Math 2423
Spring 2016
Exam 3
4/22/16
Time Limit: 50 Minutes

Name (Print): Answer Key

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- Partial credits apply. Part of an important step in mathematics is to write what you think in a coherent way, even when it does not yield fruitful results.
- A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- You may use the following identities and formulas:

$$\sin^2 x + \cos^2 x = 1 \tag{1}$$
$$\tan^2 x + 1 = \sec^2 x \tag{2}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2} \tag{3}$$
$$\cos^2 x = \frac{1 - \sin 2x}{2} \tag{4}$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{5}$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{6}$$
$$\int \tan x \, dx = -\ln|\cos x| + C \tag{7}$$
$$\int \cot x \, dx = \ln|\sin x| + C \tag{8}$$
$$\int \sec x \, dx = \ln|\sec x + \tan x| + C \tag{9}$$
$$\int \csc x \, dx = \ln|\csc x - \cot x| + C \tag{10}$$

Problem	Points	Score
1	12	
2	10	
3	15	
4	23	
5	20	
6	20	
Total:	100	

1. (12 points) Use the theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \tag{11}$$

Evaluate the derivative $\frac{\mathrm{d}}{\mathrm{d}x} \cos^{-1}(x)$ (Giving only the answer by memorization without any justification will result in little credit.)

$$\frac{d}{dx} \cos^{-1}(x) = \frac{1}{\cos'(\cos^{-1}(x))}$$

$$= \frac{1}{-\sin(\cos^{-1}(x))}$$

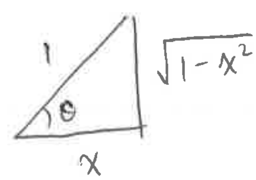
$$= \frac{1}{-\sqrt{1-x^2}}$$

either: $\sin^2\theta + \cos^2\theta = 1$

$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-x^2}$

(where $x = \cos\theta$)

or:



2. (10 points) Use L'Hospital's Rule to evaluate

$$\lim_{x\to\infty}(1+\frac{1}{x})^x \tag{12}$$

$y = (1+\frac{1}{x})^x$ (2')

$\ln y = x \ln(1+\frac{1}{x})$ (2')

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln(1+\frac{1}{x})$

("$\infty \cdot 0$")

$= \lim_{x\to\infty} \dfrac{\ln(1+\frac{1}{x})}{\frac{1}{x}}$ ("$\frac{0}{0}$") (2')

$= \lim_{x\to\infty} \dfrac{\frac{1}{1+\frac{1}{x}} \cdot (-\frac{1}{x^2})}{-\frac{1}{x^2}}$ (L'Hospital's Rule) (4')

$= \lim_{x\to\infty} \dfrac{1}{1+\frac{1}{x}} = 1$

$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y}$

$= e^{\lim_{x\to\infty} \ln y}$

$= e^1 = e$

3. (15 points) Use integration by parts, evaluate the integral

$$\int e^x \sin x \, dx \tag{13}$$

$u = e^x$ $\quad$ $v = -\cos x$

$du = e^x\,dx$ $\quad$ $dv = \sin x\,dx$ $\quad$ (+4)

(~~The~~ Using $u = \sin x$ and $dv = e^x dx$ correctly will receive full credit.)

$$\int e^x \sin x dx = e^x(-\cos x) - \int (-\cos x) e^x dx$$

$$= -e^x \cos x + \int \cos x e^x dx \quad (+4)$$

$u = e^x$ $\quad$ $v = \sin x$

$du = e^x dx$ $\quad$ $dv = \cos x dx$

$$\int \cos x e^x = e^x \sin x - \int e^x \sin x dx \quad (+4)$$

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

$$2\int e^x \sin x dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x dx = \frac{-e^x \cos x + e^x \sin x}{2} + C \quad (+3)$$

4. (a) (13 points) Evaluate the integral

$$\int \sin^2 x \cos x \, dx \tag{14}$$

$u = \sin x$

$du = \cos x \, dx$ +4

$\int \sin^2 x \cos x \, dx$

$= \int u^2 \, du$ +4

$= \frac{u^3}{3} + C$ +4

$= \frac{\sin^3 x}{3} + C$ +1

(Using integration by parts correctly, or trig identities correctly, will receive full credit.)

5. Find the partial fraction decomposition of

$$\frac{5x^2+13x+10}{(x+1)^2(x+2)} \tag{16}$$

(a) (5 points) Write out the decomposition, according to the general formula, with undetermined coefficients.

$$\frac{5x^2+13x+10}{(x+1)^2(x+2)} = \frac{\lambda_1}{x+1} + \frac{\lambda_2}{(x+1)^2} + \frac{\lambda_3}{x+2}$$

(b) (15 points) Determine the coefficients and write down the final decomposition.

Multiply both sides by $(x+1)^2(x+2)$,

$$5x^2+13x+10 = \lambda_1(x+1)(x+2) + \lambda_2(x+2) + \lambda_3(x+1)^2 \quad (3')$$

$$= \lambda_1(x^2+3x+2) + \lambda_2(x+2) + \lambda_3(x^2+2x+1) \quad (3')$$

$$= \lambda_1 x^2 + 3\lambda_1 x + 2\lambda_1 \quad (1')$$

$$+ \lambda_2 x + 2\lambda_2 \quad (1')$$

$$+ \lambda_3 x^2 + 2\lambda_3 x + \lambda_3 \quad (1').$$

(1') ① $5 = \lambda_1 + \lambda_3$

(1') ② $13 = 3\lambda_1 + \lambda_2 + 2\lambda_3$

(1') ③ $10 = 2\lambda_1 + 2\lambda_2 + \lambda_3$

① - ② + ③: $2 = \lambda_2$

$\lambda_2 = 2$ (1')

plug in ②: $13 = 3\lambda_1 + \overset{2}{\lambda_2} + 2\lambda_3$

④ $11 = 3\lambda_1 + 2\lambda_3$

④ - 2 × ①:

$1 = \lambda_1$ (1')

plug in ①: $5 = 1 + \lambda_3$

$4 = \lambda_3$ (1')

$$\frac{5x^2+13x+10}{(x+1)^2(x+2)} = \frac{1}{x+1} + \frac{2}{(x+1)^2} + \frac{4}{(x+2)}$$

6. Use trigonometric substitution, find the integral

$$\int \frac{x}{x^2+2x+2}\,dx \tag{17}$$

(a) (5 points) Complete the square so that $x^2+2x+2=(x+a)^2+b^2$ for some real numbers a and b.

$$x^2+2x+2 = (x+1)^2+1$$

(b) (5 points) Make a trigonometric substitution based on your answer in part a).

$$\tan\theta = x+1$$

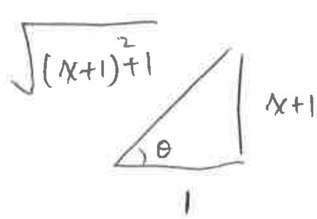
~~$\tan\theta =$~~

$$\sec^2\theta\, d\theta = dx$$

(c) (10 points) Use part a) and b) to finish the integral. Your final answer should be in terms of x.

$$\int \frac{\tan\theta - 1}{\tan^2\theta+1}\sec^2\theta\, d\theta$$

$$= \int \frac{\tan\theta-1}{\cancel{\sec^2\theta}}\cancel{\sec^2\theta}\, d\theta$$



$$= \int (\tan\theta - 1)\, d\theta$$

$$= -\ln|\cos\theta| - \theta + C$$

$$= -\ln\left|\sqrt{(x+1)^2+1}\right| - \arctan(x+1) + C$$