Homework 21

4.3 # 8, 10, 12, 20, 28, 34, 60

Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

8) $g(x) = \int_1^x (2+t^4)^5 \, dt$

$g'(x) = (2+x^4)^5$

10) $g(r) = \int_0^r \sqrt{x^2+4} \, dx$

$g'(r) = \sqrt{r^2+4}$

12) $G(x) = \int_x^1 \cos\sqrt{t} \, dt$

$= -\int_1^x \cos\sqrt{t} \, dt$

$= -\cos\sqrt{x} \, dt$

Evaluate the integral

20) $\int_{-1}^{1} x^{100} \, dx$

$= \frac{x^{101}}{101} \Big|_{-1}^{1} = \frac{(1)^{101}}{101} - \frac{(-1)^{101}}{101} = \frac{1}{101} + \frac{1}{101} = \boxed{\frac{2}{101}}$

28) $\int_0^4 (4-t)\sqrt{t} \, dt$

$= \int_0^4 (4\sqrt{t} - t\sqrt{t}) \, dt$

$= \int_0^4 (4t^{\frac{1}{2}} - t^{\frac{3}{2}}) \, dt$

$= \frac{8}{3}t^{\frac{3}{2}} - \frac{2}{5}t^{\frac{5}{2}} \Big|_0^4 = \left[\frac{8}{3}(8) - \frac{2}{5}(32)\right] - 0 = \boxed{\frac{128}{15}}$

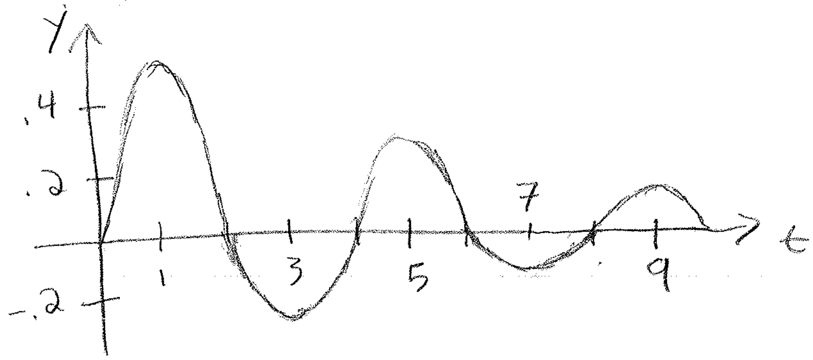
34) $\int_1^2 \frac{s^4+1}{s^2}\,ds$

$= \int_1^2 \left(s^2 + \frac{1}{s^2}\right) ds$

$= \int_1^2 \left(s^2 + s^{-2}\right) ds$

$= \frac{1}{3}s^3 - s^{-1}\Big|_1^2 = \left(\frac{1}{3}(2)^3 - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right) = \frac{17}{6}$

60) Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.



a) At what values of x do the local maximum and minimum values occur?

$g'(x) = f(x)$, by the Fundamental Theorem of Calculus Part 1.

Hence, when $f(x)$ goes from positive to negative, there is a maximum.

Also, when $f(x)$ goes from negative to positive, there is a minimum.

Maximums: $x = 2, 6$

Minimums: $x = 4, 8$

b) Where does g achieve an absolute maximum?

We can see that $\left|\int_0^2 f(t)\,dt\right| > \left|\int_2^4 f(t)\,dt\right| > \left|\int_4^6 f(t)\,dt\right| > \left|\int_6^8 f(t)\,dt\right|$
$> \left|\int_8^{10} f(t)\,dt\right|$,

Also, $g(2) = \left|\int_0^2 f\,dt\right|$

$g(6) = g(2) - \left|\int_2^4 f\,dt\right| + \left|\int_4^6 f\,dt\right|$

$g(10) = g(6) - \left|\int_6^8 f\,dt\right| + \left|\int_8^{10} f\,dt\right|$

From the inequality, we can see that we are subtracting more each time then we are adding.

So, $g(2)$ is the global maximum

c) On what intervals are g concave downward?

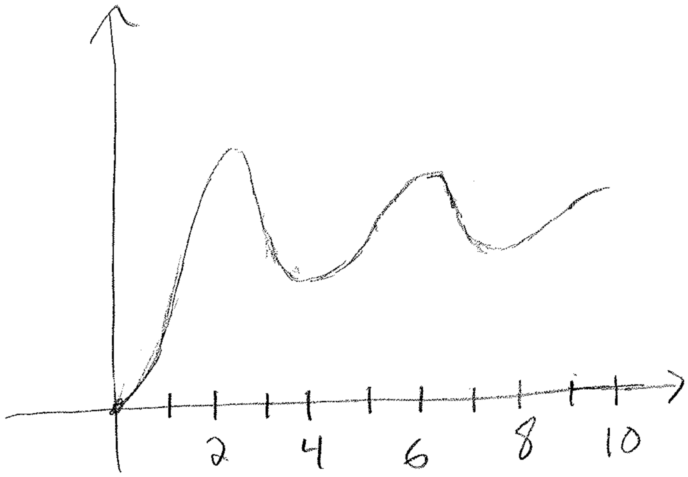
g is concave downward when $g'' < 0$.

$g' = f$, so g is concave downward when $f' < 0$.

i.e. when f is decreasing.

Hence, g is concave downward on $(1,3)$, $(5,7)$, and $(9,10)$.

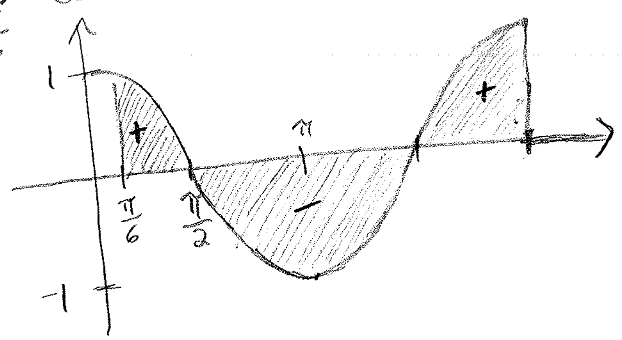
d)



4.3 # 48, 52, 62, 66 ; 4.4 # 6, 10, 33, 41, 46

Evaluate the integral and interpret it as a difference of areas. Illustrate with a sketch.

48) $\int_{\frac{\pi}{6}}^{2\pi} \cos x \, dx$



$$\int_{\frac{\pi}{6}}^{2\pi} \cos x \, dx = \sin x \Big|_{\frac{\pi}{6}}^{2\pi}$$

$$= \sin(2\pi) - \sin\left(\frac{\pi}{6}\right)$$

$$= 0 - \frac{1}{2} = \boxed{-\frac{1}{2}}$$

Find the derivative of the function.

52) $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}} \, dt$

$$= \int_{\tan x}^{0} \frac{1}{\sqrt{2+t^4}} \, dt + \int_{0}^{x^2} \frac{1}{\sqrt{2+t^4}} \, dt$$

$$= -\int_{0}^{\tan x} \frac{1}{\sqrt{2+t^4}} \, dt + \int_{0}^{x^2} \frac{1}{\sqrt{2+t^4}} \, dt$$

$$= -\frac{1}{\sqrt{2+\tan^4 x}} (\tan x)' + \frac{1}{\sqrt{2+(x^2)^4}} (x^2)'$$

$$= \boxed{\frac{-\sec^2 x}{\sqrt{2+\tan^4 x}} + \frac{2x}{\sqrt{2+x^8}}}$$

Evaluate the limit by first recognizing the sum as a Reimann sum for a function defined on $[0,1]$.

62) $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\dots+\sqrt{\frac{n}{n}}\right)$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}$$

$\Delta x = \frac{1}{n}$ $\quad x_i = \frac{i}{n}$

$b-a=1$ $\quad a=0$

$b=1$

$$\int_0^1 \sqrt{x}\,dx = \int_0^1 x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}(1)^{\frac{3}{2}}-0 = \boxed{\frac{2}{3}}$$

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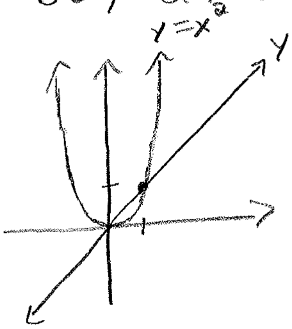
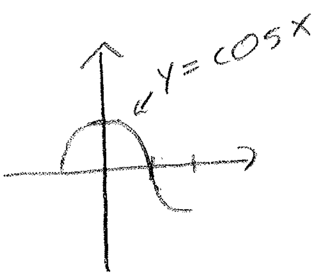
66) a) Show that $\cos(x^2) \geq \cos x$ for $0 \leq x \leq 1$



On $0 \leq x \leq 1$:

$x^2 \leq x$

$\cos x^2 \geq \cos x$



cos x is decreasing on $[0,1]$, so we will flip the inequality.

b) Deduce $\int_0^{\frac{\pi}{6}} \cos x^2\,dx \geq \frac{1}{2}$.

$\cos x^2 \geq \cos x$

So, $\int_0^{\frac{\pi}{6}} \cos x^2\,dx \geq \int_0^{\frac{\pi}{6}} \cos x\,dx$ $\quad = \sin x \Big|_0^{\frac{\pi}{6}}$

$= \sin\frac{\pi}{6} - \sin 0$

$= \frac{1}{2}$

So, $\int_0^{\frac{\pi}{6}} \cos x^2\,dx \geq \frac{1}{2}$

Find the general indefinite integral.

6) $\int (\sqrt{x^3} + \sqrt[3]{x^2})\,dx$

$= \int (x^{\frac{3}{2}} + x^{\frac{2}{3}})\,dx$

$= \boxed{\frac{2}{5}x^{\frac{5}{2}} + \frac{3}{5}x^{\frac{5}{3}} + C}$

10) $\int v(v^2+2)^2\,dv$

$= \int v(v^4+4v^2+4)\,dv$

$= \int (v^5 + 4v^3 + 4v)\,dv$

$= \boxed{\frac{1}{6}v^6 + v^4 + 2v^2 + C}$

Find the integral.

33) $\int_0^{\frac{\pi}{4}} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta$

$= \int_0^{\frac{\pi}{4}} (\sec^2\theta + 1)\,d\theta$

$= (\tan\theta + \theta)\Big|_0^{\frac{\pi}{4}}$

$= (\tan\frac{\pi}{4} + \frac{\pi}{4}) - (\tan 0)$

$= \boxed{1 + \frac{\pi}{4}}$

41) $\int_{-1}^{2} (x - 2|x|)\,dx$

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$= \int_{-1}^{0} (x+2x)\,dx + \int_0^2 (x-2x)\,dx$

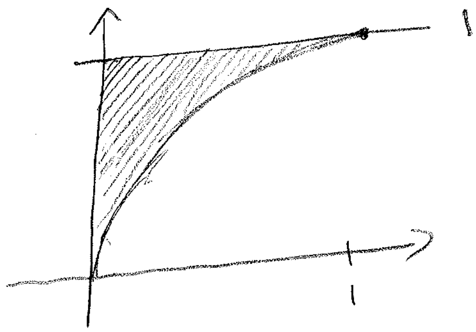
$= \int_{-1}^{0} 3x\,dx + \int_0^2 -x\,dx$

$= \frac{3}{2}x^2\Big|_{-1}^{0} + (-\frac{1}{2}x^2)\Big|_0^2$

$= (0 - \frac{3}{2}(-1)^2) + (-\frac{1}{2}(2)^2 - 0)$

$= -\frac{3}{2} - 2 = -\frac{7}{2}$

46) The boundaries of the shaded region are the y-axis, the line $y=1$, and the curve $y=\sqrt[4]{x}$. Find the area of this region by writing x as a function of y and integrating with respect to y.



$$y=\sqrt[4]{x}$$

so, $y^4=x$

$$\int_0^1 y^4\,dy$$

$$=\frac{1}{5}y^5\Big|_0^1=\frac{1}{5}(1)^5=\boxed{\frac{1}{5}}$$

4.4 #32, 42, 50, 56, 58, 59

Evaluate the integral.

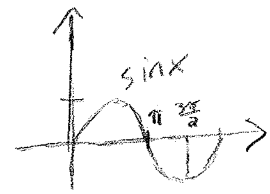
32) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc^2\theta\,d\theta$

$$=-\cot\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$=-\cot\left(\frac{\pi}{3}\right)+\cot\left(\frac{\pi}{4}\right)$$

$$=\boxed{-\frac{1}{\sqrt{3}}+1}$$

42) $\int_0^{\frac{3\pi}{2}} |\sin x|\,dx$



$$=\int_0^{\pi}\sin x\,dx+\int_{\pi}^{\frac{3\pi}{2}}-\sin x\,dx$$

$$=-\cos x\Big|_0^{\pi}+\cos x\Big|_{\pi}^{\frac{3\pi}{2}}$$

$$=(-\cos\pi+\cos 0)+(\cos\tfrac{3\pi}{2}-\cos\pi)$$

$$=(1+1)+(0+1)$$

$$=3$$

50) A honey bee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?

$100 + \int_0^{15} n'(t)\,dt$ is the bee population 15 weeks after start.

The velocity function is given for a particle moving along a line. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

56) $v(t) = t^2 - 2t - 8, \quad 1 \le t \le 6$

a) $\int_1^6 t^2 - 2t - 8\,dt$

$= \frac{1}{3}t^3 - t^2 - 8t \Big|_1^6$

$= (\frac{1}{3}(6)^3 - (6)^2 - 8(6)) - (\frac{1}{3}(1)^3 - (1)^2 - 8(1))$

$= -12 + \frac{26}{3} = \boxed{-\frac{10}{3} \text{ meters}}$

b) $\int_1^6 |t^2 - 2t - 8|\,dt$

$= \int_1^4 (-t^2 + 2t + 8)\,dt + \int_4^6 (t^2 - 2t - 8)\,dt$

$= -\frac{1}{3}t^3 + t^2 + 8t \Big|_1^4 + \frac{1}{3}t^3 - t^2 - 8t \Big|_4^6$

$= (-\frac{1}{3}(4)^3 + 4^2 + 8(4) + \frac{1}{3}(1)^3 - 1^2 - 8(1))$

$+ (\frac{1}{3}(6)^3 - 6^2 - 8(6) - \frac{1}{3}(4)^3 + 4^2 + 8(4))$

$= \frac{182}{3} + \frac{44}{3} = \boxed{\frac{226}{3} \text{ meters}}$

The acceleration function and the initial velocity are given for a particle moving along a line. Find (a) the velocity at time t and (b) the distance traveled during the given time interval.

58) $a(t) = 2t + 3$, $v(0) = -4$, $0 \le t \le 3$

a) $v(t) = \int 2t+3$

$= t^2 + 3t + c$

$v(0) = -4$

So, $0^2 + 3(0) + c = -4$

$c = -4$

$\boxed{v(t) = t^2 + 3t - 4}$

b) $\int_0^3 t^2 + 3t - 4$

$= \frac{1}{3}t^3 + \frac{3}{2}t^2 - 4t \Big|_0^3$

$= \left[\frac{1}{3}(3)^3 + \frac{3}{2}(3)^2 - 4(3)\right] - 0$

$= 9 + \frac{27}{2} - 12$

$= \boxed{10.5 \text{ m}}$

59) The linear density of a rod of length 4m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total mass of the rod.

$$\int_0^4 (9 + 2\sqrt{x})\,dx$$

$$= \int_0^4 (9 + 2x^{\frac{1}{2}})\,dx$$

$$= 9x + \frac{4}{3}x^{\frac{3}{2}} \Big|_0^4$$

$$= \left[9(4) + \frac{4}{3}(4)^{\frac{3}{2}}\right] - 0$$

$$= 36 + \frac{32}{3} = \boxed{\frac{140}{3}}$$

4.5: 6, 16, 18, 28, 38, 42, 48, 60

Evaluate the integral by making the given substitution.

6) $\int \frac{\sec^2(\frac{1}{x})}{x^2}\,dx$, $u = \frac{1}{x}$

$du = -\frac{1}{x^2}\,dx$

$= -\int \sec^2(u)\,du$

$= -\tan(u) + c$

$= \boxed{-\tan(\frac{1}{x}) + c}$

Evaluate the indefinite integral.

16) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2du = \frac{1}{\sqrt{x}}\,dx$

$= 2\int \sin u\,du$

$= -2\cos u + c$

$= \boxed{-2\cos\sqrt{x} + c}$

18) $\int \cos^4\theta \sin\theta\,d\theta$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$-du = \sin\theta\,d\theta$

$= -\int u^4\,du$

$= -\frac{1}{5}u^5 + c$

$= \boxed{-\frac{1}{5}\cos^5\theta + c}$

28) $\int x^2 \sqrt{2+x}\, dx$ $\qquad u = 2+x, \quad x = u-2$
$\qquad du = dx$

$= \int (u-2)^2 \sqrt{u}\, du$

$= \int (u^2-4u+4)\sqrt{u}\, du$

$= \int (u^2\sqrt{u} - 4u\sqrt{u} + 4\sqrt{u})\, du$

$= \int (u^{2+\frac{1}{2}} - 4u^{1+\frac{1}{2}} + 4u^{\frac{1}{2}})\, du$

$= \int (u^{\frac{5}{2}} - 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}})\, du$

$= \frac{2}{7}u^{\frac{7}{2}} - \frac{8}{5}u^{\frac{5}{2}} + \frac{8}{3}u^{\frac{3}{2}} + C$

$= \frac{2}{7}(2+x)^{\frac{7}{2}} - \frac{8}{5}(2+x)^{\frac{5}{2}} + \frac{8}{3}(2+x)^{\frac{3}{2}} + C$ This answer is also fine.

$= \boxed{\frac{2}{7}\sqrt{(2+x)^7} - \frac{8}{5}\sqrt{(2+x)^5} + \frac{8}{3}\sqrt{(2+x)^3} + C}$

Evaluate the definite integral.

38) $\int_0^{\sqrt{\pi}} x\cos(x^2)\, dx$ $\qquad u = x^2$
$\qquad du = 2x\, dx$
$\qquad \frac{1}{2}du = x\, dx$

$= \frac{1}{2}\int_{0^2}^{(\sqrt{\pi})^2} \cos u\, du$

$= \frac{1}{2}\int_0^{\pi} \cos u\, du$

$= \frac{1}{2}\sin u \Big|_0^{\pi}$

$= \frac{1}{2}\sin \pi - \frac{1}{2}\sin 0$

$= \frac{1}{2}(0) - \frac{1}{2}(0) = \boxed{0}$

42) $\int_0^{\frac{\pi}{2}} \cos x \sin(\sin x)\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$

$$= \int_{\sin(0)}^{\sin(\frac{\pi}{2})} \sin(u)\,du$$

$$= \int_0^1 \sin u\,du$$

$$= -\cos u \Big|_0^1 = -\cos(1) + \cos(0)$$

$$= -\cos(1) + 1 = \boxed{1-\cos 1}$$

48) $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$ $\quad u = 1+2x$, $du = 2\,dx$, $\frac{1}{2}du = dx$, $x = \frac{1}{2}(u-1)$

$$= \frac{1}{2}\int_{1+2(0)}^{1+2(4)} \frac{\frac{1}{2}(u-1)}{\sqrt{u}}\,du$$

$$= \frac{1}{4}\int_1^9 \frac{u-1}{\sqrt{u}}\,du = \frac{1}{4}\int_1^9 \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)du$$

$$= \frac{1}{4}\int_1^9 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du$$

$$= \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)\Big|_1^9$$

$$= \frac{1}{4}\left(\frac{2}{3}(9)^{\frac{3}{2}} - 2(9)^{\frac{1}{2}}\right) - \frac{1}{4}\left(\frac{2}{3}(1)^{\frac{3}{2}} - 2(1)^{\frac{1}{2}}\right)$$

$$= \frac{1}{4}(18-6) - \frac{1}{4}\left(\frac{2}{3} - 2\right)$$

$$= 3 + \frac{1}{3} = \boxed{\frac{10}{3}}$$

60) If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 x f(x^2)\,dx$.

$$\int_0^3 x f(x^2)\,dx \qquad u = x^2$$

$$du = 2x\,dx$$

$$= \frac{1}{2}\int_{0^2}^{3^2} f(u)\,du \qquad \frac{1}{2}du = x\,dx$$

$$= \frac{1}{2}\int_0^9 f(u)\,du$$

$$= \frac{1}{2}(4) = \boxed{2}$$