Introduction to Modern Quantum Field Theory

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Recall Einstein's famous equation,

$$E^2 = (Mc^2)^2 + (c\vec{p})^2,$$

where c is the speed of light, M is the classical rest mass, and $\vec{p}$ is the classical three momentum. From now on, we take $c = 1$.

- One of the main contributions Einstein made to physics was placing time on equal footing with the three spatial dimensions. In fact, the quantity above is the Euclidean norm of the four-component momentum on space-time $p_\alpha = (M, \vec{p})^t$, where $\alpha = 0, 1, 2, 3$.
- Mathematically, Einstein embedded the classical three-dimensional physics inside of Minkowski space, $\mathbb{R}^{1,3}$, with the "Minkowski congugate" g such that $\mathrm{diag}(g) = (+1, -1, -1, -1)$.

$$E^2 = (M)^2 + \vec{p} \cdot \vec{p},$$

- If $E^2 > 0$, then we change the basis of our coordinate system until the particle appears to be at rest, so $E^2 = M^2$ and $\vec{p} = 0$, which fixes a cannonical choice of presentation for $p_\alpha = (M, 0, 0, 0)^t$ and motivates the terminalogy *a massive particle at rest*.
- If $E^2 = 0$, then we have one of two cases: the zero momentum case and the massless (light-like) case.

1. In the zero momentum case, $p_\alpha = (0, 0, 0, 0)^t$.
2. In the massless case, the square of the particle's energy equals the square of the particle's momentum: $E^2 - \vec{p} \cdot \vec{p} = M^2 = 0$. In other words, all of the particle's energy is in its (3-dimensional) momentum. Hence, the cannonical choice of presentation for $p_\alpha = (\omega, \omega, 0, 0)^t$.

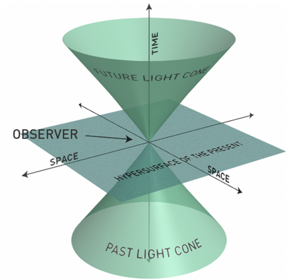
Remark

To recapitulate the above, the four-momentum p_α of a particle is of one of the standard forms:

i $p_\alpha = (0,0,0,0)^t$ (zero momentum),

ii $p_\alpha = (\omega,\omega,0,0)^t$ (massless), or

iii $p_\alpha = (M,0,0,0)^t$ (massive).



Definition

The Lorentz algebra has six generators subject to the followng relations:

$$[J^i, J^j] = \sqrt{-1}\epsilon^{ijk} J^k, \tag{1}$$

$$[K^i, K^j] = -\sqrt{-1}\epsilon^{ijk} J^k, \text{ and} \tag{2}$$

$$[J^i, K^j] = \sqrt{-1}\epsilon^{ijk} K^k. \tag{3}$$

This is the Lie algebra, $\mathfrak{so}(1,3)$, of the Lie group $SO(1,3)$.

Given the above set of relations, we set out to decouple the algebra into two separate (commuting) algebras.
Let $J_L^i = \frac{1}{2}(J^i - \sqrt{-1}K^i)$ and $J_R^i = \frac{1}{2}(J^i + \sqrt{-1}K^i)$.
Then we have

$$[J_L^i, J_L^j] = \sqrt{-1}\epsilon^{ijk} J_L^k,$$
$$[J_R^i, J_R^j] = \sqrt{-1}\epsilon^{ijk} J_R^k, \text{ and}$$
$$[J_L^i, J_R^j] = 0.$$

Remark

Thus, $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ as algebras.

In this context, we may describe the states of a relativistic quantum particle in terms of two angular momenta (quantum numbers) (j, j'), which correspond to the eigenvalues of the operators J_L^i and J_R^i respectively.

Given the Lorentz algebra above, we construct the corresponding Lie group called the Lorentz group. A general element of the Lorentz algebra is

$$\vec{\theta} \cdot \vec{J} + \vec{\omega} \cdot \sqrt{-1}\vec{K},$$

so a general element of the Lorentz group is of the form

$$\Lambda = exp(\vec{\theta} \cdot \vec{J} + \vec{\omega} \cdot \sqrt{-1}\vec{K}).$$

Definition

More invariantly, we can define the *Lorentz group* as

$$O(1,3) = \{\Lambda : \mathbb{R}^{1,3} \to \mathbb{R}^{1,3} \mid \langle x, x \rangle = \langle \Lambda x, \Lambda x \rangle\},$$

where the angle brackets are the natural pairing in Minkowski space.

Definition

- The transformations such that $\det(\Lambda) = 1$ form a subgroup of $O(1,3)$, which is denoted by $SO(1,3)$. Such transformations are called the *proper Lorentz transformations*.
- The proper Lorentz group has two disconnected components, namely, transformations such that $\Lambda^0_0 \geq 0$ or such that $\Lambda^0_0 \leq 0$; these transformations are called *orthochronous* or *non-orthochronous*, respectively.
- We will denote the proper orthochronous Lorentz transformations as $SO^+(1,3)$.

The transformations such that $\det(\Lambda^{\alpha}_{\beta}) = -1$ can occur in three different ways which generate the remaining three component from the component containing the identity element:

- T such that $(t, x, y, z) \mapsto (-t, x, y, z)$ is called the time reversal operation,
- Transformations such that $(t, x, y, z) \mapsto (t, -x, y, z)$ or $(t, x, -y, z)$ or $(t, x, y, -z)$ (i.e.reflections about a single spatial axis), and
- P such that $(t, x, y, z) \mapsto (t, -x, -y, -z)$ is called the *spatial parity operator*,

Remark

We use representations from the connected component which contains the identity element (the proper orthochronous elements). Then we extend the representation of the restricted Lorentz group to a representation of the Lorentz group by acting on the restricted representation by the operations P, T, and PT.

We consider representations of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ and their respective Lie groups.

Definition

To this end we introduce the *Pauli matrices*:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \text{and } \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $[\sigma_i, \sigma_j] = 2\sqrt{-1}\epsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$.

In addition to the three Pauli matrices above, we introduce a fourth matrix, $\sigma_0 = I_2$.

Remark

Then the fact that $(\sigma_0)^2 = I_2$ and $(\sigma_i)^2 = -I_2$ motivates our choice of signature for the Minkowski space, which is isomorphic to the group of quaternions with unit Euclidean length.

Definition

- We say that a square matrix is *Hermitian* when it is invariant under the action of the adjoint operator: $A = \overline{A^t}$.
- We let $\mathbb{H}(2) \subset M_{\mathbb{C}}(2)$ denote the space of 2×2 Hermitian matrices.

Then we use the injective vector space homomorphism $\mathbb{R}^{1,3} \cong \mathbb{R}^4 \to \mathbb{H}(2)$ such that $e_\eta \mapsto \sigma_\eta$:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_0 + x_3 & x_1 - \sqrt{-1}x_2 \\ x_1 + \sqrt{-1}x_2 & x_0 - x_3 \end{bmatrix}.$$

The natural pairing in the Minkowski space under this homomorphism becomes the determinant of the 2×2 matrix presentation of the vector in Hermitian space.
Consider the orthogonal transformation $\Lambda \in O(1,3)$ in the σ basis: $\Lambda \in \mathbb{H}(2)$.
Then the action of $SL_{\mathbb{C}}(2)$ by congugation is

$$\Lambda \mapsto X\Lambda X^{-1} = X\Lambda\overline{X}^t,$$

which leaves the natural product on Minkowski space invariant.

Definition

The above homomorphism is called the *spinor representation of* $SO(1,3)$ *in* $GL_{\mathbb{C}}(2)$.

Remark

The homomorphism $\mathbb{H}(2) \to SO^{+}(1,3)$ is a double cover.

Remark

The kernel of this homomorphism are $\{X \in SL_{\mathbb{C}}(2) \mid \Lambda X = X\Lambda\}$.

Now that we have discussed the spinor representation of $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, we recall that $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$, so we define an algebra on the direct sum of two spinor representations.

We do this explicitly in block form, or, equivalently, using a chiral basis.

Definition

The γ-*matrices* are given as:

$$\gamma^0 = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}, \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}$$

where $i =$1,2, or 3.

Consider an arbitrary element $\Lambda \in O(1,3)$ such that

$$\gamma' = \Lambda\gamma.$$

There is an analogous homomorphism called the *spinor representation of the group* $O(1,3)$ *in* $GL_{\mathbb{C}}(4)$.

Definition

The 4-dimensional complex space, $\mathbb{C}^4$, with the above spinor representation acting on $\mathbb{C}^4$,
is called the *space of (4-component) spinors*.
The elements of this space are column vectors and are called *spinors*.

We decompose $\mathbb{C}^4$ into $\mathbb{C}^2 \oplus \mathbb{C}^2$. Then the spinors become

$$\psi = \begin{bmatrix} \xi \\ \chi \end{bmatrix},$$

where $\psi \in \mathbb{C}^4$, $\xi \in \mathbb{C}^2$, and $\chi \in \mathbb{C}^2$.
The spatial parity operator, in this basis, is such that $\chi \mapsto \eta$ and $\eta \mapsto \chi$.

Remark

Note that the (2-component) *spinors* are a complex ordered pair on a unit sphere.

We introduce one more γ-matrix,

$$\gamma^5 = \sqrt{-1}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}.$$

The operator $\pi_{L/R} = \frac{1}{2}(I_4 \pm \gamma^5)$ is called the *Lorentz invariant projection operator*.
We can use $\pi_{L/R}$ to separate the (4-component) spinors into their two (2-component) spinors:

$$\psi_{L/R} = \pi_{L/R}\psi.$$

For $\mu, \nu = 0, 1, 2, 3$, the γ^μ generate a Clifford algebra on four generators

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

($\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ whenever $\mu \neq \nu$).
This Clifford algebra was discovered by P.A.M. Dirac as necessary and sufficient conditions for the Klein-Gordon equation,

$$-\left(g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} + M^2\right)\psi = 0,$$

to be "factored" into two (congugate) first order equations:

$$\left(\sqrt{-1}\gamma^\alpha\frac{\partial}{\partial x^\alpha} + M\right)\left(\sqrt{-1}\gamma^\beta\frac{\partial}{\partial x^\beta} - M\right)\psi = 0.$$

Definition

The *Dirac equation* (for the 4-spinor ψ) is

$$\left(\sqrt{-1}\gamma^{\beta}\frac{\partial}{\partial x^{\beta}} - m\right)\psi = 0, \tag{4}$$

while the *conjugate Dirac equation* is
$\left(\sqrt{-1}\gamma^{\alpha}\frac{\partial}{\partial x^{\alpha}} + m\right)\overline{\psi} = 0,$
where $\overline{\psi} = \psi^{*}\gamma^{0}$ is the *Dirac conjugate*.

Remark

This is the relativistic wave equation for massive, spin 1/2 (anti-)particles.

Remark

- We have now shone the spherical symmetry and harmonic nature of spinors; it should come as no surprise when we encounter sherical harmonics in our further investigations into this algebra.
- Further, anyone with a background in quantum mechanics will notice this process as defining the "raising" and "lowering" operators.
- It is the raising operators which generate our Clifford algebra.

In full generalization of the relativity described by the Euclidean group,
Weyl introduced the generators of space-time translations,

$$P_\alpha = \sqrt{-1}\frac{\partial}{\partial x_\alpha},$$

as a symmetry of the special theory of relativity.
In this way, a generic translation operator in the Poincaré group is of the form

$$\exp(-\sqrt{-1}P^\alpha a_\alpha),$$

where a_α is a four component position vector and P^α is the four component energy-momentum operators associated to the vector x^α which is being translated.

Definition

The Poincaré algebra is given as:

$$[J^i, J^j] = \sqrt{-1}\epsilon^{ijk} J^k, \; [K^i, K^j] = -\sqrt{-1}\epsilon^{ijk} J^k,$$
$$[J^i, K^j] = \sqrt{-1}\epsilon^{ijk} K^k, \; [J^i, P^j] = \sqrt{-1}\epsilon^{ijk} P^k,$$
$$[K^i, P^j] = \sqrt{-1} P^0 \delta^{ij} I_3, \; [P^i, P^j] = 0,$$
$$[J^i, P^0] = 0, \; [P^i, P^0] = 0,$$
$$[K^i, P^0] = \sqrt{-1} P^i.$$