Summaries of Papers and Preprints

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1 Preprints

On stable commutator length of non-filling curves in surfaces, Max Forester and Justin Malestein

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We give a new proof of rationality of stable commutator length of certain elements in surface groups: those represented by curves that do not fill the surface. Such elements always admit extremal surfaces for scl. These results also hold more generally for non-filling 1–chains.

Arithmetic Quotients of the Automorphism Group of a Right-Angled Artin Group

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It was previously shown by Grunewald and Lubotzky that the automorphism group of a free group, $\mathrm{Aut}(F_n)$, has a large collection of virtual arithmetic quotients. Analogous results were proved for the mapping class group by Looijenga and by Grunewald, Larsen, Lubotzky, and Malestein. In this paper, we prove analogous results for the automorphism group of a right-angled Artin group A_Γ for a large collection of defining graphs Γ. The graphs are required to satisfy certain constraints that ensure the automorphism group has a significant subgroup of transvections supported on an independent set in $V(\Gamma)$. As a consequence of the method of proof, we also find arithmetic quotients of many relative outer automorphism groups of free groups. These are groups which preserve a subset of free factors up to conjugation. The corresponding virtual arithmetic quotients (viewed as groups acting on a vector space) then have invariant subspaces in direct correspondence to the invariant free factors. Like the other papers mentioned above, all virtual arithmetic quotients arise from actions on $\mathrm{H}_1(R)$ where R is a finite index subgroup of F_n or A_Γ.

As a corollary of our methods, we also produce new virtual arithmetic quotients of $\mathrm{Aut}(F_n)$ for $n \geq 4$. Grunewald and Lubtotzky studied the action of $\mathrm{Aut}(F_n)$ specifically on $\mathrm{H}_1(R)$ where R is *redundant*, i.e. R contains a primitive element of F_n. In this paper, we determine (up to finite index) the image of the virtual representation of $\mathrm{Aut}(F_n)$ obtained from the action on some isotypic components of $\mathrm{H}_1(R)$ for some nonredundant R. As above, the images are arithmetic groups. Using previous results by Malestein and Putman, some of these representations are known to contain, for some fixed k depending on the representation, kth powers of all transvections relative to any basis of F_n. Then, for some values of k, we deduce that the quotient of $\mathrm{Out}(F_n)$ by the subgroup generated by kth powers of transvections contains nonabelian free groups. This expands on results of Malestein and Putman which showed these quotients often have infinite order elements and of Bridson and Vogtmann which showed these quotients are often infinite.

Ultrarigid Periodic Frameworks, Justin Malestein and Louis Theran

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A number of recent papers have studied the question of generic rigidity of a symmetric bar-joint framework. These are bar-joint frameworks in $\mathbb{R}^d$ that are invariant under the action of some subgroup of the isometry group of $\mathbb{R}^d$. Symmetric frameworks have been studied from the perspective of both "forced symmetry" and "incidental symmetry". In the former case, the question is whether the framework has nontrivial motions while retaining the symmetry and in the latter case whether the framework has nontrivial motions while breaking symmetry. If one wants to model a physical system, then the incidental symmetry model is probably more realistic, but it is also much more challenging to understand. In our previous paper "Generic combinatorial rigidity of periodic frameworks", we deduced a combinatorial criterion for a generic periodic bar-joint framework in $\mathbb{R}^2$ to be rigid in the forced symmetry model. A periodic bar-joint framework is a bar-joint framework in $\mathbb{R}^2$ that is invariant under the action of some lattice Λ of translations. In this paper, we study a kind of middle ground between the forced symmetry and incidental symmetry models. A rigid motion is a motion which preserves the lengths of bars and is periodic with respect to some sublattice of Λ. A framework is then ultrarigid if it has no nontrivial such motion for any sublattice of Λ.

We prove a number of results, some of them foundational. A number of fundamental facts which hold for rigidity in other situations either don't hold here or are not straighforward to show. E.g. it is not clear if ultrarigidity and infinitesimal ultrarigidity coincide generically. In fact, we give an example in the case where we require the area of the lattice to be constant where this fails. Since the set of graphs which are generically ultrarigid is an intersection of infinitely many matroids, it is unclear and probably false that the set is a matroid. Moreover, it is not obvious if there is an algorithm to check the infinitesimal ultrarigidity of a framework. A priori, this must be checked for each of infinitely many sublattices. Nevertheless, we are able to establish results along these lines. We show that infinitesimal ultrarigidity of a framework can be checked in finite time. Moreover, we show that, given any framework, there is some sufficiently large integer N, depending on the framework, such that it is ultrarigid if it is rigid at the level of the sublattice $N\Lambda$. For frameworks with the minimum possible number of edges for rigidity, we characterize combinatorially which ones generically infinitesimal ultrarigid (in the case of a flexible lattice, fixed lattice, or fixed area lattice).

2 Papers

Simple closed curves, finite covers of surfaces, and power subgroups of $\mathrm{Out}(F_n)$, with Andrew Putman, published in Duke Math Journal, 2019

We construct examples of finite covers of punctured surfaces where the first rational homology is not spanned by lifts of simple closed curves. More generally, for any set $\mathcal{O} \subset F_n$ which is contained in the union of finitely many $\mathrm{Aut}(F_n)$-orbits, we construct finite-index normal subgroups of F_n whose first rational homology is not spanned by powers of elements of $\mathcal{O}$. These examples answer questions of Farb–Hensel, Kent, Looijenga, and Marché. We also show that the quotient of $\mathrm{Out}(F_n)$ by the subgroup generated by k^{th} powers of transvections often contains infinite order elements, strengthening a result of Bridson–Vogtmann saying that it is often infinite. Finally, for any set $\mathcal{O} \subset F_n$ which is contained in the union of finitely many $\mathrm{Aut}(F_n)$-orbits, we construct integral linear representations of free groups that have infinite image and map all elements of $\mathcal{O}$ to torsion elements.

Pseudo-Anosov dilatations and the Johnson filtration, with Andrew Putman, published in Groups, Geometry, and Dynamics, 2016

Answering a question of Farb, Leininger, and Margalit, we give explicit lower bounds for the dilatations of pseudo-Anosov mapping classes lying in $\mathcal{N}_k(\Sigma)$, the k^{th} term of the Johnson filtration of the mapping class group of a closed surface Σ. Farb, Leininger, and Margalit showed that the smallest dilatation of a pseudo-Anosov mapping class in $\mathcal{N}_k(\Sigma)$ goes to infinity with k, and moreover there is a uniform lower bound for all genus. Our explicit lower bound also depends only on k and not the genus of the surface. Specifically, we show that log of the smallest dilatation of a pseudo-Anosov class in $\mathcal{N}_k(\Sigma)$ is at least roughly $\log(k)$. The heart of the argument is Theorem B which states that if two distinct (isotopy classes of) simple closed curves are conjugate modulo the kth term of the lower central series of $\pi_1(\Sigma)$, then their geometric intersection number is bounded below by $O(k^c)$ for some explicit constant c. The argument of this theorem proceeds by contradiction. If the geometric intersection is too small, then we show that there is a relatively small nilpotent cover where the curves can be distinguished homologically. This cover is constructed by taking a tower of covers where each time a definite fraction of the intersections between the curves is resolved when lifting. Each cover in the tower is not particularly easy to construct explicitly, so we use a probabilistic argument to show the cover exists at each stage in the tower.

Frameworks with forced symmetry I: Reflections and rotations, with Louis Theran, published in Discrete & Computational Geometry, 2015

We give a combinatorial characterization of generic frameworks that are minimally rigid under the additional constraint of maintaining symmetry with respect to a finite order rotation or a reflection. To prove the theorem, we use direction networks, expanding upon our methodology in "Generic combinatorial rigidity of periodic frameworks". As in that paper, we define the characterization in terms of the quotient of the framework by the symmetry group with edges labelled by the symmetry group. In the case of a rotation, this required defining the right notion of rotationally symmetric direction network and a different proof that rotationally-symmetric direction networks have faithful realizations when the quotient graph is "cone-Laman". In the case of a reflection, the situation is much more delicate. Unlike the cases of periodic and rotational symmetry, it is not true that if a framework faithfully realizes a direction network, then its rigidity matrix has the same rank as the direction network. To overcome this difficulty, we have to carefully analyze the reflection-Laman and reflection-(2,2) sparsity matroids to go from the rank of the direction network to the rank of the rigidity matrix of its realization. Using completely different methods, specifically inductive constructions, Jordán, Kaszanitsky, and Tanigawa also prove similar results and additionally generalize the results to frameworks with odd dihedral group symmetry.

Arithmetic quotients of the mapping class group, with Fritz Grunewald, Michael Larsen, and Alexander Lubotzky, published in GAFA, 2015

In this paper, we show that $\mathrm{Mod}(\Sigma_g)$, the mapping class group of a closed orientable genus g surface for $g \geq 4$ has a rich infinite collection of virtual arithmetic quotients. A virtual arithmetic quotient is a surjective representation from a finite index subgroup to an arithmetic group. By inducing to the larger group, we can obtain a representation on the entire mapping class group. These virtual representations are defined in a natural way by taking the action on the first homology of some finite index regular covers of Σ_g. The virtual arithmetic quotients we obtain are the following.

To every $\mathbb{Q}$-irreducible representation r of a finite group H, there corresponds a simple factor A of $\mathbb{Q}[H]$ with an involution τ. To this pair (A, τ), we associate an arithmetic group Ω consisting of all $(2g-2) \times (2g-2)$ matrices over a natural order $\mathfrak{O}^{op} \subset A^{op}$ which preserve a natural skew-Hermitian sesquilinear form on A^{2g-2}. We show that if H is generated by less than g elements, then Ω is a virtual quotient of the mapping class group $\mathrm{Mod}(\Sigma_g)$. As one can see, then, the classical quotient $\mathrm{Sp}(2g, \mathbb{Z})$ is just a first case, in a long and complex list, corresponding to the trivial group H and the trivial representation. Other pairs of H and r give rise to many new arithmetic quotients of $\mathrm{Mod}(\Sigma_g)$ which are defined over various subfields of cyclotomic fields and are of type $\mathrm{Sp}(2m), \mathrm{SO}(2m, 2m)$, and $\mathrm{SU}(m, m)$ for arbitrarily large m. Previously, Looijenga proved an analogous theorem for finite abelian H and produced virtual surjective representations of $\mathrm{Mod}(\Sigma_g)$ to products of $\mathrm{SU}(g-1, g-1, \mathbb{Z}[\zeta])$ where ζ is a primitive root of unity.

Frameworks with forced symmetry II: Orientation-preserving crystallographic groups, with Louis Theran, published in Geom. Ded., 2014

We give a combinatorial characterization of minimally rigid planar frameworks with orientation-preserving crystallographic symmetry, under the constraint of forced symmetry. The main theorems are proved by extending the methods of the first paper in this sequence from groups generated by a single rotation to groups generated by translations and rotations. The proofs make use of a new family of matroids defined on crystallographic groups and associated submodular functions.

Generic rigidity with forced symmetry and sparse colored graphs, with Louis Theranc, chapter in Rigidity and Symmetry, Fields Institute Communications Volume 70, 2014

We review some recent results in the generic rigidity theory of planar frameworks with forced symmetry, giving a uniform treatment to the topic. We also give new combinatorial characterizations of minimally rigid periodic frameworks with fixed-area fundamental domain and fixed-angle fundamental domain.

Topological designs, with Igor Rivin and Louis Theran, published in Geom. Ded., 2014

We give an exponential upper and a quadratic lower bound on the number of pairwise non-isotopic simple closed curves that can be placed on a closed surface of genus g such that any two of the curves intersect at most once. Although the gap is large, both bounds were the best known at the time. Of particular interest is the quadratic lower bound in rather stark contrast to the well-known linear bound for disjoint simple closed curves. In genus one and two, we solve the problem exactly. Our methods generalize to variants in which the allowed number of pairwise intersections is odd, even, or bounded, and to surfaces with boundary components.
Various subsequent papers have sharpened some of these results considerably. Most notably, Przytycki showed that the maximum size of a collection of simple closed curves on a genus g surface intersecting at most once is at most $O(g^3)$. Moreover, in the case of arcs on a punctured surface intersecting at most once, he finds the exact number for a maximal set. For sets of curves or arcs allowing at most k intersections, he provides polynomial upper bounds.

Generic combinatorial rigidity of periodic frameworks, with Louis Theran, published in Advances in Math., 2013

We give a combinatorial characterization of generic minimal rigidity for planar periodic bar-joint frameworks. A periodic graph is a graph $\tilde{G}$ equipped with a free action by $\mathbb{Z}^2$. A planar periodic framework is a placement of the vertices or joints $\mathbf{p} : V(\tilde{G}) \to \mathbb{R}^2$ and a homomorphism $\mathbf{L} : \mathbb{Z}^2 \to \mathbb{R}^2$ such that $\mathbf{p}$ is equivariant with respect to the $\mathbb{Z}^2$-action. Roughly speaking, generic here means generic apart from the requirements of periodicity. In our model, a motion of the bar-joint framework is a continuous change of both $\mathbf{p}$ and $\mathbf{L}$ such that lengths of bars and periodicity are preserved throughout the motion. The characterization is in terms of the quotient graph $G = \tilde{G}/\mathbb{Z}^2$ with edges labelled by $\mathbb{Z}^2$. The necessary and sufficient condition on G, which we call "periodic-Laman", is a requirement that certain subgraphs be sufficiently sparse. One key feature of the argument is the discovery of sufficiently tight degree-of-freedom counts on subgraphs which are embodied in the sparsity condition. The characterization we prove is a true analogue of the Maxwell-Laman Theorem from rigidity theory; it is stated in terms of a finite combinatorial object and the conditions are checkable by polynomial time combinatorial algorithms. As a corollary, we also rederive a combinatorial characterization, previously proven by Ross, for rigidity of a generic periodic framework where the lattice $\mathbf{L}$ is required to remain fixed under motions.
To prove our rigidity theorem we introduce and develop periodic direction networks and $\mathbb{Z}^2$-graded-sparse colored graphs. A periodic graph is an infinite graph $\tilde{G}$ equipped with a free action by $\mathbb{Z}^2$ and a direction network on $\tilde{G}$ is a choice of direction for each edge such that all edges in a $\mathbb{Z}^2$-orbit have the same direction. A realization of the network is a map $\mathbf{p} : V(\tilde{G}) \to \mathbb{R}^2$ and homomorphism $\mathbf{L} : \mathbb{Z}^2 \to \mathbb{R}^2$ such that realized edges are in the chosen direction and $\mathbf{p}$ is $\mathbb{Z}^2$-equivariant. The main steps, similar to those in "Slider-pinning Rigidity: a Maxwell-Laman-type Theorem" by Streinu and Theran, are showing that generic periodic direction networks have faithful realizations $(\mathbf{p}, \mathbf{L})$ (no collapsed edges) if G is periodic-Laman and that the rigidity matrix of $(\mathbf{p}, \mathbf{L})$ has the same rank as the direction network. However, the details in proving the steps are quite different. E.g., at one point, we use Edmonds' theorem on matroid unions to decompose "periodic-(2,2)" graphs into an edge-disjoint union of two subgraphs each of which are a tree plus two edges with certain extra conditions on the labelling of their cycles.

On Genericity of Pseudo-Anosovs in the Torelli Group, with Juan Souto, published in IMRN, 2013

We show that, for any (symmetric) finite generating set of the Torelli group of a closed surface, the probability that a random word is not pseudo-Anosov decays exponentially in terms of the length of the word. We do this as follows. First, we consider the action of the Torelli group on the first homology of all index 2 covers of the surface. We prove a new Casson-like criterion for a mapping class to be pseudo-Anosov. A new criterion other than Casson's criterion is required since any mapping class acts reducibly on the homology of an index 2 cover and in fact the entire mapping class group preserves a splitting into a summand of two subspaces. Our criterion is based on the action on one of these summands. The fact that the Torelli group acts on this summand essentially as $\mathrm{Sp}(2g-2,\mathbb{Z})$ and results of Rivin then establishes the result.

On the self-intersections of curves deep in the lower central series of a surface group, with Andrew Putamn, published in Geom. Ded., 2010

We give various estimates of the minimal number of self-intersections of a nontrivial element of the k^{th} term of the lower central series and derived series of the fundamental group of a surface. Specifically, we show for compact surfaces with boundary that the number of self-intersections must grow linearly with k with coefficients depending on rank of the fundamental group. For all surfaces including closed surfaces, we obtain a logarithmic lower bound independent of the surface. For the derived series, we find an exponential lower bound which is essentially sharp. As an application, we obtain a new topological proof of the fact that free groups and fundamental groups of closed surfaces are residually nilpotent. Along the way, we prove that a nontrivial element of the k^{th} term of the lower central series of a free group must have word length at least k in a free generating set. Rather surprisingly, we also show that there are words in the kth term of the lower central series which only grow like k^4. Subsequent papers have improved on the upper bound, but the exact growth rate is still unknown.

Pseudo-Anosov homeomorphisms and the lower central series of a surface group, published in Algebraic and Geometric Topology, 2007

In this paper, I prove an algebraic criterion for a mapping class in the Torelli group of a surface S with one boundary component to be pseudo-Anosov. Given a mapping class f in the Torelli group, we use the interpretation of its image $\tau(f)$ under the Johnson homomorphism as a derivation on the Lie algebra associated to the lower central series of $\pi_1(S)$. Using this interpretation, we are able to construct an endomorphism $\Psi_k(f) \in \mathrm{End}(\mathrm{H}_1(S;\mathbb{Z}))$. We then show that if the characteristic polynomial of $\Psi_k(f)$ is irreducible, then f is pseudo-Anosov. A similar construction of an endomorphism $\Psi_k(f)$ is carried out for any f in some term of the Johnson filtration, and we prove similarly that f is pseudo-Anosov if $\Psi_k(f)$ is irreducible. Some explicit mapping classes are shown to be pseudo-Anosov.