Additional Problems

HW #5

March 9, 2016

Problem 1. In class we were trying to show that

$$\frac{\text{the number of ways to choose m things out of (m+n) things}}{(m+n)!} = \frac{1}{m!n!} \tag{1}$$

Please show that

$$\text{the number of ways to choose m things out of (m+n) things} = \frac{(m+n)!}{m!n!} \tag{2}$$

More specifically, all the things are **distinct**, you can imagine choosing an m person team(without a captain) out of $m+n$ people.

Proof. A very well written solution can be found at: `http://betterexplained.com/articles/easy-permutations-and-combinations/`. This does require some original insights, and this problem is now graded as a bonus problem. □

Problem 2. The purpose of this problem is that using $\exp(x+y)$ =$\exp(x)\exp(y)$, one can show exp is indeed an exponential function. We proceed by steps:
1) Show that for a positive integer n, $\exp(n) = (\exp(1))^n$.
(Hint: write $\exp(n)$ as $\exp(1 + 1 + \cdots (n \text{ times}) \cdots + 1))$
2) Using 1), show that for a rational number $\frac{a}{b}$, where both a and b are positive integers, that $\exp(\frac{a}{b}) = (\exp(1))^{\frac{a}{b}}$.

After showing 1) and 2), we have proven that exp is indeed an exponential function on all positive rational numbers. You can think about how to prove a similar statement about negative integers and negative rational numbers. After that, since all the irrational numbers can be "infinitely approximated" by rational numbers, the function exp is indeed an exponential function with base exp(1) on $\mathbb{R}$ (this part is only for you to think about.)

Proof. 1)

$$\begin{aligned}\exp(n) &= \exp(1 + 1 + \cdots (n \text{ times}) + \cdots + 1) &(3)\\ &= \exp(1)\exp(1)\cdots (n \text{ times}) \cdots \exp(1) &(4)\\ &= (\exp(1))^n &(5)\end{aligned}$$

2) Follow a calculation while replacing the 1's above with $\frac{a}{b}$,

$$\exp(\frac{a}{b} + \frac{a}{b} + \cdots (b \text{ times}) \cdots + \frac{a}{b}) = (\exp(\frac{a}{b}))^b \tag{6}$$

$$\exp(a) = (\exp(\frac{a}{b}))^b \tag{7}$$

$$(\exp(1))^a = (\exp(\frac{a}{b}))^b \tag{8}$$

$$\sqrt[b]{(\exp(1))^a} = \sqrt[b]{(\exp(\frac{a}{b}))^b} \tag{9}$$

$$(\exp(1))^{\frac{a}{b}} = \exp(\frac{a}{b}) \tag{10}$$

$\square$