Calculus II HW 1

App E: 8, 10, 14, 22, 30, 41 (c)

Write the sum in expanded form.

8) $\sum_{j=n}^{n+3} j^2 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$

This can be simplified to a quadratic polynomial. However, for this problem, this is sufficient.

10) $\sum_{i=1}^{n} f(x_i)\Delta x_i$

$= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n$

Write the sum in sigma notation.

14) $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \ldots + \frac{23}{27}$

$= \sum_{n=1}^{21} \frac{n+2}{n+6} = \sum_{n=3}^{23} \frac{n}{n+4} = \sum_{n=7}^{27} \frac{n-4}{n}$

This form will be most common since it starts at $n=1$. However, there are many ways to write this. These are just three examples.

Find the value of the sum.

22) $\sum_{i=3}^{6} i(i+2) = 3(5) + 4(6) + 5(7) + 6(8)$

$= 15 + 24 + 35 + 42$

$= 116.$

30) $\sum_{i=1}^{n} (2-5i) = \left(\sum_{i=1}^{n} 2\right) + \left(\sum_{i=1}^{n} -5i\right)$

$= \left(\sum_{i=1}^{n} 2\right) - 5\left(\sum_{i=1}^{n} i\right)$

$= 2n - 5\frac{n(n+1)}{2}$ This is fine, but you can go further

$= \boxed{2n - \frac{5}{2}n(n+1)} = 2n - \frac{5}{2}n^2 - \frac{5}{2}n = -\frac{5}{2}n^2 - \frac{1}{2}n$

Evaluate each telescoping sum.

41(c) $\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

Examine the first few terms:

$$\left(\frac{1}{3} - \underbrace{\frac{1}{4}\right) + \left(\frac{1}{4}}_{\text{these cancel}} - \underbrace{\frac{1}{5}\right) + \left(\frac{1}{5}}_{\text{these cancel}} - \frac{1}{6}\right)$$

Hence, we get

$$\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{3} - \frac{1}{99+1} = \frac{1}{3} - \frac{1}{100} = \frac{97}{300}$$

App E: 43, 44; 4.1: 4(a), 20, 22, 24

Find the limit.

43) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \frac{i^2}{n^2}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{1}{6} \cdot \frac{n}{n} \xrightarrow{} 1 \cdot \frac{n+\cancel{1}}{n} \xrightarrow{} 1 \cdot \frac{2n+\cancel{1}}{n} \xrightarrow{} 2$$

$$= \frac{1}{6} \cdot 1 \cdot 1 \cdot 2$$

$$= \frac{1}{3}$$

44) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 1\right]$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\frac{i^3}{n^3} + 1\right]$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^3}{n^4} + \frac{1}{n}\right)$

$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{i^3}{n^4} + \sum_{i=1}^{n} \frac{1}{n}\right)$

$= \lim_{n\to\infty} \left(\frac{1}{n^4} \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} \frac{1}{n}\right)$

$= \lim_{n\to\infty} \left(\frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 + n \cdot \frac{1}{n}\right)$

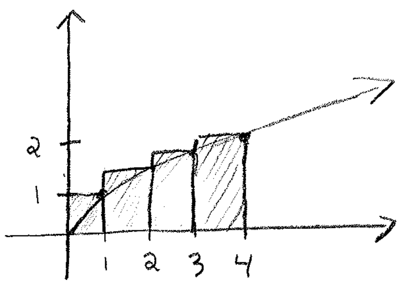
$= \lim_{n\to\infty} \left(\frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] + 1\right)$

$= \lim_{n\to\infty} \left(\frac{1}{4} \cdot \cancel{\frac{n}{n}}^{1} \cdot \cancel{\frac{n}{n}}^{1} \cdot \cancel{\frac{n+1}{n}}^{1} \cdot \cancel{\frac{n+1}{n}}^{1} + 1\right)$

$= \frac{1}{4} \cdot 1 \cdot 1 \cdot 1 \cdot 1 + 1$

$= \frac{1}{4} + 1 = \frac{5}{4}$

4(a) Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?



width of each rectangle = 1
height of each rectangle = $\sqrt{x_i}$

$$\text{Sum} = 1 \cdot \sqrt{1} + 1 \cdot \sqrt{2} + 1 \cdot \sqrt{3} + 1 \cdot \sqrt{4}$$
$$= 1 + \sqrt{2} + \sqrt{3} + 2$$
$$= \boxed{3 + \sqrt{2} + \sqrt{3}}$$

From the graph, we can see it is an overestimation.

Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

Definition 2 The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

- $$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x \right]$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

- $\Delta x = \dfrac{b-a}{n}$; $x_i = a + i\Delta x = a + i\left(\dfrac{b-a}{n}\right)$

20) $f(x) = x^2 + \sqrt{1+2x}$, $4 \le x \le 7$

$$\Delta x = \frac{7-4}{n} = \frac{3}{n} \; ; \; x_i = 4 + \frac{3i}{n}$$

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(4+\frac{3i}{n}\right)^2 + \sqrt{1+2\left(4+\frac{3i}{n}\right)} \right] \frac{3}{n}}$$

Note: $f(x_i) = x_i^2 + \sqrt{1+2x_i}$

So, above we replaced each x_i with $4 + \frac{3i}{n}$

Determine a region whose area is equal to the given limit. Do not evaluate the limit.

22) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$

$\Delta x = \frac{2}{n}$, so $b - a = 2$

$f(x_i) = \left(5 + \frac{2}{n} i\right)^{10}$, so $x_i = 5 + \frac{2}{n} i$ and $a = 5$

So, $\boxed{f(x) = x^{10} \text{ and } a = 5, b = 7}$

24) a) Use Definition 2 to find an expression for the area under the curve $y = x^3$ from 0 to 1 as a limit.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$$

$$f(x_i) = x_i^{\,3} = \left(\frac{i}{n}\right)^3$$

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right)}$$

b) Evaluate the limit in part (a).

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^3}\,\frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty} \frac{1}{n^4}\,\frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty} \frac{1}{4} \cdot \cancelto{1}{\frac{n}{n}} \cdot \cancelto{1}{\frac{n}{n}} \cdot \cancelto{1}{\frac{n+1}{n}} \cdot \cancelto{1}{\frac{n+1}{n}}$$

$$= \boxed{\frac{1}{4}}$$

4.2: 20, 23, 27, 33, 34, 37

Express the limit as a definite integral on the given interval.

20) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2+4}\,\Delta x$, $[1,3]$

$$= \boxed{\int_1^3 \frac{x}{x^2+4}\,dx}$$

Use the form of the definition of the integral given in Theorem 4 to evaluate the integral.

23) $\int_{-2}^{0} (x^2+x)\,dx$

$$\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = -2 + i\left(\frac{2}{n}\right) = -2 + \frac{2i}{n}$$

$$\int_{-2}^{0} (x^2+x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2 + x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-2+\frac{2i}{n}\right)^2 + \left(-2+\frac{2i}{n}\right)\right]\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[4 - \frac{8i}{n} + \frac{4i^2}{n^2} - 2 + \frac{2i}{n}\right]\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 - \frac{6i}{n} + \frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n} - \frac{12i}{n^2} + \frac{8i^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{4}{n} - \sum_{i=1}^{n} \frac{12i}{n^2} + \sum \frac{8i^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{4}{n} - \frac{12}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty} \left[\frac{4}{n}\cdot n - \frac{12}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty} \left(4 - 6\cdot\cancel{\frac{n}{n}}^{1}\cdot\cancel{\frac{n+1}{n}}^{1} + \frac{8}{6}\cdot\cancel{\frac{n}{n}}^{1}\cdot\cancel{\frac{n+1}{n}}^{1}\cdot\cancel{\frac{2n+1}{n}}^{2}\right)$$

$$= 4 - 6 + \frac{8}{3} = \boxed{\frac{2}{3}}$$

27) Prove that $\int_a^b x\,dx = \frac{b^2-a^2}{2}$

$$\int_a^b x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} x_i\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[a+i\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)$$

$\Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{a(b-a)}{n} + i\,\frac{(b-a)^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{a(b-a)}{n} + \sum_{i=1}^{n} i\,\frac{(b-a)^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{a(b-a)}{n} + \frac{(b-a)^2}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty}\left(a(b-a) + \frac{(b-a)^2}{n^2}\,\frac{n(n+1)}{2}\right)$$

$$= \lim_{n\to\infty}\left(a(b-a) + \frac{(b-a)^2}{2}\cdot\overset{1}{\cancel{\frac{n}{n}}}\cdot\overset{1}{\cancel{\frac{n+1}{n}}}\right)$$

$$= a(b-a) + \tfrac{1}{2}(b-a)^2$$

$$= (b-a)\left(a+\tfrac{1}{2}(b-a)\right)$$

$$= (b-a)\left(a+\tfrac{1}{2}b-\tfrac{1}{2}a\right)$$

$$= (b-a)\left(\tfrac{1}{2}a+\tfrac{1}{2}b\right)$$

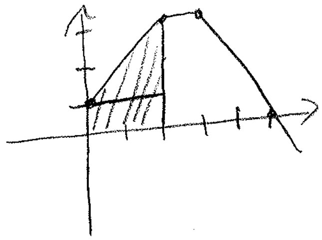
$$= \tfrac{1}{2}(b-a)(b+a)$$

$$= \tfrac{1}{2}(b^2-a^2)$$

$$= \frac{b^2-a^2}{2}$$

33) The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

a) $\int_0^2 f(x)\,dx$



Area of rectangle = width $\cdot$ height

$= 2 \cdot 1$

$= 2$

Area of triangle $= \frac{1}{2} b \cdot h$

$= \frac{1}{2}(2)(2)$

$= \boxed{2}$

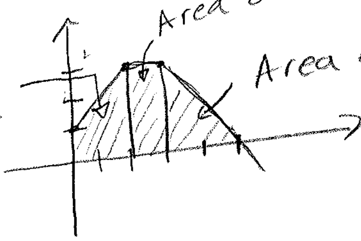
$\int_0^2 f(x)\,dx = 2 + 2 = 4$

b) $\int_0^5 f(x)\,dx$

Area of rectangle $= 1 \cdot 3 = 3$

Area of triangle $= \frac{1}{2} bh = \frac{1}{2}(2)(3) = 3$

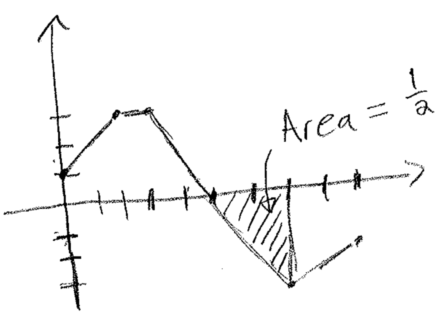
4 from part (a).



$\int_0^5 f(x)\,dx = 4 + 3 + 3 = \boxed{10}$

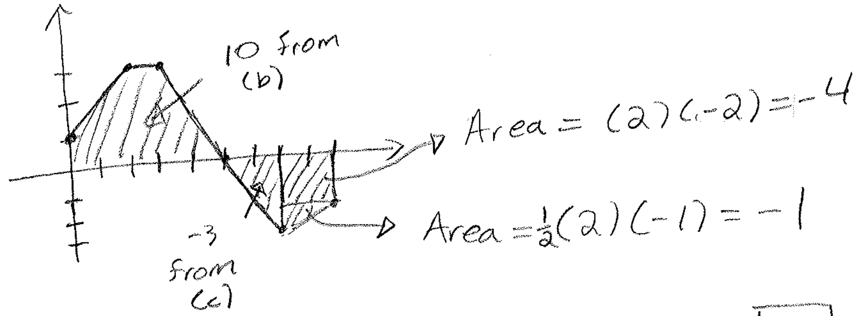
c) $\int_5^7 f(x)\,dx$

Area $= \frac{1}{2} bh = \frac{1}{2}(2)(-3) = -3$



$\int_5^7 f(x)\,dx = \boxed{-3}$

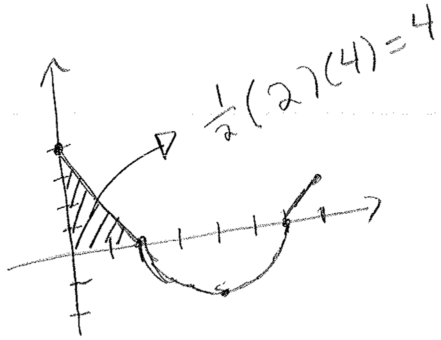
d) $\int_0^9 f(x)\,dx$



$\int_0^9 f(x)\,dx = 10 - 3 - 4 - 1 = \boxed{2}$

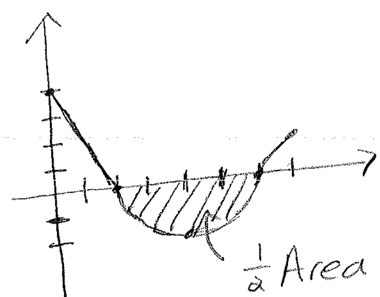
34) The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.

a) $\int_0^2 g(x)\,dx$



$\int_0^2 g(x)\,dx = 4$

b) $\int_2^6 g(x)\,dx$



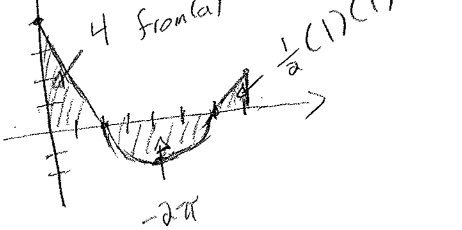
$\frac{1}{2}$ Area of a circle

$= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2 = 2\pi$

$\int_2^6 g(x)\,dx = -2\pi$

Don't forget the negative! This semicircle is below the x-axis.

c) $\int_0^7 g(x)\,dx$

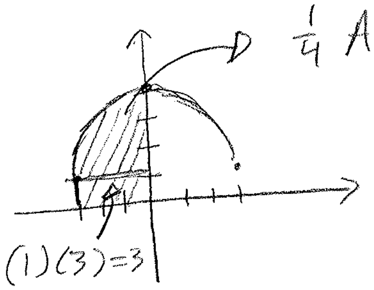


$\int_0^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = \boxed{\frac{9}{2} - 2\pi}$

37) Evaluate the integral by interpreting it in terms of area.

37) $\int_{-3}^{0} (1+\sqrt{9-x^2})\,dx$

$\frac{1}{4}$ Area of circle $= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(3)^2$
$= \frac{9}{4}\pi$

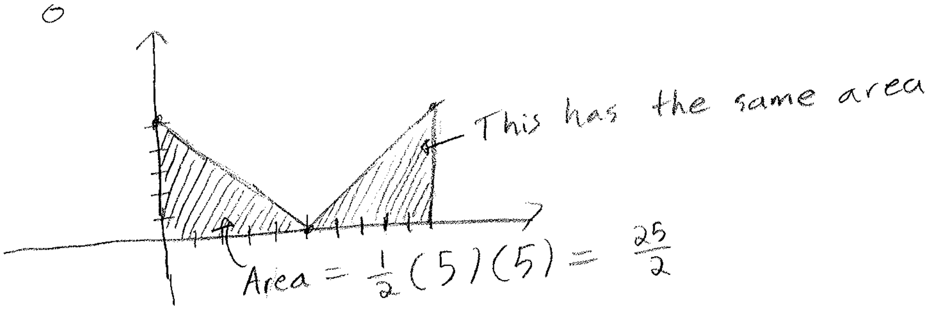


$\int_{-3}^{0} (1+\sqrt{9-x^2})\,dx = \boxed{3+\frac{9}{4}\pi}$

4.2: 40, 48, 50, 55, 58, 61, 72

Evaluate the integral by interpreting it in terms of area.

40) $\int_{0}^{10} |x-5|\,dx$



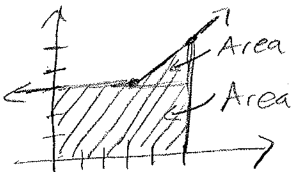
$\int_{0}^{10} |x-5|\,dx = \frac{25}{2} + \frac{25}{2} = \boxed{25}$

48) If $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 3.6$, find $\int_1^4 f(x)\,dx$.

$$\int_1^4 f(x)\,dx = \int_1^5 f(x)\,dx - \int_4^5 f(x)\,dx = 12 - 3.6$$
$$= \boxed{8.4}$$

50) Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$$



$$\int_0^5 f(x)\,dx = 2 + 15 = 17$$

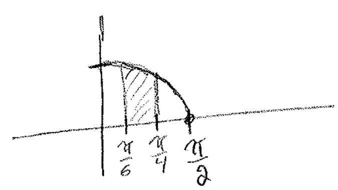
Use the properties of integrals to verify the inequality without evaluating the integrals.

55) $\int_0^4 (x^2 - 4x + 4)\,dx \geq 0$

$x^2 - 4x + 4 = (x-2)^2 \geq 0$ on $[0,4]$, so

Property 6 $\int_0^4 (x^2 - 4x + 4)\,dx \geq 0$

58) $\frac{\sqrt{2}\pi}{24} \leq \int_{\pi/6}^{\pi/4} \cos x\,dx \leq \frac{\sqrt{3}\pi}{24}$



If $\frac{\pi}{6} \leq x \leq \frac{\pi}{4}$, then $\cos\frac{\pi}{4} \leq \cos x \leq \cos\frac{\pi}{6}$ since $\cos x$ is decreasing.

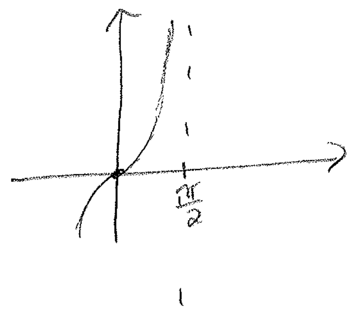
So $\frac{\sqrt{2}}{2} \leq \cos x \leq \frac{\sqrt{3}}{2}$

Property 8 $\frac{\sqrt{2}}{2}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \leq \int_{\pi/6}^{\pi/4} \cos x\,dx \leq \frac{\sqrt{3}}{2}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$\frac{\sqrt{2}\pi}{24} \leq \int_{\pi/6}^{\pi/4} \cos x\,dx \leq \frac{\sqrt{3}\pi}{24}$$

Use Property 8 to estimate the value of the integral.

61) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \, dx$



tan x is increasing!

$\frac{\pi}{4} \le x \le \frac{\pi}{3}$

$\tan \frac{\pi}{4} \le \tan x \le \tan \frac{\pi}{3}$

$1 \le \tan x \le \sqrt{3}$

$1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \, dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$$\boxed{\frac{\pi}{12} \le \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \, dx \le \frac{\sqrt{3}\pi}{12}}$$

Express the limit as a definite integral.

72) $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(\frac{i}{n})^2}$

$x_i = \frac{i}{n}$, so $a = 0$

$\Delta x = \frac{1}{n} = \frac{b-0}{n}$, so $b = 1$

$$\boxed{\int_0^1 \frac{1}{1+x^2}\,dx}$$