Homework 3 (problems assigned on Feb 2,4,6, due Feb 11/12)

The problems are solved in the same order as they are assigned on the class web-site.

1) $\int_0^1 (3t-1)^{50}\,dt = \int_{-1}^{2} u^{50}\,\frac{1}{3}\,du$

$[u=3t-1,\ du=3dt,\ t=0\to u=-1,\ t=1\to u=2]$

$= \frac{1}{3}\frac{u^{51}}{51}\Big|_{u=-1}^{2} = \frac{2^{51}-(-1)^{51}}{3\cdot 51} = \frac{2^{51}+1}{3\cdot 51}$

2) $\int_{-\pi/4}^{\pi/4} (x^3+x^4\tan x)\,dx = 0$ because the function $f(x)=x^3+x^4\tan x$ is odd:

$f(-x)=(-x)^3+(-x)^4\tan(-x)$

$= -x^3+x^4(-\tan x) = -f(x)$,

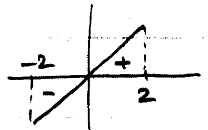
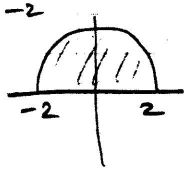
and we are integrating it over the interval $[-\frac{\pi}{4},\frac{\pi}{4}]$, which is symmetric around 0.

3) $\int_{1/2}^{1} \frac{\cos(x^{-2})}{x^3}\,dx = \int_4^1 \cos u\left(-\frac{1}{2}\right)du$

$[u=x^{-2},\ du=-\frac{2}{x^3}dx,\ x=\frac{1}{2}\to u=4,\ x=1\to u=1]$

$= -\frac{1}{2}\sin u\Big|_{u=4}^{1} = \frac{1}{2}(\sin 4-\sin 1)$

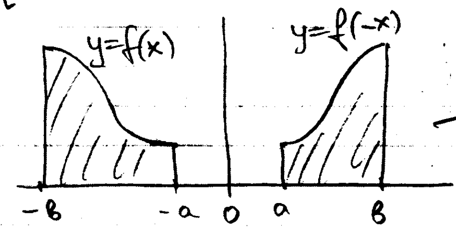
4) $\int_{-2}^{2}\left(x+3\sqrt{4-x^2}\right)dx = \int_{-2}^{2} x\,dx + 3\int_{-2}^{2}\sqrt{4-x^2}\,dx$

$=$  $+\ 3$ 

$= 0+3\cdot\frac{1}{2}\pi\cdot 2^2 = 6\pi$

5) $\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)\,du$

$[u=-x,\ du=-dx,\ x=a\to u=-a,\ x=b\to u=-b]$



— equal areas

6) $f(x)=\sqrt{x^2+3}$ is a monotonically increasing function, so

$f(1)\le f(x)\le f(3)$ for $x\in[1,3]$

$\Rightarrow\ 2=\sqrt{1^2+3}\le\sqrt{x^2+3}\le\sqrt{3^2+3}=\sqrt{12}=2\sqrt{3}$

$\Rightarrow$ by Comparison Property 8 on page 305,

$4=2(3-1)\le\int_1^3\sqrt{x^2+3}\,dx\le 2\sqrt{3}\,(3-1)=4\sqrt{3}$

7) (a) $C'(x) = \cos\left(\frac{1}{2}\pi x^2\right)$ is positive when

$\frac{1}{2}\pi x^2 \in \left(0, \frac{\pi}{2}\right)$, or $\frac{1}{2}\pi x^2 \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$,

or $\frac{1}{2}\pi x^2 \in \left(\frac{7\pi}{2}, \frac{9\pi}{2}\right)$, or $\frac{1}{2}\pi x^2 \in \left(\frac{11\pi}{2}, \frac{13\pi}{2}\right), \ldots$

i.e., when $x^2 \in (0,1)$ or $x^2 \in (3,5)$ or $x^2 \in (7,9), \ldots$

i.e., when

$x \in (-1,1)$ or $x \in (\sqrt{3}, \sqrt{5})$ or $x \in (-\sqrt{5}, -\sqrt{3})$

or $x \in (\sqrt{7}, \sqrt{9})$ or $x \in (-\sqrt{9}, \sqrt{7})$ or...

Since in these intervals $C'(x) > 0$, the function C is increasing there.

(b) C is concave up wherever $C'' > 0$, i.e., for those x for which

$C''(x) = -\sin\left(\frac{1}{2}\pi x^2\right) \cdot \frac{1}{2}\pi 2x > 0.$

8) Differentiate both sides and use FTC:

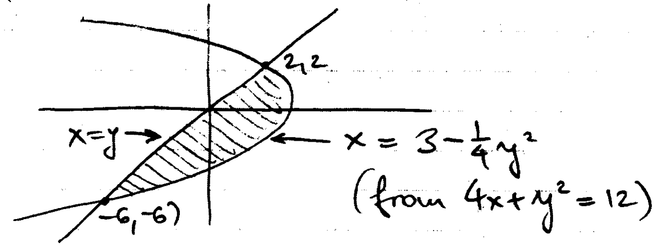
$2f(x) = 2\cos x \Rightarrow f(x) = \cos x.$

To find a, set $x = a$ in $2\int_a^x f(t)\,dt = 2\sin x - 1$ to get $0 = 2\sin a - 1 \Rightarrow \sin a = \frac{1}{2}$, $a = \frac{\pi}{6}$.

9) First find the intersection points of $4x + y^2 = 12$ and $x = y$: exclude y to get

$4x + x^2 = 12 \Rightarrow x_{1,2} = \{-6, 2\}$,

from $y = x$ we obtain that the intersection points are $(-6,-6)$ and $(2,2)$.



$A = \int_{-6}^{2} \left(3 - \frac{1}{4}y^2 - y\right) dy$

$= \left(3y - \frac{1}{12}y^3 - \frac{1}{2}y^2\right)\Big|_{y=-6}^{2}$

$= 3\cdot 2 - \frac{1}{12}\cdot 2^3 - \frac{1}{2}\cdot 2^2 - \left(3\cdot(-6) - \frac{1}{12}(-6)^3 - \frac{1}{2}(-6)^2\right)$

$= 22 - \frac{2}{3} = \frac{64}{3}$

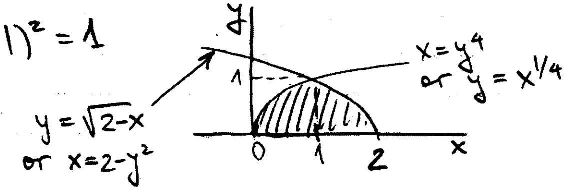
10) $\begin{cases} x = y^4 \\ y = \sqrt{2-x} \end{cases} |^2 \Rightarrow y^2 = 2 - x = 2 - y^4$

$\rightarrow y^4 + y^2 - 2 = 0,$

set $z = y^2$: $z^2 + z - 2 = 0$, $z_{1,2} = \{1, -2\}$

$\Rightarrow y = \pm 1$ (the root $z = -2$ doesn't work)

$x = (\pm 1)^2 = 1$



$$A=\int_0^1\left[(2-y^2)-y^4\right]dy=\left(2y-\frac{y^3}{3}-\frac{y^5}{5}\right)\Big|_{y=0}^{1}=\frac{22}{15}$$

or, alternatively,

$$A=\int_0^1 x^{1/4}\,dx+\int_1^2\sqrt{2-x}\,dx$$

$$=\frac{x^{5/4}}{5/4}\Big|_{x=0}^{1}+\int_1^0 u^{1/2}(-du)$$

$$\left[u=2-x,\ du=-dx,\ x=1\to u=1,\ x=2\to u=0\right]$$

$$=\frac{4}{5}+\frac{u^{3/2}}{3/2}\Big|_{u=0}^{1}=\frac{4}{5}+\frac{2}{3}=\frac{22}{15}$$

11) $$A=\int_0^1\left[\sqrt{y}-(y^2-1)\right]dy=\int_0^1\left(y^{1/2}-y^2+1\right)dy$$

$$=\left(\frac{y^{3/2}}{3/2}-\frac{y^3}{3}+y\right)\Big|_{y=0}^{1}=\frac{2}{3}-\frac{1}{3}+1=\frac{4}{3};$$

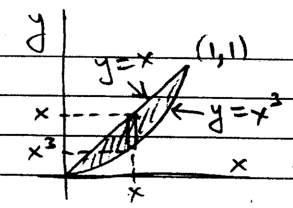
alternatively,

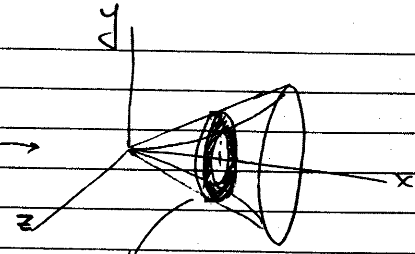
$$A=\int_{-1}^0\sqrt{1+x}\,dx+\int_0^1\left(1-x^2\right)dx \qquad \begin{matrix}u=1+x\\ du=dx\end{matrix}$$

$$=\int_0^1 u^{1/2}\,du+\left(x-\frac{x^3}{3}\right)\Big|_{x=0}^{1}$$

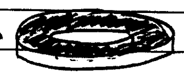
$$=\frac{u^{3/2}}{3/2}\Big|_{u=0}^{1}+\left(1-\frac{1}{3}\right)=\frac{2}{3}+1-\frac{1}{3}=\frac{4}{3}$$

12)





outer radius x,
inner radius x^3,
thickness Δx

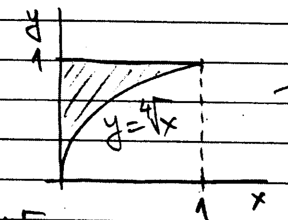


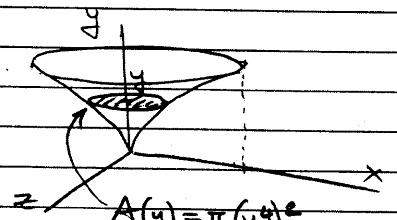
$$\text{Volume}=\underbrace{\left[\pi(x)^2-\pi(x^3)^2\right]}_{A(x)}\Delta x$$

$$\text{Total volume}=\int_0^1 A(x)\,dx$$

$$=\int_0^1\left(\pi x^2-\pi x^6\right)dx=\pi\left(\frac{x^3}{3}-\frac{x^7}{7}\right)\Big|_{x=0}^{1}=\frac{4}{21}\pi$$

13)



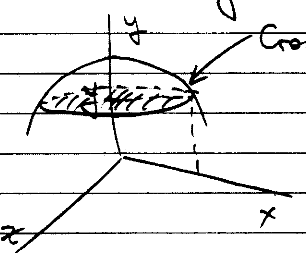


$y=\sqrt[4]{x}$ is the same as $x=y^4$

$$A(y)=\pi(y^4)^2=\pi y^8$$

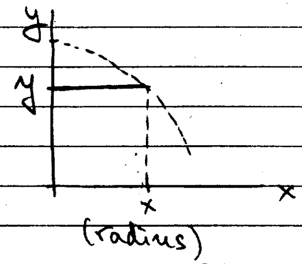
$$V=\int_0^1 A(y)\,dy=\int_0^1\pi y^8\,dy=\frac{\pi y^9}{9}\Big|_0^1=\frac{\pi}{9}$$

14) Clearly, $\pi \int_{-1}^{1} (1-y^2)^2 \, dy$ represents a solid of rotation obtained by rotation of a curve in the (x,y)-plane about the y-axis.



Cross-sectional area $A(y) = \pi (1-y^2)^2$

$= \pi \, (\text{radius})^2$



$\Rightarrow \quad x = 1 - y^2$

$\Rightarrow \pi \int_{-1}^{1} (1-y^2)^2 \, dy$ describes the volume of the solid obtained by rotating the region

$R = \{(x,y) \mid -1 \le y \le 1, \; 0 \le x \le 1 - y^2\}$

of the (x,y)-plane about the y-axis.