very much like a Cantor set. Namely, any clopen subset $U \subset E$ containing a proper subset of the maximal ends is homeomorphic to its complement U^c. This gives rise to the notion of a *half-space* in a uniformly self-similar surface Σ, which is a subsurface $H \subset \Sigma$ with a single connected, compact boundary component, such that $\overline{H^c}$ is also a half-space and homeomorphic to H. (The exact definition is different and appears as Definition 3.2). We then find an *H–translation*, that is, a homeomorphism ϕ such that $\{\phi^n(H)\}_{n\in\mathbb{Z}}$ are all disjoint. This is a key step in the proof and requires putting the surface Σ into a particular form that reflects its symmetry. By our construction, the H–translation ϕ is a product of two conjugate involutions. Then, using a standard trick, we write every $f \in \mathrm{Homeo}(H,\partial H)$ as a commutator of the form $f = [\hat{f},\phi]$ for some $\hat{f}$. The final step is to show $\mathrm{Homeo}^+(\Sigma)$ is the normal closure of $\mathrm{Homeo}(H,\partial H)$, and so it is normally generated by ϕ and hence by a single involution. The other statements are achieved by keeping track of the number of commutators or involutions needed at each step.

Many of our steps above carry over to the case of equipping Σ with a marked point $*$. The key difference is now one can find a curve $\alpha \subset \Sigma$ which is not contained in any half-space H of Σ. Thus, it is no longer immediate that the Dehn twist about α can be generated by elements supported on H and their conjugates. To deal with this issue, we invoke the lantern relation. Using self-similarity, we can find an appropriate lantern, i.e. a 4-holed sphere bounding α with the other boundary components lying in half-spaces. Once we get all Dehn twists, then combining our previous method together with the fact that mapping class groups of compact surfaces are generated by Dehn twists, we obtain the following theorem.

Theorem B. *Let Σ be a uniformly self-similar surface with a marked point $* \in \Sigma$. Then,* $\mathrm{MCG}(\Sigma,*)$ *is perfect, generated by involutions, and normally generated by a single involution.*

Because we used the lantern relation, our proof does not apply to the homeomorphism group. For a different argument that the mapping class group of the marked sphere minus a Cantor set is perfect, see [18]. Theorem B is sharp in the sense that we cannot expect a statements about uniform perfection or a bound on the number of involutions, due to the fact that the marked sphere minus a Cantor set provides a counterexample, by [3].

It is not possible for all big mapping class groups to be generated by torsion elements or be perfect, even among the class of self-similar surfaces. One counterexample is the infinite genus surface with one end. This is a self-similar surface, but, by Domat and Dickmann [7], the abelianization of its mapping class group contains $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$ as a summand.

Using their results and a covering trick, we can show the mapping class group of the surface $\mathbb{R}^2 \setminus \mathbb{N}$ has similarly large abelianization. Note that this surface is also self-similar but not uniformly. On the other hand, using a method similar to our proof of Theorem A, we can also show $\mathrm{MCG}(\mathbb{R}^2 \setminus \mathbb{N})$ is *topologically* generated by involutions. Since any homomorphism from a Polish group to $\mathbb{Z}$ is always continuous, this makes its first cohomology group vanish, in contrast with homology.

Theorem C. *The group* $\mathrm{MCG}(\mathbb{R}^2 \setminus \mathbb{N})$ *surjects onto* $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$. *In particular, it is not perfect or generated by torsion elements. On the other hand,* $\mathrm{MCG}(\mathbb{R}^2 \setminus \mathbb{N})$ *is topologically generated by involutions, so* $\mathrm{H}^1(\mathrm{MCG}(\mathbb{R} \setminus \mathbb{N}), \mathbb{Z}) = 0$.

The statement of topological generation by involutions also extends to the mapping class group of the one-ended infinite genus surface Σ_L. Additionally, we can get infinitely many examples of surfaces whose mapping class groups have similarly large abelianization, by considering appropriate maps to Σ_L or $\mathbb{R}^2 \setminus \mathbb{N}$. Many of these examples are self-similar but not uniformly. Another application of our result beyond the ones mentioned is the *automatic continuity property*. Recall a Polish group G has the automatic continuity property if every homomorphism from G to a separable topological group is necessarily continuous. The family of surfaces we construct also gives rise to a large family of homeomorphism groups or big mapping class groups that do not have this property. This gives some progress towards answering [14, Question 2.4]. We highlight the following examples and refer to Theorem 5.3 and Corollary 5.5 for the full technical statement.

Theorem D. *Let* $\Sigma = S^2 \setminus E$, *where* S^2 *is the 2-sphere and* E *is a countable closed subset of the Cantor set homeomorphic to the ordinal* $\omega^\alpha + 1$, *where* α *is a countable successor ordinal. Let* $G(\Sigma)$ *be either* $\mathrm{Homeo}^+(\Sigma)$ *or* $\mathrm{MCG}(\Sigma)$. *Then* $G(\Sigma)$ *is not perfect, is not generated by torsion elements, and does not have the automatic continuity property.*

One may wonder what happens in the case of positive genus, rather than 0 or infinite genus. Our methods do not extend to these surfaces. However, for a surface Σ obtained by removing a Cantor set from a surface of genus g with 0 or 1 puncture, Chen and Calegari [6] showed various results for $\mathrm{MCG}(\Sigma)$ including that it is generated by torsion. Additionally, Mann [14] showed $G(\Sigma)$ has the automatic continuity property. It would be interesting to know if their techniques extend to uniformly self-similar ends spaces. We refer to their papers for more details.

As many cases still remain open, we invite the reader to explore other classes of surfaces of infinite type which may verify the properties in Theorem A or admit an obstruction. It would also be interesting to find other natural generating sets for big mapping class groups or homeomorphism groups. Similar questions can also be asked for the homeomorphism groups of ends spaces.

Here is a brief outline of the paper. In Section 2, we introduce ends spaces and the classification of surfaces of infinite type. Following [15], we define self-similar ends spaces and surfaces and a partial order on ends spaces. We also observe some nice properties about self-similar ends spaces that lead to the definition of half-spaces in uniformly self-similar surfaces. The proof of Theorem A is contained in Section 3, and the proof of Theorem B in Section 4. The two parts of Theorem C appear in Section 5 as Proposition 5.1 and Theorem 5.7. Theorem D follows from Corollary 5.5 as a special case.

2 Preliminaries

2.1 Partial order on ends spaces

An *ends space* is a pair (E,F), where E is a totally disconnected, compact, metrizable space and $F \subset E$ is a (possibly empty) closed subspace. For simplicity, we will often suppress the notation F, but by convention, all homeomorphisms of E will be relative to F. For instance, to say $C \subset E$ is homeomorphic to $D \subset E$ means there is a homeomorphism from $(C, C \cap F)$ to $(D, D \cap F)$. We denote by $\mathrm{Homeo}(E,F)$ the group of homeomorphisms of E preserving F.

The assumptions on E imply it is homeomorphic to a closed subspace of the standard Cantor set (see [17, Proposition 5]). We will often view E as this subspace (and F as a further closed subspace).

Definition 2.1. An ends space (E,F) is called *self-similar* if for any decomposition of $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$ into pairwise disjoint clopen sets, then there exists some clopen set D contained in some E_i such that $(D, D \cap F)$ is homeomorphic to (E,F).

Following [15], given an ends space (E,F), define a preorder $\preceq$ on E where for $x, y \in E$, we say $x \preceq y$ if every neighborhood U of y contains some homeomorphic copy of a neighborhood V of x. *Here and throughout the paper, a neighborhood in an end space will always be a clopen neighborhood.* We say x and y are equivalent, and write $x \sim y$, if $x \preceq y$ and $y \preceq x$. This defines an equivalence relation on E. For $x \in E$, denote by $E(x)$ the equivalence class of x, and denote by $[E]$ the set of equivalence classes. From this we get a partial order $\prec$ on $[E]$, defined by $E(x) \prec E(y)$ if $x \preceq y$ and $x \nsim y$. Note that by definition, if $x \preceq y$, then y is either locally homeomorphic to x or an accumulation point of homeomorphic images of x under $\mathrm{Homeo}(E,F)$. One easily verifies that, since $F \subset E$ is closed, either $E(x) \cap F = \emptyset$ or $E(x) \cap F = E(x)$.

We say a point $x \in E$ is *maximal* if $E(x)$ is maximal with respect to $\prec$. Observe that when $x \in E$ is maximal and $F \neq \emptyset$, then $E(x) \cap F = E(x)$. Denote by $M(E)$ the set of maximal elements in E.

Proposition 2.2 ([15])**.** *Let (E,F) be an ends space. The following statements hold.*

- *The set $M(E)$ of maximal elements under the partial order $\prec$ is non-empty.*
- *For every $x \in M(E)$, its equivalence class $E(x)$ is either finite or homeomorphic to a Cantor set.*
- *If (E,F) is self-similar, then $M(E)$ is a single equivalence class $E(x)$, and $E(x)$ is either a singleton or homeomorphic to a Cantor set.*

2.2 Classification of infinite-type surfaces

By a *surface* we always mean a connected, orientable 2–manifold. A surface has *finite type* if its fundamental group is finitely generated; otherwise, it has infinite type. In this paper, we are primarily interested in surfaces of infinite type. We refer to [17] for details.

The collection of compact sets on a surface Σ forms a directed set by inclusion. The *space of ends* of Σ is

$$E(\Sigma) = \varprojlim \pi_0(\Sigma \setminus K),$$

where the inverse limit is taken over the collection of compact subsets $K \subset \Sigma$. Equip each $\pi_0(\Sigma \setminus K)$ with the discrete topology. Then the limit topology on $E(\Sigma)$ is a totally disconnected, compact, and metrizable. An element of $E(\Sigma)$ is called an *end* of Σ.

An end $e \in E(\Sigma)$ is *accumulated by genus* if for all subsurface $S \subset \Sigma$ with $e \in E(S)$, then S has infinite genus; otherwise, e is called *planar*. Let $E^g(\Sigma)$ be the subset of $E(\Sigma)$ consisting of ends accumulated by genus. This is always a closed subset of $E(\Sigma)$, with $E(\Sigma) = \emptyset$ if and only if Σ has finite genus. Hence the pair $(E(\Sigma), E^g(\Sigma))$ forms an ends space. Conversely, by [17, Theorem 2], every ends space (E, F) can be realized as the space of ends of some surface Σ, with $(E, F) = (E(\Sigma), E^g(\Sigma))$.

Infinite type surfaces are completely classified by the following data: the genus (possibly infinite), and the homeomorphism type of the ends space $(E(\Sigma), E^g(\Sigma))$. More precisely:

Theorem 2.3 ([17, Theorem 1])**.** *If Σ and Σ' are surfaces of the same (possibly infinite) genus. Then, Σ and Σ' are homeomorphic if and only if there is a homeomorphism between $(E(\Sigma), E^g(\Sigma))$ and $(E(\Sigma'), E^g(\Sigma'))$.*

We remark that although Richards' classification of infinite type surfaces is only stated for boundaryless surfaces, it easily extends to surfaces with finitely many boundary components. That is, two surfaces with the same genus, same number of (finitely many) boundary components, and homeomorphic end space pairs (E, E^g) are in fact homeomorphic.

Fix an orientation on a surface Σ and set $(E, F) = (E(\Sigma), E^g(\Sigma))$. Let $\mathrm{Homeo}^+(\Sigma)$ be the group of orientation preserving homeomorphisms of Σ. This is a topological group equipped with the compact open topology, and moreover it is a Polish group. The connected component of the identity is a closed normal subgroup $\mathrm{Homeo}_0(\Sigma)$ comprised of homeomorphisms isotopic to the identity. The quotient group

$$\mathrm{MCG}(\Sigma) = \mathrm{Homeo}^+(S) / \mathrm{Homeo}_0(S)$$

is called the *mapping class group* of Σ. When Σ has finite type, then $\mathrm{MCG}(\Sigma)$ is discrete and finitely presented. When Σ has infinite type, then $\mathrm{MCG}(\Sigma)$ is a non locally-compact Polish group.

Every homeomorphism of Σ induces a homeomorphism of its ends space (E, F), and two homotopic homeomorphisms of Σ induce the same map on (E, F). This gives a continuous homomorphism $\Phi : \mathrm{Homeo}^+(\Sigma) \to \mathrm{Homeo}(E, F)$ that factors through $\mathrm{MCG}(\Sigma)$. By [17], the map Φ is also surjective.

As noted in [15, Section 4], we also know the preorder $\preceq$ on E is equivalent to: $x \preceq y$ if and only if for every neighborhood U of y there is a neighborhood V of x and $f \in \mathrm{Homeo}^+(\Sigma)$ such that $\Phi(f)(V) \subset U$.

Definition 2.4. A surface Σ is called *self-similar* if its space of ends $(E(\Sigma), E^g(\Sigma))$ is self-similar and Σ has genus 0 or infinite genus.

Note that when Σ is self-similar and has infinite genus, then each maximal end of $E(\Sigma)$ must be accumulated by genus.

Remark 2.5. We point out our definition of self-similar surfaces is equivalent to another notion. First, following [15], a subset A of a surface Σ is called *non-displaceable* if $f(A) \cap A \neq \emptyset$ for every $f \in \mathrm{Homeo}(S)$. Then, Σ is self-similar if and only if Σ has self-similar ends space and no non-displaceable compact subsurfaces. One direction is clear: if Σ has finite positive genus, then Σ has a compact non-displaceable subsurface. The other direction is observed by [1, Lemma 5.9 and 5.13].

2.3 Stable neighborhoods of ends and self-similarity

We now collect some facts about self-similar ends spaces. The key take away of this section is that self-similar ends spaces with infinitely many maximal ends behave very much like a Cantor set.

Definition 2.6. Given $x \in E$, a neighborhood U of x is called *stable* if any smaller neighborhood $V \subset U$ contains a homeomorphic copy of U. (Recall that this means that $(V, V \cap F)$ contains a homeomorphic copy of $(U, U \cap F)$).

Lemma 2.7 ([1, Lemma 5.4])**.** *If (E,F) is self-similar, then for all maximal element $x \in E$, the set E is a stable neighborhood of x.*

The following statement is reminiscent of the statement of [15, Lemma 4.17], but stronger than what the latter implies, though our proof is modeled after theirs.

Lemma 2.8. *Suppose (E,F) is self-similar. Then for all maximal points $x, y \in M(E)$ and all clopen neighborhoods U, V resp. of x, y, there exists a homeomorphism $\varphi : (U, U \cap F) \to (V, V \cap F)$ such that $\varphi(x) = y$.*

Proof. The proof follows a back-and-forth argument. As usual, we will suppress F, so all maps below are maps of pairs relative to F.

Let $U_0 = U$ and $V_0 = V$. We define the homeomorphism from U to V inductively on clopen subsets exhausting $U \setminus \{x\}, V \setminus \{y\}$. For convenience, we choose some metric on E. We choose $U_1 \subseteq U_0$ to be a proper neighborhood of x of diameter less than 1. Since E is a stable neighborhood of y by Lemma 2.7, and $U_0 \setminus U_1$ is clopen, there is a continuous map

$$f_0 : U_0 \setminus U_1 \to V_0$$

which is a homeomorphism onto a clopen image. We can choose f_0 such that $\mathrm{im}(f_0) \subseteq V_0 \setminus \{y\}$ for the following reasons. If $M(E) = \{y\}$, this is automatic. If $M(E)$ is a Cantor set, then $V_0 \setminus \{y\}$ contains some $z_0 \in M(E)$, and Lemma 2.7 ensures we can map $U_0 \setminus U_1$ homeomorphically into a sufficiently small neighborhood of z_0 which avoids y. Since $\mathrm{im}(f_0)$ is clopen, we can choose a proper clopen subset $V_1 \subseteq V_0 \setminus \mathrm{im}(f_0)$ of y which has diameter less than 1. By the same token, we can find a map

$$g_0 : V_0 \setminus (V_1 \cup \mathrm{im}(f_0)) \to U_1 \setminus \{x\}$$

which is a homeomorphism onto a proper clopen image. We similarly define a proper clopen neighborhood $U_2 \subseteq U_1 \setminus \mathrm{im}(g_0)$ of x which has diameter less than $\frac{1}{2}$.

Inductively, suppose $U_0, \ldots, U_{n+1}, V_0, \ldots, V_n$ have been constructed as well as maps which are homeomorphic onto their image

$$f_i : U_i \setminus (U_{i+1} \cup \mathrm{im}(g_{i-1})) \to V_i \setminus V_{i+1}$$

$$g_i : V_i \setminus (V_{i+1} \cup \mathrm{im}(f_i)) \to U_{i+1} \setminus U_{i+2}$$

for $0 \le i \le n-1$. Using Lemma 2.7 as above, we then define a map which is a homeomorphism onto its image

$$f_n : U_n \setminus (U_{n+1} \cup \mathrm{im}(g_{n-1})) \to V_n \setminus \{y\}$$

and choose a proper clopen neighborhood $V_{n+1} \subseteq V_n \setminus \mathrm{im}(f_n)$ of y, of diameter less than $\frac{1}{n+1}$. Similarly, we define a map which is a homeomorphism onto its image

$$g_n : V_n \setminus (V_{n+1} \cup \mathrm{im}(f_n)) \to U_{n+1} \setminus \{x\}$$

and choose a proper clopen neighborhood $U_{n+2} \subseteq U_{n+1} \setminus \mathrm{im}(g_n)$ of x, of diameter less than $\frac{1}{n+2}$. We thereby inductively construct such a sequence of maps $f_0, f_1, \ldots$ and $g_0, g_1, \ldots$.

Now, restrict target spaces of f_i, g_i to their images. Then, by construction, the domains and images of the f_i and g_i^{-1} are disjoint and their respective unions are $U \setminus \{x\}$ and $V \setminus \{y\}$. Thus, by taking the union of f_i and g_i^{-1}, we obtain a continuous bijection $\psi : U \setminus \{x\} \to V \setminus \{y\}$ since their domains are open subsets. Similarly, we can define the continuous inverse of ψ with the f_i^{-1} and g_i. Moreover, we can extend ψ to a homeomorphism $\varphi : U \to V$ by mapping x to y. □

3 Generation of the homeomorphism group

Our proof of Theorem A in the case of an unmarked surface proceeds via the following steps. First, we define the notion of a *half-space* of Σ and show that the normal closure of a single involution contains an H-translation for some half-space H. Formally, if H is a half-space, we say a homeomorphism φ is an *H-translation* if $\{\varphi^n(H)\}_{n\in\mathbb{Z}}$ are all pairwise disjoint. We then show that the normal closure of such a φ contains all homeomorphisms supported on H and that half-space supported homeomorphisms generate $\mathrm{Homeo}^+(\Sigma)$.

Definition 3.1. A self-similar ends space (E,F) is *uniformly* self-similar if $M(E)$ is one equivalence class homeomorphic to a Cantor set. A surface Σ is called *uniformly self-similar* if $(E(\Sigma), E^g(\Sigma))$ is uniformly self-similar and Σ has genus 0 or infinity.

Definition 3.2. For a uniformly self-similar surface Σ, we will define a *half-space* to be a subsurface $H \subset \Sigma$ such that

(i) H is a closed subset of Σ.

(ii) H has a single connected, compact boundary component.

(iii) $E(H)$ and $E(\overline{H^c})$ both contain a maximal end of $E(\Sigma)$.

The following statement seems to be well known, and we state it without proof.

Lemma 3.3. *Let Σ be a surface. Every clopen set U of $E(\Sigma)$ is induced by a connected subsurface of Σ with a single boundary circle. Consequently, if Σ is uniformly self-similar, and both U, U^c contain maximal ends, then this subsurface is a half-space.*

The following corollary follows easily from the above Lemma and the fact that $E(\Sigma)$ is a subspace of a Cantor set. Note that the sequence will be finite in the case x is an isolated end, but otherwise the sequence is countably infinite.

Corollary 3.4. *Let Σ be a uniformly self-similar surface, and let $x \in E(\Sigma)$. There exists a sequence of nested half-spaces $S_1 \supseteq S_2 \supseteq \dots$ such that $\{x\} = \bigcap_i E(S_i)$ and ∂S_i is compact and connected for all i.*

Lemma 3.5. *Let Σ be a uniformly self-similar surface. Then, there exists a half-space $H \subseteq \Sigma$, an involution τ, and $\varphi \in \mathrm{Homeo}^+(\Sigma)$ such that $\left\{\varphi^n(H)\right\}_{n\in\mathbb{Z}}$ are all pairwise disjoint and $\varphi \in \langle\langle \tau \rangle\rangle$. Moreover, φ is a product of two conjugates of τ, and we can choose τ and φ to fix some point in the complement of $\bigcup_{n\in\mathbb{Z}} \varphi^n(H)$.*

Proof. We first construct a somewhat explicit surface which is homeomorphic to Σ. Let $E = E(\Sigma)$ and $M = M(E)$. Let $y, z \in M$ be distinct points. Since E is homeomorphic to a subspace of the Cantor set, we can find disjoint clopen subsets $\{U_i \mid i \in \mathbb{Z}\}$ such that

- $U_i \cap M \neq \emptyset$
- $E = \left\{y, z\right\} \cup \bigcup_i U_i$
- y is an accumulation point of $\{U_i \mid i \le 0\}$ but not $\{U_i \mid i \ge 0\}$, and z is an accumulation point of $\{U_i \mid i \ge 0\}$ but not $\{U_i \mid i \le 0\}$.

By Lemma 3.3, there is a half-space $\Sigma_0 \subseteq \Sigma$ where $E(\Sigma_0) = U_0$. We let S_i be a copy of Σ_0 for each $i \in \mathbb{Z}$, and we let $\underline{S}$ be the (oriented) infinite cylinder with countably many disjoint open discs removed in a periodic fashion. (We make sure to choose discs with disjoint closures.) See Figure 1. Let $\{C_i \mid i \in \mathbb{Z}\}$ be the boundary components of $\underline{S}$. Let S be the surface obtained by gluing C_i to ∂S_i via some homeomorphism $\psi_i : C_i \to \partial S_i$ which respects orientation of the surfaces.

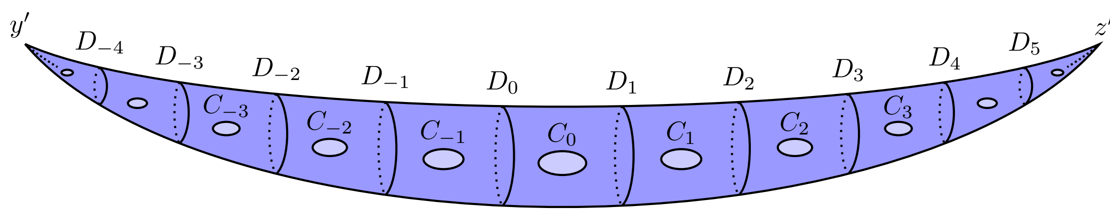

Figure 1: The surface $\underline{S}$.

We first claim that $S \cong \Sigma$. By Theorem 2.3, we need only prove that S, Σ have the same genus and that there is a homeomorphism of end spaces mapping $E^g(S)$ to $E^g(\Sigma)$. We will implicitly use some results from [17] without referencing them. Recall that Σ has genus 0 or ∞, and in the latter case, a maximal end must be accumulated by genus. Thus S_i and S have infinite genus if and only if Σ does. By Lemma 2.8, we have that $E(S_i) \cong U_0 \cong U_i$ (respecting genus accumulation). By construction of S, all of $E(S_i)$ are clopen subsets of $E(S)$. Moreover, for $i \in \mathbb{Z}$, let D_i be disjoint curves in the cylinder $\underline{S}$ such that they are all translates of each other, separate the two ends of the cylinder, and the two sides of D_i contain $\{C_j \mid j < i\}$ and $\left\{C_j \mid j \ge i\right\}$. See Figure 1. Let P_i^+, P_i^- be the subsurfaces of S on either side of D_i. Then, P_i^+ for $i \ge 0$ (resp. P_i^- for $i \le 0$) defines an end z' (resp. y') of S. Then, for $n \in \mathbb{N}$,

$$E(S) = E(P_{-n}^-) \cup E(P_n^+) \cup \bigcup_{i=-n}^{n-1} E(S_i),$$

and since only y', z' are in all $E(P^-_{-n})$ and $E(P^+_n)$ resp., we have $E(S) = \{y', z'\} \cup \bigcup_i E(S_i)$. Moreover, it's clear that $E(S_i)$ accumulate to y' but not z' as $i \to -\infty$ and $E(S_i)$ accumulates to z' but not y' as $i \to \infty$. Therefore, we may define a homeomorphism $E(S) \to E(\Sigma)$ mapping $E(S_i) \to U_i$ and $\{y', z'\} \to \{y, z\}$.

Recall that the homeomorphism $E(S_i) \cong U_i$ maps ends accumulated to genus by ends accumulated by genus. If S has infinite genus, then every S_i has infinite genus and so y', z' are accumulated by genus (as must y, z as they are maximal in $E(\Sigma)$). Consequently, $E(S) \cong E(\Sigma)$ maps $E^g(S)$ to $E^g(\Sigma)$ and only $E^g(S)$ to $E^g(\Sigma)$. Consequently, $S \cong \Sigma$.

We now construct an explicit involution τ of S which normally generates the desired φ. First, we define an involution $\underline{\tau}$ on $\underline{S}$. We simply take the "rotation" about an axis piercing D_0 and interchanging the ends y', z'. This induces a homeomorphism between pairs of curves C_i, C_j. For $i < j$ where $\underline{\tau}(C_i) = C_j$, define a homeomorphism $\tau_{i,j} : S_i \to S_j$ such that

$$\tau_{i,j}|_{\partial S_i} = \psi_j \circ \underline{\tau} \circ \psi_i^{-1}.$$

For the same pair, define $\tau_{j,i} : S_j \to S_i$ as the inverse of $\tau_{i,j}$. Note that

$$\tau_{j,i}|_{\partial S_j} = \psi_i \circ \underline{\tau}^{-1} \circ \psi_j^{-1} = \psi_i \circ \underline{\tau} \circ \psi_j^{-1}$$

since $\underline{\tau}$ has order 2. Thus, the $\tau_{i,j}$ agree on the overlap with $\underline{\tau}$, and so we extend $\underline{\tau}$ to a homeomorphism τ on all of S via the $\tau_{i,j}$. It is clear that τ has order 2.

We can similarly define an involution σ which is a "rotation" with axis piercing D_1. Then, $\sigma(\tau(S_i)) = S_{i+2}$. I.e. $\varphi = \sigma \circ \tau$ is the desired H-translation where $H = S_0$. This establishes the lemma.

To show that we can choose τ and φ to also fix a point outside the S_i, we do the following. We homotope D_0 and D_1 within $\underline{S}$ towards each other until they meet tangentially at one point. One can choose an involution τ that permutes the C_i in the same manner as above, maps D_0 to itself and fixes the common point of $D_0 \cap D_1$. Similarly, σ may be chosen to map D_1 to itself and fix the common point of $D_0 \cap D_1$. Since the new τ and σ each permute the C_i in the same manner as before, the rest of the argument goes through, and φ will fix the same point. $\square$

Remark 3.6. Note that in the above proof, the translation φ (in the version without a fixed point) verifies that a surface Σ with uniformly self-similar ends space with 0 or infinite genus has no non-displaceable surfaces. This also follows from [1, Lemma 5.9, Lemma 5.13] which prove it in the case where $E(\Sigma)$ is merely self-similar, but our construction gives a different perspective.

We now show that an H-translation normally generates $\mathrm{Homeo}(H, \partial H) < \mathrm{Homeo}^+(\Sigma)$ for some half-space H. The proof technique is sometimes referred to as a "swindle".

Lemma 3.7. *Let Σ be uniformly self-similar, and let φ have the properties as described in Lemma 3.5. Then, $\langle\langle\varphi\rangle\rangle$ contains* $\mathrm{Homeo}(H, \partial H)$. *Moreover, every element of* $\mathrm{Homeo}(H, \partial H)$ *is a product of φ and a conjugate of φ^{-1}.*

Proof. Let $f \in \mathrm{Homeo}(H, \partial H) \subseteq \mathrm{Homeo}^+(\Sigma)$. We let $\hat{f} = \prod_{i=0}^{\infty} \varphi^{-i} f \varphi^i$. This is well-defined since $\varphi^{-i} f \varphi^i$ is supported on $\varphi^{-i}(H)$ and these are pairwise disjoint for all $i \geq 0$ by assumption. Then, we have the following computation, which, again, is valid because of disjoint supports.

$$[\hat{f}, \varphi^{-1}] = \hat{f}\varphi^{-1}\hat{f}^{-1}\varphi = \left(\prod_{i=0}^{\infty} \varphi^{-i} f \varphi^i\right) \varphi^{-1} \left(\prod_{i=0}^{\infty} \varphi^{-i} f^{-1} \varphi^i\right) \varphi = \left(\prod_{i=0}^{\infty} \varphi^{-i} f \varphi^i\right) \left(\prod_{i=1}^{\infty} \varphi^{-i} f^{-1} \varphi^i\right) = f. \quad \square$$

3.1 Half-space homeomorphisms generate

Our proof that homeomorphisms of half-spaces generate $\mathrm{Homeo}^+(\Sigma)$ relies on a few key facts about half-spaces in uniformly self-similar surfaces. We record these as lemmas.

Lemma 3.8. *Let $H_1, H_2 \subset \Sigma$ be two half-spaces. Then, one of $H_1 \cap H_2^c, H_1^c \cap H_2^c$ contains a half-space.*

Lemma 3.9. *If H_1, H_2 are two distinct half-spaces contained in a third distinct half-space H_3 and both are disjoint from a fourth half-space $H_4 \subset H_3$, then there exists $\varphi \in \mathrm{Homeo}^+(\Sigma)$ supported on H_3 such that $\varphi(H_1) = H_2$.*

Lemma 3.10. *If $H \subseteq \Sigma$ is a half-space, then so is $\overline{H^c}$. All half-spaces are homeomorphic via an ambient homeomorphism of Σ. Every half-space contains two disjoint half-spaces.*

Using these three lemmas we can prove one of our main results, namely, that half-space homeomorphisms generate $\mathrm{Homeo}^+(\Sigma)$.

Theorem 3.11. *Let Σ be a uniformly self-similar surface, and let $H \subseteq \Sigma$ be a half-space. Then,* $\mathrm{Homeo}^+(\Sigma)$ *is the normal closure of the subgroup* $\mathrm{Homeo}(H, \partial H)$*. Furthermore, every element of* $\mathrm{Homeo}^+(\Sigma)$ *is a product of at most* 3 *homeomorphisms, each of which is conjugate to an element of* $\mathrm{Homeo}(H, \partial H)$.

Proof. Let $f \in \mathrm{Homeo}^+(\Sigma)$. First, note that by Lemma 3.10, any half-space supported homeomorphism is conjugate into $\mathrm{Homeo}(H, \partial H)$. Thus, it suffices to show f is a product of at most 3 half-space supported homeomorphisms.

Let $H_1 = H$ and $H_2 = f(H)$. We now apply Lemma 3.8, and first consider the case where $H_1^c \cap H_2^c$ contains a half-space. By Lemma 3.10, $H_1^c \cap H_2^c$ contains two disjoint half-spaces H_3, H_4. Applying Lemma 3.9 to $H_1, H_2, \overline{H_3^c}$, and H_4, we see that there is a homeomorphism φ_1, supported on $\overline{H_3^c}$, such that $\varphi_1(H_2) = \varphi_1(f(H_1)) = H_1$. By further composing by some φ_2 supported on H_1, we can ensure $\varphi_2 \circ \varphi_1 \circ f$ restricts to the identity on H_1. Finally, composing by an appropriate third homeomorphism φ_3 supported on $\overline{H_1^c}$, we obtain $\varphi_3 \circ \varphi_2 \circ \varphi_1 \circ f = \mathrm{Id}$. Note that $\varphi_2 \circ \varphi_1$ is supported on $\overline{H_3^c}$, so in this case we only require two half-space supported homeomorphisms.

Now, suppose we are in the case where $H_1 \cap H_2^c$ contains a half-space. By Lemma 3.10, we can assume $H_1 \cap H_2^c$ contains three disjoint half-spaces H_3, H_4, H_5. By Lemma 3.9 applied to $H_2, H_4, \overline{H_3^c}$, and H_5, there exists a homeomorphism ψ supported on $\overline{H_3^c}$ such that $\psi(H_2) = H_5$. Since $H_5 \subset H_1$, the subsurface $H_1^c \cap \psi(f(H_1))^c = H_1^c$ contains a half-space, and we are reduced to the previous case. In this case, we see that f is a product of three half-space supported homeomorphisms. □

We now prove the required lemmas.

Proof of Lemma 3.8. By definition, $E(\overline{H_2^c})$ contains some maximal end x. By Corollary 3.4, there exist nested half-spaces $S_1 \supset S_2 \supset \dots$ such that $\bigcap_i E(S_i) = \{x\}$. Since these half-spaces leave every compact set, eventually some S_i does not intersect $\partial H_1 \cup \partial H_2$. Since $x \in E(\overline{H_2^c})$ and the S_i are connected, either $S_i \subseteq H_1 \cap H_2^c$ or $S_i \subseteq H_1^c \cap H_2^c$. □

Proof of Lemma 3.10. The first statement follows immediately from the definition of half-space. Since a half-space has a maximal end of Σ, it has the same genus as Σ. Thus, by the classification of surfaces and Lemma 2.8, any two half-spaces are homeomorphic. Since the (closures of) the complements are half-spaces too, we can map the complement to the complement and extend the homeomorphism to all of Σ.

By assumption, $E(H)$ has some maximal end x of $E(\Sigma)$. Since $E(H)$ is a clopen in $E(\Sigma)$ and the set of maximal ends is a Cantor set, $E(H)$ contains another distinct maximal end y. Using Corollary 3.4 for x and y each and compactness of boundaries of half spaces, we can easily deduce the existence of the required half-spaces. $\square$

Proof of Lemma 3.9. The presence of the half-space H_4 guarantees that $E(\overline{H_3 \setminus H_1})$ and $E(\overline{H_3 \setminus H_2})$ both contain a maximal end. Thus, by Lemma 2.8,

$$(E(\overline{H_3 \setminus H_1}), E^g(\overline{H_3 \setminus H_1})) \cong (E(\overline{H_3 \setminus H_2}), E^g(\overline{H_3 \setminus H_2})).$$

Clearly, the two subsurfaces have the same genus and finite number of boundary components, and so $\overline{H_3 \setminus H_1} \cong \overline{H_3 \setminus H_2}$. Similarly, by Lemma 3.10, $H_1 \cong H_2$. By arranging the homeomorphisms to be identical on the overlapping boundary component, we produce a homeomorphism $H_3 \to H_3$ mapping $H_1 \to H_2$ and $\overline{H_3 \setminus H_1} \to \overline{H_3 \setminus H_2}$. $\square$

Theorem 3.12. *If Σ is uniformly self-similar, then* $\mathrm{Homeo}^+(\Sigma)$

- *is normally generated by a single involution,*
- *is normally generated by an H-translation,*
- *is uniformly perfect.*

Moreover, each element of $\mathrm{Homeo}^+(\Sigma)$ *is a product of at most* 3 *commutators,* 6 *H-translations, and* 12 *involutions.*

Proof. Combine Lemma 3.5, Lemma 3.7, and Theorem 3.11. $\square$

By considering quotients of $\mathrm{Homeo}^+(\Sigma)$ onto the mapping class group $\mathrm{MCG}(\Sigma)$ and the homeomorphism group of its ends space $(E(\Sigma), E^g(\Sigma))$, we also derive the following corollaries. Note that for a half-space $H \subset \Sigma$, $E(H)$ is a clopen set containing a non-empty proper subset of $M(E(\Sigma))$.

Corollary 3.13. *If Σ is uniformly self-similar, then the statements of Theorem 3.12 also hold for* $\mathrm{MCG}(\Sigma)$.

Corollary 3.14. *If (E, F) is uniformly self-similar, then the statements of Theorem 3.12 also hold for* $\mathrm{Homeo}(E, F)$*, where a half-space $H \subset E$ is a clopen set containing a non-empty proper subset of* $M(E)$.

Proof. By [17, Theorem 2], there is a surface Σ such that $(E(\Sigma), E^g(\Sigma)) \cong (E, F)$ and the genus is 0 if $F = \emptyset$ and infinite if $F \neq \emptyset$. Thus Σ is uniformly self-similar when (E, F) is. The corollary then follows from Theorem 3.12 and surjectivity of $\mathrm{Homeo}^+(\Sigma) \to \mathrm{Homeo}(E(\Sigma), E^g(\Sigma))$. $\square$

4 Surfaces with a marked point

The proof in the case of a marked surface is very similar to the unmarked case, and we will use some of the same lemmas. Let Σ be a uniformly self-similar surface with a fixed basepoint $* \in \Sigma$. We define half-space exactly as before, but distinguish between *marked half-spaces* which contain $*$ and *unmarked half-spaces* which don't. The main new lemma we require is the following.

Lemma 4.1. *Let Σ be a uniformly self-similar surface with a marked point $* \in \Sigma$. Let $H \subseteq \Sigma$ be an unmarked half-space. Then, every Dehn twist in* $\mathrm{MCG}(\Sigma, *)$ *is contained in the normal closure of* $\mathrm{MCG}(H, \partial H)$.

Remark 4.2. For convenience and simplicity, we will conflate half-spaces and simple closed curves with their ambient isotopy classes rel $*$ throughout this section.

Proof. Let $T_\gamma \in \mathrm{MCG}(\Sigma, *)$ be the Dehn twist about a simple closed curve γ (which avoids $*$). First, we consider the case where γ is nonseparating. Then, Σ has infinite genus. Since γ is compact, Corollary 3.4 implies that γ is contained in some half-space (or the closure of its complement which is also a half-space) which we denote by H. This case is concluded if H is unmarked. Suppose instead H is marked. Then, since γ is nonseparating, we can find a path from ∂H to $*$ which avoids γ. Deleting some small regular neighborhood of this path from H, we obtain an unmarked half-space containing γ.

Now, suppose γ is a separating curve, and let $S_1, S_2 \subseteq \Sigma$ be the two surfaces on either side of γ. If both $E(S_1), E(S_2)$ contain a maximal end of Σ, then they are both half-spaces whose mapping class groups contain T_γ, and one must be unmarked. Suppose, w.l.o.g., then that $E(S_1)$ contains no maximal ends. If S_1 is also unmarked, then we can connect it by some strip (avoiding $*$) to an unmarked half-space in S_2 to create a new unmarked half-space H' which contains S_1. Then $\mathrm{Homeo}(H', \partial H')$ contains T_γ.

The difficult case is when S_1 contains no maximal ends but is marked. Using Corollary 3.4 repeatedly, we can find three disjoint half-spaces H_1, H_2, H_3 contained in S_2. Since half-spaces have connected boundary, the complement of $H_1 \cup H_2 \cup H_3 \cup S_1$ is connected, and we may choose disjoint paths α_1, α_2 in this complement connecting $\gamma = \partial S_1$ to $\partial H_1, \partial H_2$ respectively. Let L be a regular neighborhood of $\gamma \cup \partial H_1 \cup \partial H_2 \cup \alpha_1 \cup \alpha_2$ in this complement. Then, L is a sphere with 4 boundary components, i.e. a lantern, where three boundary curves are $\gamma, \partial H_1$, and ∂H_2 and the fourth is some simple closed curve β bounding a half-space H_4 containing H_3.

We seek to use the lantern relation to show that f is a product of homeomorphisms supported on an unmarked half-space. The lantern relation implies that T_γ is equal to a word in the Dehn twists about $\partial H_1, \partial H_2, \beta$ and three other simple closed curves $\delta_1, \delta_2, \delta_3$ each of which separates L into two three-holed spheres. Thus for all $i = 1, 2, 3$, each side of δ_i must contain at least one of H_1, H_2, H_3, i.e. each δ_i separates Σ into one marked and one unmarked half-space, and thus δ_i lies in an unmarked half-space. Consequently, the twists about $\partial H_1, \partial H_2, \beta$, and the δ_i are all supported on an unmarked half-space. The lemma follows. □

In the unmarked case, we replace Lemma 3.9 with the following.

Lemma 4.3. *If H_1, H_2, and H_3 are disjoint unmarked half-spaces, then there is a homeomorphism $\varphi \in \mathrm{Homeo}(\Sigma)$ supported on some unmarked half-space H_4 such that $\varphi(H_1) = H_2$.*

Proof. By Lemma 3.10, H_3 contains two unmarked disjoint half-spaces H_3', H_3''. Since half-spaces have single boundary components, the complement of $H_1 \cup H_2 \cup H_3' \cup H_3''$ is connected, and we can attach H_1 to H_2 and H_3' by two strips disjoint from H_3'' and the marked point to create a subsurface H_4 with a single boundary circle that contains H_1, H_2, and H_3'. Since both $E(H_4) \supset E(H_1)$ and $E(H_4^c) \supset E(H_3'')$ contain a maximal end, H_4 is a half-space. We can now apply Lemma 4.3 to H_1, H_2, H_4, and H_3'. □

We can now prove the analogous theorem that half-space supported homeomorphisms generate $\mathrm{MCG}(\Sigma, *)$.

Theorem 4.4. *Let Σ be a uniformly self-similar surface with a marked point $* \in \Sigma$, and let $H \subseteq \Sigma$ be an unmarked half-space. Then,* $\mathrm{MCG}(\Sigma, *)$ *is generated by the normal closure of* $\mathrm{MCG}(H, \partial H)$.

Proof. Let $f \in \mathrm{MCG}(\Sigma, *)$. All unmarked half-spaces are the same up to $\mathrm{Homeo}(\Sigma, *)$ by an argument nearly identical to that in the proof of Lemma 3.10. Therefore, it suffices to show f is a product of mapping classes supported on unmarked half-spaces.

Let H_1 be an unmarked half-space, and let C be a simple closed curve such that C and ∂H_1 bound an annulus containing $*$ (in the interior). Let $H_2 = f(H_1)$. Then, by Lemma 3.8, either $H_1 \cap H_2^c$ or $H_1^c \cap H_2^c$ contains a half-space, which we can choose to be unmarked.

We first show that there is some mapping class g in the normal closure of $\mathrm{MCG}(H, \partial H)$ such that $g_1(f(H_1)) = H_1$ and $g_1 \circ f|_{H_1} = \mathrm{id}|_{H_1}$. Let's first consider the case where $H_1^c \cap H_2^c$ contains an unmarked half-space. Then, by Lemma 3.10, $H_1^c \cap H_2^c$ contains two disjoint unmarked half-spaces H_3, H_4. By Lemma 4.3, there are two mapping classes supported on some unmarked half-spaces, one which maps H_2 to H_3 and another which maps H_3 to H_1. By composing these maps with some appropriate third mapping class supported on H_1, we obtain the desired g_1. If instead $H_1 \cap H_2^c$ contains an unmarked half-space, then $H_1 \cap H_2^c$ contains two disjoint unmarked half-spaces H_3, H_4. By Lemma 4.3, there is some mapping class h supported on an unmarked half-space such that $h(f(H_1)) = h(H_2) = H_3 \subseteq H_1$. Thus $H_1^c \cap H_3^c = H_1^c$ contains an unmarked half-space, and we are reduced to the first case.

Let $C' = g_1(f(C))$. Then C' and ∂H_1 bound an annulus containing $*$. (Note that C' need not be C up to ambient isotopy fixing $*$.) Let $S \subseteq \Sigma$ be a compact subsurface with the following properties.

- S contains the annulus bounded by ∂H_1 and C and the annulus bounded by ∂H_1 and C'.
- S does not intersect the interior of H_1
- no boundary component bounds a disc in Σ.

Within S, each of C, C' is a separating curve which bounds an annulus with $\partial H_1 \subset \partial S$ containing a marked point. Consequently, the genus of the separating curves C, C' must be identical and they partition the boundary of S identically. Thus, there is some mapping class in $\mathrm{MCG}(S, \partial S \cup *)$ mapping C' to C. Since $\mathrm{MCG}(S, \partial S \cup *)$ is generated by Dehn twists, by Lemma 4.1, there is some g_2 in the normal closure of $\mathrm{MCG}(H, \partial H)$ such that $g_2(g_1(f(C)) = C$ and $g_2 \circ g_1 \circ f|_{H_1} = \mathrm{id}|_{H_1}$.

Let H_0 be the unmarked half-space bounded by C. Clearly, $g_2(g_1(f(H_0)) = H_0$. Since the mapping class group of the annulus with a marked point between H_0 and H_1 is generated by Dehn twists, by Lemma 4.1, we can compose by some third element g_3 in the normal closure of $\mathrm{MCG}(H, \partial H)$ such that $g_3 \circ g_2 \circ g_1 \circ f = \mathrm{id}$. □

We can now easily prove the analogous theorem for the mapping class group of a marked uniformly self-similar surface. Note that we have no statements about uniform perfection, or about a bound on the word length of an element as a product of involutions, or about the homeomorphism group. The first two are impossible by a result of J. Bavard [3] in the case of S^2 minus a Cantor set. The proof fails to show every element is a word of uniformly bounded length in involutions and half-space supported homeomorphisms only because of the step where we map C' to C. The theorem is only proven for the mapping class group and not the homeomorphism group because we use the lantern relation in the proof of Lemma 4.1.

Theorem 4.5. *Let Σ be a uniformly self-similar surface with a marked point $* \in \Sigma$. Then,* $\mathrm{MCG}(\Sigma, *)$ *is generated by involutions. Moreover,* $\mathrm{MCG}(\Sigma, *)$ *is normally generated by a single involution and is a perfect group.*

Proof. Let τ and φ be as in Lemma 3.5. Lemma 3.7 applies equally to $\mathrm{MCG}(\Sigma, *)$ (with an identical proof), and so $\langle\langle\tau\rangle\rangle$ contains $\mathrm{MCG}(H, \partial H)$ for some unmarked half-space and all elements of $\mathrm{MCG}(H, \partial H)$ are a single commutator in $\mathrm{MCG}(\Sigma)$. Theorem 4.4 finishes the proof. □

5 Self-similar but not uniformly

It is natural to wonder whether surfaces with a self-similar ends space and with genus 0 or ∞ are generated by involutions, are perfect, etc. It is already known that the mapping class group of the one-ended, infinite genus surface has abelianization containing an uncountable direct sum of $\mathbb{Q}$'s [7]. This surface fits into this category, but perhaps is not a particularly compelling example since the results of [7] are for pure mapping class groups of an infinite surfaces, and for the one-ended infinite genus surface, the mapping class group happens to coincide with the pure mapping class group. However, using a covering trick and some of the results of [7], we can prove that the abelianization of $\mathrm{MCG}(\mathbb{R}^2 \setminus \mathbb{N})$ is similarly large.

Proposition 5.1. $\mathrm{MCG}(\mathbb{R}^2 \setminus \mathbb{N})$ *surjects onto* $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$.

For the proof of the proposition, we need the following fact about abelian groups, which follows from [9, Theorem 21.3 and 23.1].

Lemma 5.2. *Let A be an abelian group. Suppose A contains $\bigoplus_I \mathbb{Q}$ for some non-empty set I. Then A surjects onto $\bigoplus_I \mathbb{Q}$.*

Proof of Proposition 5.1. Let Σ_L be the infinite genus surface with one end. This admits a 2-fold branched cover of $\mathbb{R}^2$ where $\mathbb{R}^2 = \Sigma_L / D$ and $D = \mathbb{Z}/2\mathbb{Z}$ acts by an involution. See Figure 2. More formally, one can construct this from gluing infinitely many copies of a 2-fold branch cover of an annulus by a 2-holed torus and one copy of a 2-fold branched cover of a disc by a 1-holed torus.

Let Σ_F be $\mathbb{R}^2$ with the branch points removed, i.e. $\Sigma_F \cong \mathbb{R}^2 \setminus \mathbb{N}$, and let Σ_{PL} be Σ_L with the branch points removed. Then, $\Sigma_{PL} \to \Sigma_F$ is a regular degree 2 cover with deck group D.

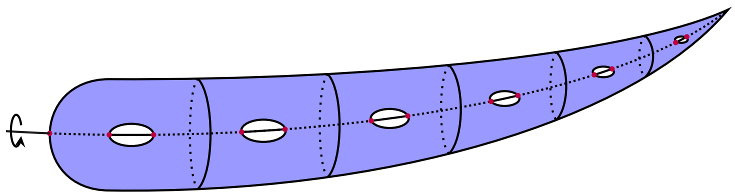

Figure 2: The surface Σ_L admits an involution which gives a degree 2 branched cover of $\mathbb{R}^2$ branched at the red points.

Choose marked points $\tilde{*} \in \Sigma_{PL}$ and $* \in \Sigma_F$. We first show that there is a lifting homomorphism $\mathrm{MCG}(\Sigma_F, *) \to \mathrm{MCG}(\Sigma_{PL}, \tilde{*})$, defined by lifting representative homeomorphisms. Since these are mapping class groups fixing marked points, it is a straightforward consequence of covering space theory that such a homomorphism exists and is unique provided the action of $\mathrm{MCG}(\Sigma_F, *)$ preserves the subgroup $K = \ker(\pi_1(\Sigma_F, *) \to D)$. Since the punctures of Σ_F came from branched points of degree 2, any simple loop in $\pi_1(\Sigma_F, *)$ which encloses a single puncture does not lift to a closed curve in Σ_{PL} and so must map to the nontrivial element of D. We can choose a basis $\{\beta_n\}_{n\in\mathbb{N}}$ of the free group $\pi_1(\Sigma_F, *)$ consisting entirely of simple loops each enclosing a single puncture. Consequently, K consists precisely of those even length words in this generating set. For any mapping class $f \in \mathrm{MCG}(\Sigma_F, *)$, the set $\{f(\beta_n)\}_{n\in\mathbb{N}}$ is also another generating set of simple loops enclosing single punctures, and for the same reasons K consists of words of even length in these generators. It is clear then that $f(K) = K$.

We now have a lifting homomorphism $\mathrm{MCG}(\Sigma_F, *) \to \mathrm{MCG}(\Sigma_{PL}, \tilde{*})$. Since the points deleted from Σ_L are isolated, $\mathrm{MCG}(\Sigma_{PL}, \tilde{*})$ preserves that set of ends, and we have a well-defined forgetful map $\mathrm{MCG}(\Sigma_{PL}, \tilde{*}) \to \mathrm{MCG}(\Sigma_L, \tilde{*})$. In [7], explicit mapping classes are constructed which project to nontrivial elements in the abelianization of $\mathrm{MCG}(\Sigma_L, \tilde{*})$. (See [7, Theorem 6.1].) Specifically, if $\{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of distinct, pairwise disjoint, separating, simple closed curves where each γ_n separates the marked point from the single end of Σ_L, then the subgroup topologically generated by the twists $\{T_{\gamma_n}\}_{n\in\mathbb{N}}$ projects to a group containing a $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$. One can easily find such γ_n which double cover simple closed curves α_n in Σ_F, and so T_{γ_n} is the lift of $T^2_{\alpha_n}$. (E.g. one can choose α_1 to be a simple closed curve bounding a disc with the marked point and three punctures and then choose α_i, α_{i+1} to always bound an annulus with two punctures. Then γ_i are the preimages of the α_i under the covering map.) Thus, $\mathrm{MCG}(\Sigma_F, \tilde{*})$ maps onto the same abelian group (generated by the T_{γ_n}). I.e. $\mathrm{H}_1(\mathrm{MCG}(\Sigma_F, \tilde{*}); \mathbb{Z})$ has a quotient A containing $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$. By Lemma 5.2, A maps onto $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$, so we also get a surjection $\varphi : \mathrm{H}_1(\mathrm{MCG}(\Sigma_F, \tilde{*}); \mathbb{Z}) \to \bigoplus_{2^{\aleph_0}} \mathbb{Q}$.

To pass to $\mathrm{MCG}(\Sigma_F)$, we borrow a technique from [7]. Consider the Birman short exact sequence (see [7])

$$1 \longrightarrow \pi_1(\Sigma_F, *) \longrightarrow \mathrm{MCG}(\Sigma_F, \tilde{*}) \longrightarrow \mathrm{MCG}(\Sigma_F) \longrightarrow 1.$$

Abelianization is right exact, so we get the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\aleph_0} \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}_1(\mathrm{MCG}(\Sigma_F, \tilde{*}); \mathbb{Z}) \arrow[r] \arrow[d, "\varphi"] & \mathrm{H}_1(\mathrm{MCG}(\Sigma_F); \mathbb{Z}) \arrow[r] \arrow[d, "\bar{\varphi}"] & 0 \\
\bigoplus_{\aleph_0} \mathbb{Z} \arrow[r] & \bigoplus_{2^{\aleph_0}} \mathbb{Q} \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

The image of $\bigoplus_{\aleph_0} \mathbb{Z}$ in $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$ still misses a copy of $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$, so the quotient P still contains a copy of $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$. The map $\bar{\varphi}$ is surjective, so we can conclude $\mathrm{H}_1(\mathrm{MCG}(\Sigma_F); \mathbb{Z})$ surjects onto $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$, again by Lemma 5.2. □

We now produce many more classes of examples by building surfaces that naturally map onto the one-ended infinite genus surface Σ_L or $\Sigma_F = \mathbb{R}^2 \setminus \mathbb{N}$.

Theorem 5.3. *Suppose Σ is a surface of one of the following two types.*

(1) $E(\Sigma)$ has exactly one end accumulated by genus.

(2) Σ has genus 0 and $E(\Sigma)$ has one maximal end y, such that in the partial order on $[E]$, the class of y has an immediate predecessor $E(x)$ with countably infinite cardinality.

Then $\mathrm{MCG}(\Sigma)$ *maps onto* $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$.

Proof. Choose a marked point $*$ on Σ. Note that by the same trick of using the Birman short exact sequence

$$1 \longrightarrow \pi_1(\Sigma, *) \longrightarrow \mathrm{MCG}(\Sigma, *) \longrightarrow \mathrm{MCG}(\Sigma) \longrightarrow 1,$$

it is enough to show the abelianization of $\mathrm{MCG}(\Sigma, *)$ maps onto $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$.

The statement for the one-ended infinite genus surface Σ_L is by [7]. The statement for $\Sigma_F = \mathbb{R}^2 \setminus \mathbb{N}$ is Proposition 5.1. For all other case, we will consider an appropriate map to one of these two surfaces.

On Σ_L, we will say a sequence of simple closed curves $\{\gamma_n\}_{n\in\mathbb{N}}$ is *good* if the curves are distinct, pairwise disjoint, separating, and each curve separates the maximal end of Σ_L from the marked point. On Σ_F, a sequence of curves $\{\alpha_n\}_{n\in\mathbb{N}}$ is *good* if under the covering map $(\Sigma_L, *) \to (\Sigma_F, *)$, each α_n is doubled covered by a curve γ_n and the sequence $\{\gamma_n\}$ is good. By [7] and the proof of Proposition 5.1, the subgroup topologically generated by Dehn twists about a good sequence of curves maps onto $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$ under the map to the abelianization of the mapping class group.

First, assume Σ is of the first type. The proof in the other case will be similar. The assumption on Σ means we have a map $(\Sigma, *) \to (\Sigma_L, *)$ by forgetting all but the only end accumulated by genus. This induces a well-defined map $\mathrm{MCG}(\Sigma, *) \to \mathrm{MCG}(\Sigma_L, *)$, since this end is invariant under $\mathrm{MCG}(\Sigma, *)$. By the previous paragraph, it is enough to exhibit a sequence of pairwise disjoint curves $\{\alpha_n\}_{n\in\mathbb{N}}$ on Σ whose image under the forgetful map forms a good sequence on Σ_L. To do this we will represent Σ in an explicit way as described below.

Identify $S^2 = \mathbb{R}^2 \cup \{\infty\}$ with base point ∞. We will construct Σ from S^2 by removing points from $\mathbb{R}^2 \subset S^2$ and attaching handles appropriately. Let $K \subset [0,1]$ be the standard Cantor set. Recall $E(\Sigma)$ is homeomorphic to a closed subset of K. Since the homeomorphism group of K acts transitively,

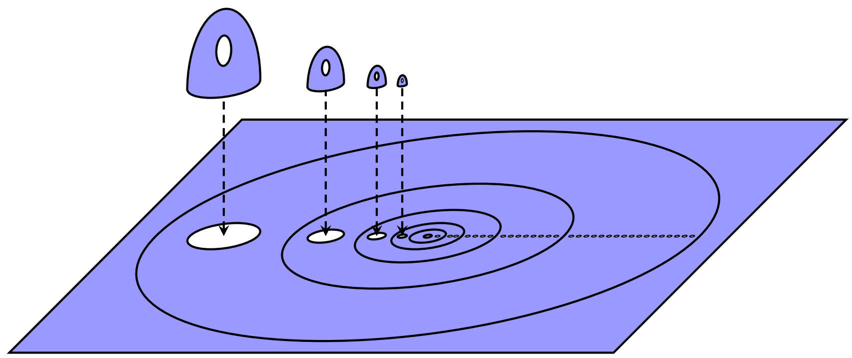

Figure 3: Building Σ of the first type.

we can realize $E(\Sigma)$ as a closed subset $E \subset K$ with the only end accumulated by genus at 0. By [17], the ends space of $S^2 - E$ is homeomorphic to E. It remains to attach handles to $\mathbb{R}^2 - E$ so that the handles will only accumulate onto the origin. To this end, choose a sequence $\{y_n\}_{n\in\mathbb{N}} \subset [0,1] - K$ such that $y_n \to 0$ monotonically. In particular, $\{y_n\} \cap E = \emptyset$. Let $d_n = y_n - y_{n+1}$. For each n, let T_n be a torus with one boundary component. Let z_n be the midpoint of $[y_{n+1}, y_n]$. In $\mathbb{R}^2$, let $p_n = (-z_n, 0)$, and B_n be the open ball of diameter $d_n/2$ centered at p_n. Now remove each B_n from $\mathbb{R}^2$ and attach T_n by gluing ∂T_n to ∂B_n. Let Σ' be the resulting surface with marked point ∞. By construction, the tori accumulate only onto the origin. It follows then by the classification of surfaces, Σ' is homeomorphic to Σ, and we can make this homeomorphism take ∞ to $*$. By filling in all of E except the origin, we get a marked surface (Σ'_L, ∞) homeomorphic to $(\Sigma_L, *)$, and a representation of the forgetful map $(\Sigma, *) \to (\Sigma_L, *)$. Using this picture, it is now easy to find the curves α_n which we take to be the circle of radius y_n centered at the origin. By construction, the circles $\{\alpha_n\}$ avoid E, are pairwise disjoint, and each separates the origin from ∞. Furthermore, since there is a handle between two consecutive circles α_n and α_{n+1}, namely T_{n+1}, these circles remain topologically distinct after filling in all of $E - \{0\}$. This finishes the proof in this case.

Now suppose Σ is of second type. We first claim that, by forgetting all but the maximal end of $E(\Sigma)$ and the class $E(x)$ of its immediate predecessor, we get a map $(\Sigma, *) \to (\Sigma_F, *)$. Taking the same approach as above, realize $E(\Sigma)$ as a closed subset $E \subset K$ with the maximal end at the origin. By Richards, the surface $(S^2 \setminus E, \infty)$ is homeomorphic to $(\Sigma, *)$. We claim the origin is the only accumulation point of $E(x)$. Since $E(x)$ has no successor except the origin, any other accumulation point of $E(x)$ must be equivalent to x. But then every point in $E(x)$ is an accumulation point of $E(x)$. This makes $E(x) \cup \{0\}$ a closed and perfect subset of K, so it is homeomorphic to K, contradicting our assumption that $E(x)$ has countable cardinality. Thus, for any compact interval $I \subset (0,1]$, $I \cap E(x)$ has finite cardinality, so we can enumerate $E(x)$ as a decreasing sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ converging to 0. This shows $\left(S^2 \setminus (E(x) \cup \{0\}), \infty\right) \cong (\Sigma_F, *)$. Since equivalence classes of ends are preserved by the mapping class group, we obtain a forgetful map $\mathrm{MCG}(\Sigma, *) \to \mathrm{MCG}(\Sigma_F, *)$. To finish, take any point $y_n \in [x_{n+1}, x_n]$ such that $\{y_n\} \cap E = \emptyset$. Then the circles $\{\alpha_n\}_{n\in\mathbb{N}}$ of radius y_n centered at the origin are pairwise disjoint and we can extract from them a subsequence that project to a good sequence of curves on Σ_F. This finishes the proof. $\square$

Remark 5.4. In our setting above, it seems plausible that the forgetful map from $\mathrm{MCG}(\Sigma)$ to either

$\mathrm{MCG}(\Sigma_L)$ or $\mathrm{MCG}(\Sigma_F)$ is surjective, but we will not pursue that statement here.

We record some consequences of Theorem 5.3.

Corollary 5.5. *Suppose Σ is a surface that satisfies one of the descriptions in Theorem 5.3. Let $* \in \Sigma$ be a marked point. Let G be either $\mathrm{Homeo}^+(\Sigma)$, $\mathrm{MCG}(\Sigma)$, $\mathrm{Homeo}^+(\Sigma, *)$, or $\mathrm{MCG}(\Sigma, *)$. Then G is not perfect, is not generated by torsion elements, and does not have the automatic continuity property.*

Proof. Since a Polish group is separable, it can have at most $\mathfrak{c} = 2^{\aleph_0}$ continuous epimorphisms to $\mathbb{Q}$. But $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$ has $2^{\mathfrak{c}}$ epimorphisms to $\mathbb{Q}$. So, by Theorem 5.3, $\mathrm{MCG}(\Sigma)$ is not perfect, is not generated by torsion elements, and does not have the automatic continuity property. These three properties are inherited by quotients, so $\mathrm{Homeo}^+(\Sigma)$ also cannot have any of these properties. The same argument applies to a marked Σ. □

Remark 5.6. If E is a countable ends space homeomorphic to $\omega^\alpha + 1$, for some countable successor ordinal α, then $\Sigma = S^2 \setminus E$ is a surface of type 2 of Theorem 5.3, by [15, Proposition 4.3]. This gives Theorem D of the introduction.

5.1 Topological generation by involutions

Theorem 5.7. *Let Σ be either $\mathbb{R}^2 \setminus \mathbb{N}$ or the infinite genus surface with one end. Then $\mathrm{MCG}(\Sigma)$ is topologically generated by involutions and is the topological closure of the normal closure of a single involution. Consequentially, $\mathrm{H}^1(\mathrm{MCG}(\Sigma), \mathbb{Z}) = 0$.*

Proof. We first focus on $\Sigma = \mathbb{R}^2 \setminus \mathbb{N}$. The beginning of the proof is very similar to that of Theorem 3.12. Note that $\mathbb{R}^2 \setminus \mathbb{N}$ is homeomorphic to $\mathbb{R}^2 \setminus \mathbb{Z}^2$. This is because both surfaces have genus 0, and their ends spaces are homeomorphic. The advantage of viewing the surface as $\mathbb{R}^2 \setminus \mathbb{Z}^2$ is as follows.

Let τ be the rotation in the plane by angle π centered at the origin, i.e. $\tau(x + iy) = e^{i\pi}(x + iy)$. We also have the translation $\phi(x + iy) = (x+1) + iy$. Both maps preserve $\mathbb{Z}^2$, so they induce homeomorphisms of Σ, where τ has order 2. One checks that $[\phi, \tau] = \phi\tau\phi^{-1}\tau = \phi^2$.

We define a *half-space* of Σ to be a closed subset $H \subset \Sigma$, such that ∂H is a properly embedded simple arc joining infinity to itself, and both H and H^c contain infinitely many punctures (isolated ends) of Σ. We will consider an explicit half-space in Σ. Let $h(x) = \sec(\pi x) - .5$ with domain $(-.5, .5)$. The graph of $h(x)$ is a convex curve that misses all of $\mathbb{Z}^2$ and is contained in the vertical strip $\left\{(x, y) : -.5 \le x \le .5\right\}$. See figure 4. The set $H = \{(x, y) \in \Sigma : y \ge h(x)\}$ is a half-space, and ϕ^2 is an H–free translation, in the sense that $\{\phi^{2n} H\}_{n \in \mathbb{Z}}$ are pairwise disjoint. Therefore, with the same swindle as before, we obtain

$$\mathrm{Homeo}(H, \partial H) \le \langle\langle \phi^2 \rangle\rangle \le \langle\langle \tau \rangle\rangle.$$

While the swindle still works, the rest of the proof for Theorem 3.12 does (and should) not work. The only statement that seems to fail is Lemma 3.8, (and so Theorem 3.11 also fails in this case).

We now move to the mapping class group. As before, to simplify the discussion, we will conflate half-spaces and simple closed curves with their ambient isotopy classes. We will keep on denoting ϕ and τ for their mapping classes.

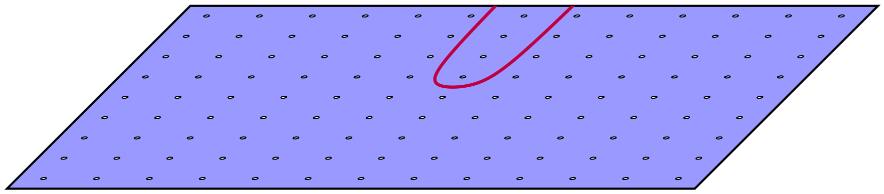

Figure 4: The half-space H in $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

Consider the short exact sequence

$$1 \to \mathrm{PMCG}(\Sigma) \to \mathrm{MCG}(\Sigma) \to \mathrm{Homeo}(E(\Sigma)) \to 1,$$

where $\mathrm{PMCG}(\Sigma)$ is called the *pure mapping class group*, i.e. the subgroup fixing each end of Σ. Since Σ has no genus, by [2], $\mathrm{PMCG}(\Sigma) = \overline{\mathrm{PMCG}_c(\Sigma)}$, where $\mathrm{PMCG}_c(\Sigma)$ is the subgroup of compactly supported mapping classes. Since Dehn twists generate the pure mapping class group of any compact surface, $\mathrm{PMCG}(\Sigma)$ is topologically generated by Dehn twists. The goal now is to show every Dehn twist in $\mathrm{MCG}(\Sigma)$ is contained in the normal closure of $\mathrm{MCG}(H, \partial H)$, and that the normal closure of $\mathrm{MCG}(H, \partial H)$ surjects onto $\mathrm{Homeo}(E(\Sigma))$

We first deal with the Dehn twists. Let $\alpha \subset \Sigma$ be any simple closed curve. Then α bounds a topological disk containing finitely many points of $\mathbb{Z}^2$. Choose a simple closed curve $\beta \subset H$ that bounds an equal number of points of $\mathbb{Z}^2$. We can find a homeomorphism $f \in \mathrm{Homeo}^+(\Sigma)$, such that $f(\alpha) = \beta$. This is simply the change-of-coordinate principle made possible by the classification of surfaces. We now have

$$T_\alpha = T_{f^{-1}(\beta)} = f^{-1} T_\beta f \in \langle\langle \mathrm{MCG}(H, \partial H) \rangle\rangle.$$

To show $\langle\langle \mathrm{MCG}(H, \partial H) \rangle\rangle$ surjects onto $\mathrm{Homeo}(E(\Sigma))$, we produce sufficiently many permutations of non-maximal ends. First note that $E(\Sigma)$ has exactly one maximal end, represented by ∞, which must be invariant under any homeomorphism. Every other end is isolated, so $\mathrm{Homeo}(E(\Sigma))$ is nothing other than the permutation group $\mathrm{Sym}(\mathbb{Z}^2)$ on $\mathbb{Z}^2$. Within H, pair off infinitely many punctures/ends $\{(x_{i,1}, x_{i,2})\}_{i \in I}\}$ such that $x_{i,2}$ is directly above $x_{i,1}$ and all the pairs are pairwise disjoint. It is clear that $\mathrm{MCG}(H, \partial H)$ contains a mapping class f which transposes all pairs simultaneously. Note that I is both infinite and co-infinite in $E(\Sigma)$. Since by [17], $\mathrm{MCG}(\Sigma)$ surjects onto $\mathrm{Homeo}(E(\Sigma)) = \mathrm{Sym}(\mathbb{Z}^2)$, the image of $\langle\langle \mathrm{MCG}(H, \partial H) \rangle\rangle$ in $\mathrm{Sym}(\mathbb{Z}^2)$ contains all order 2 permutations supported on infinite, co-infinite subsets. It is straightforward to show this set generates $\mathrm{Sym}(\mathbb{Z}^2)$. In summary, we have shown $\langle\langle \mathrm{MCG}(H, \partial H) \rangle\rangle$ topologically generates $\mathrm{PMCG}(\Sigma)$ and surjects onto $\mathrm{Homeo}(E(\Sigma))$. This yields

$$\mathrm{MCG}(\Sigma) = \overline{\langle\langle \mathrm{MCG}(H, \partial H) \rangle\rangle} = \overline{\langle\langle \phi^2 \rangle\rangle} = \overline{\langle\langle \tau \rangle\rangle}.$$

To go from $\mathbb{R}^2 \setminus \mathbb{Z}^2$ to the one-ended infinite genus surface Σ_L we observe that instead of removing the integer lattice points from $\mathbb{R}^2$, we can remove a small disk from each latter point and glue on a handle to get a surface Σ' homeomorphic to Σ_L. Furthermore, we can make sure τ and ϕ preserve Σ'. A half-space in Σ' is simply a closed component of a dividing arc that cuts off two component of infinite genus. The explicit half-space H we defined for $\mathbb{R}^2 \setminus \mathbb{Z}^2$ can also be made into a half-space here. Then running the same argument as above and observing that $\mathrm{PMCG}(\Sigma') = \mathrm{MCG}(\Sigma')$ completes the proof.

The last statement about the cohomology of these groups follows from the fact that any homomorphism from a Polish group to $\mathbb{Z}$ is automatically continuous [8]. □

References

[1] T. Aougab, P. Patel, and N. G. Vlamis. Isometry groups of infinite-genus hyperbolic surfaces. *Math. Ann.*, 2021.

[2] J. Aramayona, P. Patel, and N. G. Vlamis. The first integral cohomology of pure mapping class groups. *Int. Math. Res. Not. IMRN*, (22):8973–8996, 2020.

[3] J. Bavard. Hyperbolicité du graphe des rayons et quasi-morphismes sur un gros groupe modulaire. *Geom. Topol.*, 20(1):491–535, 2016.

[4] T. E. Brendle and B. Farb. Every mapping class group is generated by 6 involutions. *J. Algebra*, 278(1):187–198, 2004.

[5] D. Calegari. Big mapping class groups and dynamics. 2009. `http://lamington.wordpress.com/2009/06/22/big-mapping-class-groups-and-dynamics`.

[6] D. Calegari and L. Chen. Normal subgroups of big mapping class groups. 2021.

[7] G. Domat and R. Dickmann. Big pure mapping class groups are never perfect. 2020. Available at `arXiv:2007.14929 [math.GR]`.

[8] R. M. Dudley. Continuity of homomorphisms. *Duke Math. J.*, 28:587–594, 1961.

[9] L. Fuchs. *Infinite abelian groups. Vol. I.* Pure and Applied Mathematics, Vol. 36. Academic Press, New York-London, 1970.

[10] M. Kassabov. Generating mapping class groups by involutions. 2003. Available at `arXiv:0311455 [math.GR]`.

[11] M. Korkmaz. Mapping class group is generated by three involutions. *Math. Res. Lett.*, 27(4):1095–1108, 2020.

[12] J. Lanier and D. Margalit. Normal generators for mapping class groups are abundant. 2018. Available at `arXiv:1805.03666 [math.GR]`.

[13] F. Luo. Torsion elements in the mapping class group of a surface. 2000. Available at `arXiv:000.4048 [math.GR]`.

[14] K. Mann. Automatic continuity for homeomorphism groups of noncompact manifolds. 2020. Available at `arXiv:2003.01173 [math.GR]`.

[15] K. Mann and K. Rafi. Large scale geometry of big mapping class groups. 2019. Available at `arXiv:1912.10914 [math.GR]`.

[16] J. McCarthy and A. Papadopoulos. Involutions in surface mapping class groups. *Enseign. Math. (2)*, 33(3-4):275–290, 1987.

[17] I. Richards. On the classification of noncompact surfaces. *Trans. Amer. Math. Soc.*, 106:259–269, 1963.

[18] N. G. Vlamis. Three perfect mapping class groups. *New York J. Math.*, 27:468–474, 2021.

[19] O. Yildiz. Generating the mapping class group by three involutions. 2020. Available at `arXiv:2002.09151 [math.GR]`.

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