6/6. Calc III

Def A sequence (a_i) is said to have limit L if

For any given ε (positive real number)

~~there is a~~ we can find a tail of (a_i) in the

ε- nbhd of L

e.g. (a_i) $a_i = \frac{1}{10^i}$ $(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \cdots)$ has limit 0

$\varepsilon = 4$

$(a_1, a_2 \cdots)$ is in the 4- nbhd of 0 $(-4, 4)$

$\varepsilon = 10^{-9}$ 10^{-9}- nbhd of 0: $(-10^{-9}, 10^{-9})$

$a_9 = \frac{1}{10^9}$

$a_{10} = \frac{1}{10^{10}}$

$(a_{10}, a_{11}, \cdots)$ is in the 10^{-9}- nbhd of 0

Given ε. the ε- nbhd of 0 is $(-\varepsilon, \varepsilon)$

$a_i < \varepsilon$

$\frac{1}{10^i} < \varepsilon$

$1 < 10^i \varepsilon$

$\ln(1) < \ln(10^i) + \ln(\varepsilon)$

$\ln(1) < i \ln 10 + \ln \varepsilon$

$\frac{0 - \ln \varepsilon}{\ln 10} < i$

let i be an integer greater than $\frac{-\ln \varepsilon}{\ln 10}$

then $a_i < \varepsilon$ ~~then~~ $(a_i, a_{i+1}, \cdots)$ is in the ε- nbhd of 0

~~Th.~~ Def If (a_i) has limit L

denote $\lim_{i\to\infty} a_i = L$

Th if $a_i = f(i)$

then $\lim_{i\to\infty} a_i = \lim_{x\to\infty} f(x)$

e.g. $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$\infty - \infty$

$\frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = 0$

~~Th~~

Def if (a_i) has a limit, then (a_i) is said to be convergent.

~~eg~~ Th. (a_i) convergent seq. ~~with limit~~ $\lim a_i = a$

(b_i) - - - $\lim b_i = b$

① (c_i) $c_i = a_i + b_i$ $\lim_{i\to\infty} c_i = \lim a_i + \lim b_i$

$= a+b$

② (c_i) $c_i = \frac{a_i}{b_i}$ $\lim c_i = \frac{\lim a_i}{\lim b_i} = \frac{a}{b}$

③ $c_i = s a_i$ $\lim c_i = s \lim a_i$

④ $c_i = a_i^s$ $\lim c_i = (\lim a_i)^s$

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e.g. $a_{i+1} = \frac{1}{1+a_i}$

$(1, \frac{1}{2}, \cdots)$ $\qquad \lim a_i = \frac{-1+\sqrt{5}}{2}$

$(a_i) = (a_1, a_2, \cdots)$

$(b_i) = (a_2, a_3, a_4, \cdots)$

$= (\frac{1}{1+a_1}, \frac{1}{1+a_2}, \cdots)$

$b_i = \frac{1}{1+a_i}$ $\qquad \underset{\lim b_i}{\overset{\lim a_i}{\shortparallel}} = \frac{1}{\lim(1+a_i)} = \frac{1}{1+\lim a_i}$

$\lim \frac{c_i}{d_i} = \frac{\lim c_i}{\lim d_i}$ $\qquad x = \frac{1}{1+x}$

$c_i = 1 \quad (1, 1, \cdots)$ $\qquad x^2 + x = 1$

$d_i = 1 + a_i$ $\qquad x^2 + x - 1 = 0$

$x = \frac{-1 \pm \sqrt{5}}{2}$

Series

$S = 1 + 2 + 4 + 8 + \cdots$

$2S = 2 + 4 + 8 + \cdots$

$-S = 1$

$S = -1$

$a_1 + a_2 + \cdots$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

the sum is defined to be

$$\lim_{i\to\infty} S_i$$

denoted as $\sum_{i=1}^{\infty} a_i$

e.g. $(1, 2, 4, \ldots$

$$a_i = 2^{i-1}$$

$$a_1 = 2^{1-1} = 1$$

$$S_i = a_1 + \ldots + \overset{a_{i-1}}{} + a_i$$

$$= 1 + 2 + \ldots + 2^{i-2} + 2^{i-1}$$

$$2S_i = 2 + 4 + \ldots + 2^{i-1} + 2^i$$

$$2^{i-2} \cdot 2^1$$

$$2^{i-1}$$

$$-S_i = 1 - 2^i$$

$$S_i = 2^i - 1$$

$$\lim_{i\to\infty} S_i = \lim_{i\to\infty} 2^i - 1 = \infty$$

e.g. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$$\sum_{i=1}^{\infty} a_i = 2$$

Th $(a_1, ra_1, r^2a_1, \ldots)$

$\sum_{i=1}^{\infty} a_i$ is a geometric series

① $|r| \geq 1$ $\quad \sum_{i=1}^{\infty} a_i$ diverges to ∞

② $|r| < 1$ $\quad \sum_{i=1}^{\infty} a_i = \frac{a_1}{1-r}$

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$\Sigma \frac{1}{n(n+1)}$

$\frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

$\frac{1}{3}$ $\quad \frac{1}{3\cdot 4} \cdot 2\cdot 3$

$\frac{2}{4} = \frac{1}{2}$

$x(x+1) \cdot \int \frac{1}{x(x+1)} dx$

$x\cdot(x+1)\left(\frac{A}{x} + \frac{B}{x+1}\right)$

$A = 1 \quad B = -1$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\sum_{i=1}^{\infty} a_i = \lim_{i\to\infty} S_i = \lim_{i\to\infty} (a_1 + \cdots + a_i)$$

$$= \lim_{i\to\infty}\left(\left(1-\cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \tfrac{1}{3}\right) + \cdots + \left(\cancel{\tfrac{1}{i}} - \tfrac{1}{i+1}\right)\right)$$

$$\left(\frac{1}{i-1} - \cancel{\frac{1}{i}}\right)$$

$$= \lim_{i\to\infty}\left(1 - \frac{1}{i+1}\right)$$

$$= 1$$

$1 + 2 + 4 + 8 + \cdots$

Test for Divergence

Th. if $\sum_{i=1}^{\infty} a_i = \infty$ is convergent

then $\lim_{i\to\infty} a_i = 0$

Proof. $(S_i) = (S_1, S_2, \cdots$

$(S_{i+1}) = (S_2, S_3, S_4, \cdots)$

$\lim S_i = \lim S_{i+1} = \lim S_{i-1}$

$a_i = (a_1 + \cdots + a_i) - (a_1 + \cdots + a_{i-1})$

$= S_i - S_{i-1}$

$\lim a_i = \lim S_i - \lim S_{i-1} = 0$

$\Sigma \frac{1}{n}$

Corollary if ~~$\Sigma a_i \neq 0$~~ $\lim_{i\to\infty} a_i \neq 0$ or $\lim_{i\to\infty} a_i$ does not exist
then $\sum_{i=1}^{\infty} a_i$ diverges

If Alice is Oklahoman
then she is American
If Alice is not American
then she is not Oklahoman

e.g. $\Sigma \frac{n^2+1}{4n^2+7n-3}$ is divergent

$$\lim_{n\to\infty} \frac{n^2+1}{4n^2+7n-3} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

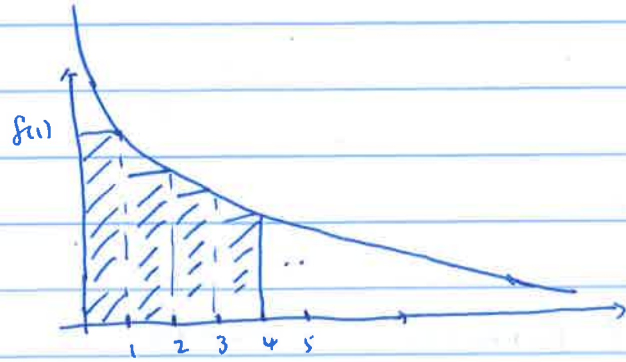
$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{4+\frac{7n}{n^2}-\frac{3}{n^2}} = \frac{1}{4} \neq 0$$

FACT 1 if $a_i > 0$

① $\sum_{i=1}^{\infty} a_i$ is convergent $\sum_{i=1}^{\infty} a_i < \infty$

② $\sum_{i=1}^{\infty} a_i$ diverges to ∞ $\sum_{i=1}^{\infty} a_i = \infty$

③ $\sum_{i=1}^{\infty} a_i$ does not exist



$i = f(i)$

$a_1 + a_2 + \cdots$

$= f(1) + f(2) + \cdots$

$= f(1)\cdot 1 + f(2)\cdot 1 + \cdots$

Integral Test

$\sum_{i=1}^{\infty} a_i \qquad a_i = f(i)$

① $f(x) > 0$

② f: decreasing

then ① if $\int_1^x f(x)\,dx < \infty$ converges. $\sum_{i=1}^{\infty} a_i$ converges

② if $\int_1^x f(x)\,dx = \infty$ $\sum_{i=1}^{\infty} a_i$ diverges

① $\int_1^{\infty} f(x)\,dx < \infty$

$\Downarrow$

$\sum_{i=1}^{\infty} a_i \le \int_0^{\infty} f(x)\,dx = \int_0^1 f(x)\,dx + \underline{\int_1^{\infty} f(x)\,dx}$ fin.

$\sum_{i=2}^{\infty} a_i \le \int_1^{\infty} f(x)\,dx$

$\sum_{i=2}^{\infty} a_i < \infty \Rightarrow \sum_{i=1}^{\infty} a_i < \infty$

$\Downarrow$

$\sum_{i=1}^{\infty} a_i < \infty$

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Integral Test

$\sum_{i=1}^{\infty} a_i \qquad f(i) = a_i$

① $f(x) > 0$ for all $x > 0$

② $f(x)$ decreasing

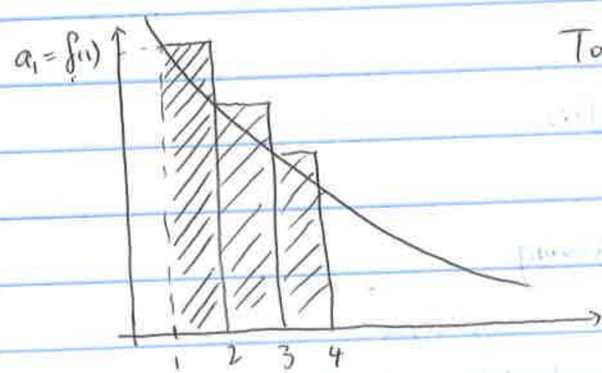
Then 1) if $\int_1^{\infty} f(x)dx < \infty$, $\sum_{i=1}^{\infty} a_i < \infty$

2) if $\int_1^{\infty} f(x)dx = \infty$, $\sum_{i=1}^{\infty} a_i = \infty$

Proof of 2). $\sum_{i=2}^{\infty} a_i < \int_1^{\infty} f(x)\,dx$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \infty$



Total area $= a_1 + a_2 + \cdots$

$\geq \int_1^{\infty} f(x)\,dx$

$\infty \qquad \infty$

$\sum_{i=1}^{\infty} a_i = \infty$

e.g. Harmonic Series

$\sum \frac{1}{n}$

$\int_1^{\infty} \frac{1}{x} dx$

$\int \frac{1}{x} dx = \ln x$

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln x \Big|_1^t$$

$$= \lim_{t\to\infty} (\ln t - \ln 1)$$

$$= \infty$$

e.g. $\sum_{n=1}^\infty \frac{1}{n^{1.1}}$

$$x^{-1.1}$$

$$\int_1^\infty \frac{1}{x^{1.1}}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^{1.1}}dx$$

$$= \lim_{t\to\infty} \frac{x^{-1.1+1}}{-1.1+1}\Big|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{t^{-0.1}}{-0.1} - \frac{1^{-0.1}}{-0.1} \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{t^{0.1}(-0.1)} - \frac{1^{-0.1}}{-0.1} \right)$$

$$= 10$$

$$\sum_{n=1}^\infty \frac{1}{n^{0.9}}$$

Th (p-series Test)

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

① $p \le 1$ $\sum \frac{1}{n^p} = \infty$

② $p > 1$ $\sum \frac{1}{n^p} < \infty$

e.g. $\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$

$$\int \frac{1}{x^2(x+1)}\,dx$$

$$x^2(x+1) \quad \frac{1}{x^2(x+1)} = \left(\frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1}\right) x^2(x+1)$$

$$1 = A(x+1) + Bx(x+1) + Cx^2$$

$$1 = Ax + \underline{A} + \underline{\underline{Bx^2}} + Bx + \underline{\underline{Cx^2}}$$

$A = 1$

$B + C = 0$

$A + B = 0 \qquad 1 + B = 0 \qquad B = -1$

$-1 + C = 0 \qquad C = 1$

$$\frac{1}{x^2(x+1)} = \frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}$$

$$\int \frac{1}{x^2(x+1)}\,dx = \frac{x^{-1}}{-1} - \ln|x| + \ln|x+1|$$

$$\int_1^{\infty} \frac{1}{x^2(x+1)}\,dx = \lim_{t\to\infty} \left(\frac{t^{-1}}{-1} - \ln|t| + \ln|t+1|\right.$$

$$\left. - \left(\frac{1^{-1}}{-1} - \ln 1 + \ln 2\right)\right)$$

$$= \lim_{t\to\infty} \frac{1}{t(-1)} + \lim_{t\to\infty}(-\ln|t| + \ln|t+1|)$$

$$- *$$

$$= 0 + \lim_{t\to\infty} \ln\left|\frac{t+1}{t}\right| - *$$

$$= \ln \lim_{t\to\infty} \left|\frac{t+1}{t}\right| - * = 0 - * < \infty$$

$$\sum \frac{1}{n^2(n+1)} < \infty$$

$$a_i - \frac{1+\sqrt{5}}{2} a_{i-1} = \frac{1-\sqrt{5}}{2}\left(a_{i-1} - \frac{1+\sqrt{5}}{2} a_{i-2}\right) = \left(\frac{1-\sqrt{5}}{2}\right)^2\left(a_{i-2} - \frac{1+\sqrt{5}}{2} a_{i-3}\right)$$

$i \rightsquigarrow i-1$

$$a_{i-1} - \frac{1+\sqrt{5}}{2} a_{i-2} = \frac{1-\sqrt{5}}{2}\left(a_{i-2} - \frac{1+\sqrt{5}}{2} a_{i-3}\right)$$

$i \rightsquigarrow i-1$

$$a_{i-2} - \frac{1+\sqrt{5}}{2} a_{i-3} = \frac{1-\sqrt{5}}{2}\left(a_{i-3} - \frac{1+\sqrt{5}}{2} a_{i-4}\right)$$

$$= \left(\frac{1-\sqrt{5}}{2}\right)^3\left(a_{i-3} - \frac{1+\sqrt{5}}{2} a_{i-4}\right)$$

$$\vdots$$

$$= \left(\frac{1-\sqrt{5}}{2}\right)^{i-2}\left(a_2 - \frac{1+\sqrt{5}}{2} a_1\right)$$

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FACT $a_i > 0$

① $\sum a_i = \infty$

② $\sum a_i$ converges $(\sum a_i < \infty)$

FACT $\sum_{i=1}^{\infty} a_i < \infty$ then $\sum_{i=k}^{\infty} a_i < \infty$

FACT $\sum a_i < \infty$ $\sum c a_i = c \sum a_i < \infty$

The Comparison Test

$a_i > 0$ $b_i > 0$

"bad series" "good series"

i.e. we don't know how to deal with it

① $a_i \le b_i$ for all i's

$\sum b_i < \infty \Rightarrow \sum a_i < \infty$

② $a_i \geq b_i$ for all i's

$\sum b_i = \infty$ $\sum a_i = \infty$

e.g. $\sum \frac{n+1}{n^2 - 2n - 4} = \sum a_n$ diverges

want $\sum b_n$. $a_n \geq b_n$

$\sum b_n = \infty$

$b_n = \frac{n+1}{n^2}$

$\frac{n+1}{n^2 - 2n - 4} \geq \frac{n+1}{n^2}$

$\sum \frac{n+1}{n^2}$

$$b_n = \frac{1}{n+1}$$

?

$$(n+1)(n^2-2n-4)\frac{n+1}{n^2-2n-4} \geq \frac{1}{n+1}(n+1)(n^2-2n-4)$$

$$(n+1)^2 \geq n^2-2n-4$$

$$n^2+2n+1 \geq n^2-2n-4 \quad \checkmark$$

$$\sum_{n=1}^{\infty} \frac{1}{n+1} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

$$= \sum_{n=2}^{\infty} \frac{1}{n} \qquad \text{p-series with } p=1$$

$$\infty$$

Harmonic Series

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Th. Limit Comparison Test

$a_i > 0 \quad b_i > 0$

$$\lim_{i\to\infty} \frac{a_i}{b_i} = c \neq 0 \quad \left((c_i) \quad c_i = \frac{a_i}{b_i}. \quad \lim_{i\to\infty} c_i = c \right)$$

Then $\sum b_i = \infty \Rightarrow \sum a_i = \infty$ (**)

$\sum b_i < \infty \Rightarrow \sum a_i < \infty$ (*)

<u>Proof of (*)</u> $\lim c_i = c$ For any ε $(\varepsilon > 0)$

A tail of (c_i) is in an ε-nbhd of c

$(c_k, c_{k+1}, \ldots \quad)$ is in $(c-\varepsilon, c+\varepsilon)$

i.e. $\quad c-\varepsilon < c_k < c+\varepsilon$

$$c-\varepsilon < c_{k+1} < c+\varepsilon$$

$$\vdots$$

$$c-\varepsilon < c_i < c+\varepsilon \qquad i=k, k+1, \dots$$

$$\frac{a_i}{b_i} < c+\varepsilon$$

$$a_i < (c+\varepsilon) b_i \qquad i=k, k+1, \dots$$

$$\sum_{i=k}^{\infty} (c+\varepsilon) b_i = (c+\varepsilon) \sum_{i=k}^{\infty} b_i < \infty$$

$$\sum_{i=k}^{\infty} a_i < \infty$$

therefore $\quad \sum_{i=1}^{\infty} a_i < \infty$

e.g. $\quad \sum \frac{n+1}{n^2-2n-4} = \sum a_n$

$$b_n = \frac{n}{n^2} = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^2-2n-4} \cdot n$$

$$= \lim_{n\to\infty} \frac{n^2+n}{n^2-2n-4} = 1$$

$$\sum b_n = \infty$$

~~thefo~~ therefore $\sum a_n = \infty$

e.g. $\sum \frac{2^n+3^n}{4^n+n} = \sum a_n$

$$b_n = \frac{3^n}{4^n}$$

$$\lim \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n+3^n}{4^n+n} \cdot \frac{4^n}{3^n}$$

$$= \lim_{n\to\infty} \frac{2^n+3^n}{4^n+n} \cdot \frac{\cancel{4^n}}{3^n} \cdot \frac{\frac{1}{\cancel{4^n}}}{\frac{1}{4^n}}$$

$$= \lim_{n\to\infty} \frac{2^n+3^n}{\left(1+\frac{n}{4^n}\right)\cdot 3^n}$$

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$\sum \frac{2^n+3^n}{4^n+n}$ $\quad b_n = \frac{3^n}{4^n}$

$$\frac{a_n}{b_n} = \frac{2^n+3^n}{4^n+n} \cdot \frac{4^n}{3^n}$$

$$= \frac{2^n+3^n}{4^n+n} \cdot \frac{\cancel{4^n}}{3^n} \cdot \frac{\frac{1}{\cancel{4^n}}}{\frac{1}{4^n}}$$

$$= \frac{2^n+3^n}{\left(1+\frac{n}{4^n}\right)} \cdot \frac{1}{3^n}$$

$$= \frac{2^n+3^n}{1+\frac{n}{4^n}} \cdot \frac{1}{\cancel{3^n}} \cdot \frac{\frac{1}{3^n}}{\frac{1}{\cancel{3^n}}}$$

$$= \frac{(2^n+3^n) \cdot \frac{1}{3^n}}{1+\frac{n}{4^n}}$$

$$= \frac{2^n \cdot \frac{1}{3^n} + 3^n \cdot \frac{1}{3^n}}{1+\frac{n}{4^n}}$$

$$= \frac{\left(\frac{2^n}{3^n}\right) + 1}{1+\frac{n}{4^n}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{\frac{2^n}{3^n}+1}{1+\frac{n}{4^n}} = \frac{0+1}{1+0} = 1$$

$$\lim \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

$$\lim \frac{n}{4^n} = 0$$

Geometric Series

$\sum \left(\frac{3}{4}\right)^n < \infty$

$\left(\frac{3}{4}, \frac{3^2}{4^2}, \frac{3^3}{4^3}, \cdots \right)$

$r = \frac{3}{4}$

$(a_1, ra_1, r^2a_1, \cdots$

$(7, 10, 13, \cdots)$

$\frac{10}{7} \quad \frac{13}{10}$

e.g. $\sum \frac{\sin \frac{1}{n}}{\sqrt{n^2+4n}}$

$$b_n = \frac{\frac{1}{n}}{\sqrt{n^2}} = \frac{\frac{1}{n}}{n} = \frac{1}{n^2}$$

$$\sin \frac{1}{n} \overset{?}{\approx} \frac{1}{n}$$

$n \to \infty \quad \frac{1}{n} \to 0$

$\sin x \overset{?}{=} \qquad x \to 0$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

$$\sin x \approx x$$

$$\frac{a_n}{b_n} = \frac{\sin \frac{1}{n}}{\sqrt{n^2+4n}} \cdot n^2 = \frac{\sin \frac{1}{n}}{\sqrt{n^2+4n}} \cdot \frac{\sqrt{n^2}}{\frac{1}{n}}$$

$$= \frac{\sin \frac{1}{n}}{\frac{1}{n}} \cdot \frac{\sqrt{n^2}}{\sqrt{n^2+4n}}$$

$$= \frac{\sin \frac{1}{n}}{\frac{1}{n}} \cdot \frac{n}{\sqrt{n^2+4n}}$$

$$= \frac{\sin \frac{1}{n}}{\frac{1}{n}} \frac{\cancel{n} \cdot \frac{1}{\cancel{n}}}{\sqrt{n^2+4n} \cdot \sqrt{\frac{1}{n^2}}}$$

$$= \frac{\sin\frac{1}{n}}{\frac{1}{n}} \cdot \frac{1}{\sqrt{(n^2+4n)\frac{1}{n^2}}}$$

$$= \frac{\sin\frac{1}{n}}{\frac{1}{n}} \cdot \frac{1}{\sqrt{1+\frac{4}{n}}}$$

$$\lim \frac{\sin\frac{1}{n}}{\frac{1}{n}} \cdot \frac{1}{\sqrt{1+\frac{4}{n}}} = 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

$(1, -1, 1, -1, \dots) = \Sigma (-1)^{n+1}$

$(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots) = \Sigma (-1)^{n+1} \cdot \frac{1}{n}$ $<$ $\Sigma \frac{1}{n} = \infty$

is convergent.

Def An alternating series is a series of the form

$$\Sigma (-1)^{n+1} a_n$$

$a_n > 0$

$0 < S_{2n} = (1-\frac{1}{2}) + (\frac{1}{3}-\frac{1}{4}) + \cdots + (\frac{1}{2n-1} - \frac{1}{2n})$

$\wedge$

$S_{2n+2} = (1 \quad) + \cdots + (\quad) + (\frac{1}{2n+1} - \frac{1}{2n+2})$

Σa_n $\quad \Sigma (-1)^{n+1} a_n \quad a_n > 0$

if ① a_n : weakly decreasing
(i.e. $a_1 \geq a_2 \geq a_3 \geq \cdots$)

② $\lim_{n \to \infty} a_n = 0$

then $\Sigma (-1)^{n+1} a_n$ converges

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$\Sigma(-1)^{n+1} a_n \quad a_n > 0$

① $a_1 \geq a_2 \geq a_3 \geq \cdots$

(a_n's weakly decreasing)

② $\lim_{n\to\infty} a_n = 0$

then $\Sigma(-1)^{n+1} a_n$ is convergent

e.g. $\Sigma(-1)^{n+1}\left(n^{\frac{1}{n}} - 1\right)$

$(0, -(\sqrt{2}-1), (\sqrt[3]{3}-1), \cdots)$

Any series with alternating signs (starting with a positive term) can be written in the form $\Sigma(-1)^{n+1} a_n$

$\sum_{n=3}^{\infty} (-1)^{n+1}\left(n^{\frac{1}{n}} - 1\right)$

for a_n: weakly decreasing

we want

$f(n) = \left(n^{\frac{1}{n}} - 1\right)$

$f(x) = x^{\frac{1}{x}} - 1$

to be a decreasing function

$f'(x) = \left(x^{\frac{1}{x}}\right)'$

$y = x^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln x$

$\frac{1}{y}\cdot y' = \frac{1}{x}\cdot\frac{1}{x} + \left(\frac{1}{x}\right)'\ln x$

$\frac{1}{y}\cdot y' = \frac{1}{x^2} - \frac{1}{x^2}\ln x$

$y' = x^{\frac{1}{x}} \frac{1}{x^2}(1 - \ln x)$

we need $\ln x < 1$

$$\underset{0}{\overset{}{\ln 1}} \quad \ln 2 \quad \underset{\ln 2.712}{\overset{1}{\ln e}} \quad \ln 3 > 1$$

② $\cancel{\lim_{x \to \infty} f(x)} =$

$$\lim_{n \to \infty}\left(n^{\frac{1}{n}} - 1\right) = \left(\lim_{n \to \infty} n^{\frac{1}{n}}\right) - 1$$

$$\cancel{\lim} \quad y = n^{\frac{1}{n}}$$

$$\ln y = \frac{1}{n} \ln n$$

$$y = e^{\ln y} = e^{\frac{1}{n}\ln n}$$

$$\lim y = \cancel{\lim y} = \lim e^{\ln y} = \lim e^{\frac{1}{n}\ln n}$$

$$e^{\lim \ln y} = e^{\lim \frac{1}{n}\ln n}$$

$$= e^{\lim \frac{\ln n}{n}}$$

$$= e^{\lim \frac{\frac{1}{n}}{1}} = e^0 = 1$$

$$\left(\lim n^{\frac{1}{n}}\right) - 1 = 1 - 1 = 0$$

e.g. $(-1)^{n+1} \cdot \dfrac{2n+3}{6n+7}$ a_n

$$f(x) = \frac{2x+3}{6x+7}$$

$$f'(x) = \frac{2(6x+7)-(2x+3)\cdot 6}{(6x+7)^2} = \frac{12x+14-(12x+18)}{(6x+7)^2}$$

$$= \frac{-4}{(6x+7)^2} < 0$$

$$\lim f(x) = \frac{1}{3}$$

Corollary. (Test for Divergence)

if Σb_n if $\lim b_n \neq 0$ or $\lim b_n$ does not exist then Σb_n is divergent

Integral Test — p-series

~~Ratio~~ Comparison / — geometric series

limit Comparison test

$$b_n = (-1)^{n+1} \frac{2n+3}{6n+7}$$

HW #10 $\lim b_n$ does not exist

The Root Test

$$\sum \left(\frac{3}{4}\right)^n$$

$$\sqrt[n]{a_n} = (a_n)^{\frac{1}{n}} = \frac{3}{4}$$

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = r \quad (r \text{ is a constant})$$

then

"almost" $\left(\frac{3}{4}\right)^n$

① $r<1$ $\quad \sum a_n$ is convergent $\quad \sum \frac{1}{n^2}$

② $r>1$ $\quad \sum a_n$ is divergent

$$\sum \frac{1}{n}.$$

$$\lim \left(\frac{1}{n}\right)^{\frac{1}{n}} = \lim \frac{1}{n^{\frac{1}{n}}} = \frac{1}{\lim n^{\frac{1}{n}}} = 1$$

<u>Proof of ①</u> $\quad r = \frac{3}{4}$. $\quad \sum a_n$ is convergent

$$\lim a_n^{\frac{1}{n}} = \frac{3}{4}.$$

A tail $\left(a_k^{\frac{1}{k}}, a_{k+1}^{\frac{1}{k+1}}, \dots \right)$

is in the ε-nbhd $(0.75-\varepsilon,\ 0.75+\varepsilon)$.

$\varepsilon = 0.1 \quad (0.65,\ 0.85)$

$a_i^{\frac{1}{i}} < 0.85 \quad$ for $i = k, k+1, \dots$

$$a_i < 0.85^i$$

$$\sum_{i=k}^{\infty} a_i < \sum_{i=k}^{\infty} 0.85^i$$

$\sum_{i=k}^{\infty} a_i$ is convergent. $\quad$ therefore $\sum_{i=1}^{\infty} a_i$ is convergent

6/15 Thur

The Root Test $\quad \Sigma a_n$ with $a_n > 0$

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = r$$

① $r < 1 \quad \Sigma a_n$ is convergent

② $r > 1 \quad \Sigma a_n = \infty$

e.g. $\quad \Sigma a_n = \Sigma \dfrac{1}{(1+\frac{1}{n})^n}$

$\quad \Sigma b_n = \Sigma \dfrac{1}{(1+\frac{1}{n})^{n^2}}$

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{(1+\frac{1}{n})^n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\left((1+\frac{1}{n})^n\right)^{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$$\lim \frac{a_n}{b_n} = \frac{1}{(1+\frac{1}{n})^n}(1+\frac{1}{n})^{n^2} = (1+\frac{1}{n})^{n^2-n}$$

$$\lim_{n\to\infty} b_n^{\frac{1}{n}} = \frac{1}{\left((1+\frac{1}{n})^{n^2}\right)^{\frac{1}{n}}} = \frac{1}{(1+\frac{1}{n})^n}$$

$$\lim b_n^{\frac{1}{n}} = \frac{1}{\lim_{n\to\infty}(1+\frac{1}{n})^n} \quad \left(= \frac{1}{e}\right) < 1$$

$$\lim \left(1+\frac{1}{x}\right)^{x}$$

$$\ln\left(1+\frac{1}{x}\right)^{x} = x \ln\left(1+\frac{1}{x}\right)$$

$$\lim x \ln\left(1+\frac{1}{x}\right) = \lim \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \frac{0}{0}$$

$$\infty \cdot \ln(1)$$

$$\infty \cdot 0$$

$$= \lim \frac{\frac{1}{1+\frac{1}{x}} \cdot \left(1+\frac{1}{x}\right)'}{-\frac{1}{x^2}}$$

$$= \lim \frac{\frac{1}{1+\frac{1}{x}} \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} = 1$$

$$e^{\lim \ln\left(1+\frac{1}{x}\right)^{x}} = e^{1}$$

$$\lim \left(1+\frac{1}{x}\right)^{x} = e$$

$$\lim a_n = \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

$$c_n = \frac{1}{e} \qquad \lim \frac{a_n}{c_n} = \cancel{\lim} \frac{\lim a_n}{\lim c_n} = \frac{\frac{1}{e}}{\frac{1}{e}} = 1$$

$$\sum c_n = \frac{1}{e} + \frac{1}{e} + \cdots \qquad = \infty$$

$\sum a_n$ geometric

$$\frac{a_{n+1}}{a_n} = r$$

The Ratio Test

$\sum a_n$ is a series with $a_n > 0$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = r$$

① if $r < 1$ then $\sum a_n$ is convergent

② if $r > 1$ then $\sum a_n = \infty$

e.g. $\sum \frac{1}{n!}$ $\quad n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

$1! = 1$

$2! = 2 \cdot 1$

$\vdots$

$a_1 = \frac{1}{1}$

$a_2 = \frac{1}{2}$

$a_3 = \frac{1}{2 \cdot 3}$

$a_4 =$

$$\frac{a_3}{a_2} = \frac{1}{6} \cdot 2 = \frac{1}{3}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1) \cdot \cancel{n} \cdot \cancel{(n-1)} \cdots \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} \cdot \cancel{n} \cdot \cancel{(n-1)} \cdots \cancel{3} \cdot \cancel{2} \cdot \cancel{1}$$

$$= \frac{1}{n+1}$$

$$\lim_{n \to \infty} \frac{1}{n+1} = 0$$

$\sum a_n$ is convergent

e.g. $\sum \frac{n^n}{n! \cdot 4^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)! \cdot 4^{n+1}} \cdot \frac{n! \cdot 4^n}{n^n}$$

$$= \frac{n!}{(n+1)!} \cdot \frac{(n+1)^{n+1} \cdot \cancel{4^n}}{4^{\cancel{n+1}} \cdot n^n}$$

$$= \frac{1}{n+1} \cdot \frac{1}{4} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{1}{4} \cdot \frac{(n+1)^n}{n^n}$$

$$= \frac{1}{4} \left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{4} \left(1+\frac{1}{n}\right)^n$$

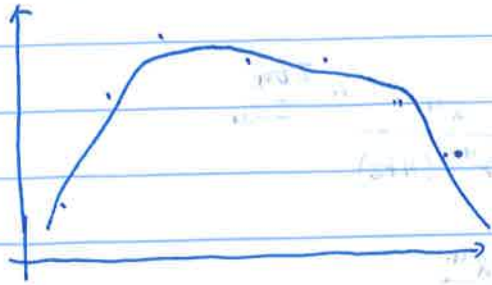
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{4} \left(1+\frac{1}{n}\right)^n$$

$$= \frac{1}{4} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \frac{1}{4} e < 1$$

Tue 6/20

Motivation

①



$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

$n \nearrow$ higher accuracy

$$y' = n c_n x^{n-1} + (n-1) c_{n-1} x^{n-2} + \cdots + c_1$$

$$y'' = n(n-1) c_n x^{n-2} + (n-1)(n-2) x^{n-3} + \cdots + c_2$$

② $y = f(x)$ -

$$y' y'' + y x + y^{(3)} e^x = 0$$

$$\left(n c_n x^{n-1} + (n-1) c_{n-1} x^{n-2} + \cdots \right)\left(n(n-1) c_n x^{n-2} + \cdots \right) + \cdots = 0$$

$$n^2 (n-1) c_n^2 x^{2n-3} + \qquad + \_\_ \; x^{2n-3}$$

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

(centered at $x=0$)

Def A power series is a function of the form

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$f(0) = c_0$$

In general,

A power series centered at $x=a$ is a function of the form

$$f(x) = d_0 + d_1 (x-a) + d_2 (x-a)^2 + \cdots$$

$$f(a) = d_0$$

e.g. Find all values of x for which the following series is convergent

$$\sum \frac{x^n}{4^{n+1}(n+2)} \quad \text{" } \sum a_n \quad \sum b_n$$

$$\sum b_n = \frac{x^n}{4^n}$$

$$\frac{x^n}{4^{n+1}(n+2)} \leq \frac{x^n}{4^n}$$

$$\left(\frac{x^n}{4^n}\right)^{\frac{1}{n}} = \frac{x}{4}$$

when $x < 4$ $\sum b_n$ is convergent

$\sum a_n$ is convergent

Ratio Test

$$\sum |a_n| = \sum \frac{|x^n|}{|4^{n+1}(n+2)|}$$

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{x^{n+1}}{4^{n+2}(n+3)} \cdot \frac{4^{n+1}(n+2)}{x^n} \right|$$

$$= \left| \frac{x}{4} \cdot \frac{n+2}{n+3} \right|$$

$$\lim \frac{|a_{n+1}|}{|a_n|} = \lim \left|\frac{x}{4}\right| \cdot \lim \left|\frac{n+2}{n+3}\right|$$

$$= \left|\frac{x}{4}\right|$$

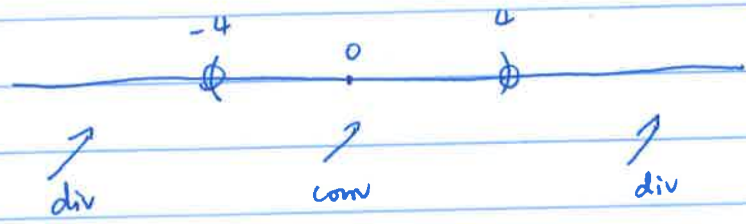
when $\left|\frac{x}{4}\right| < 1$ $\quad \Sigma |a_n|$ is convergent

$|x| < 4$ $\quad \Sigma a_n$ is convergent.

(by Absolute convergence Test)

when $\left|\frac{x}{4}\right| > 1$ $\quad \Sigma |a_n|$ is divergent

(Σa_n is also divergent)



$x = 4$ $$\Sigma a_n = \Sigma \frac{4^n}{4^{n+1}(n+2)} = \Sigma \frac{1}{4(n+2)}$$

$$= \frac{1}{4} \Sigma \frac{1}{n+2} = \infty$$

$x = -4$ $$\Sigma a_n = \Sigma \frac{(-4)^n}{4^{n+1}(n+2)}$$

$$= \Sigma \frac{((-1)\,4)^n}{4^{n+1}(n+2)}$$

$$= \Sigma \frac{(-1)^n \cdot 4^n}{4^{n+1}(n+2)} = \Sigma \frac{(-1)^n}{4(n+2)}$$ is convergent

All values of x for which the series Σa_n is convergent

form an interval $[-4, 4)$

we will call it

<u>Interval of Convergence</u>

Wed 6/21

Ratio Test

① $\lim \frac{|a_{n+1}|}{|a_n|} = r < 1$ $\quad \Sigma |a_n|$ is convergent

Σa_n is convergent

② $\lim \frac{|a_{n+1}|}{|a_n|} = r > 1$ $\quad \Sigma |a_n|$ is divergent

Σa_n is divergent

($\lim a_n = \infty$ or $\lim a_n$ does not exist)

③ $r = 1$

if $\lim \frac{|a_{n+1}|}{|a_n|}$

Ratio Test is inconclusive

e.g $\quad \Sigma \frac{(n!)^2 (x-1)^n}{(2n)!}$

Find the interval of convergence

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{((n+1)!)^2 (x-1)^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2 (x-1)^n} \right|$$

$$= \left| \frac{1}{(2n+2)(2n+1)} \cdot \frac{x-1}{1} \cdot \frac{(n+1)^2}{1} \right|$$

$$\frac{(2n)!}{(2(n+1))!} = \frac{(2n)!}{(2n+2)!} = \frac{2n(2n-1)\cdot}{(2n+2)(2n+1)\cdot 2n\cdot(2n-1)\cdot}$$

$$= \frac{1}{(2n+2)(2n+1)}$$

$$\frac{(n!)^2}{((n+1)!)^2} = \frac{n! \cdot n!}{(n+1)! \cdot (n+1)!}$$

$$\frac{((n+1)!)^2}{(n!)^2} = \frac{(n+1)! \cdot (n+1)!}{n! \cdot n!}$$

$$\lim \frac{|a_{n+1}|}{|a_n|} = \lim \frac{(n+1)^2}{(2n+2)(2n+1)} \cdot \lim |x-1|$$

$$= \lim \frac{n^2+2n+1}{4n^2+6n+2} |x-1| = \frac{1}{4}|x-1| < 1$$

(for Σa_n to be convergent)

$|x-1| < 4$

$\cancel{|x| < 5}$ $\qquad -4 < x-1 < 4$

$-3 < x < 5$

$x = 5$

$$\Sigma \frac{(n!)^2 \cdot 4^n}{(2n)!}$$

$$= \Sigma \frac{n! \cdot n! \; 4^n}{2n \cdot (2n-1)(2n-2)(2n-3)(2n-4) \cdots 1}$$

$$= \Sigma \frac{n! \; n! \; 4^n}{2 \cdot n \; (2n-1) \; 2(n-1) \; (2n-3) \; 2(n-2) \cdots 1}$$

$$= \Sigma \frac{n! \cdot 4^n}{(2n-1)(2n-3)(2n-5) \cdots 1 \cdot 2^n}$$

$$= \Sigma \frac{n! \cdot 2^n}{(2n-1)(2n-3) \cdots 1}$$

$$= \sum \frac{2\cdot n \cdot 2(n-1)\cdot 2(n-2)\cdots \quad 1}{(2n-1)\,(2n-3)\,(2n-5)\cdots \quad 1}$$

$$= \sum \frac{2n}{2n-1} \cdot \frac{2(n-1)}{2n-3} \cdot \frac{2(n-2)}{2n-5} \cdots 1$$

Σa_n $\quad a_n > 0 \quad a_n$: increasing

($\lim a_n \neq 0$ or $\lim a_n$ does not exist)

Σa_n is divergent

$a_n < a_{n+1}$

$$a_{n+1} = \frac{2(n+1)}{2(n+1)-1} \cdot \frac{2n}{2(n+1)-3} \cdot \frac{2(n-2)}{n-5} \cdots 1$$

$$= \frac{2(n+1)}{2(n+1)-1} \cdot a_n > a_n$$

Similarly, $\Sigma \frac{(n!)^2 x^n}{(2n)!}$ is divergent at $x=-3$

The interval of convergence is $(-3, 5)$

e.g. $\quad \Sigma \frac{x^n}{n!}$

$$x^n \cdot x$$

$$\lim \frac{|a_{n+1}|}{|a_n|} = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right|$$

$$\lim \frac{|a_{n+1}|}{|a_n|} = \lim \left| \frac{x}{n+1} \right| = 0$$

$\Sigma \frac{x^n}{n!}$ is convergent for all real numbers

e.g $\Sigma n^n x^n$

$\lim \frac{|a_{n+1}|}{|a_n|} = \infty.$

① $\Sigma C_n (x-a)^n$ is convergent on

$(a-r, a+r)$, $(a-r, a+r]$, $[a-r, a+r)$, $[a-r, a+r]$

② $\Sigma C_n (x-a)^n$ is convergent on all real numbers

③ $\Sigma C_n (x-a)^n$ is convergent only at $x=a$

Q $2+x+2x^2+x^3+2x^4+x^5+\cdots$

has an interval of convergence

Thur. 6/22

$\Sigma c_n x^n$

$$\lim \frac{|a_{n+1}|}{|a_n|} = \lim \left|\frac{c_{n+1}}{c_n}\right| \cdot |x| < 1$$

$|x| < r$

radius of convergence

$(1-x)(1+x) = 1-x^2$

$(1-x)(1+x+\cdots+x^n) = 1-x^{n+1}$

$|x|<1$

$(1-x)(1+x+x^2+\cdots \quad) = 1$

$$\boxed{1+x+x^2+\cdots = \frac{1}{1-x}}$$

$\overset{\shortparallel}{\sum_{n=0}^{\infty} x^n} \qquad \overset{\shortparallel}{(1-x)^{-1}}$

$(-1)(1-x)^{-2}(-1)$

$\shortparallel$

$$\boxed{\frac{1}{(1-x)^2} = 1+2x+3x^2+\cdots}$$

alternatively,

$$= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=\phi}^{\infty} nx^{n-1}$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)x^n}{nx^{n-1}}\right|$$

$$= \lim \left|\frac{n+1}{n}\right| \cdot |x| = |x| < 1$$

$$\cancel{c} + \boxed{x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\ln|1-x|}$$

$x=1$

$$c + 1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad \text{divergent}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{convergent}$$

$x=0$

$$c = -\ln|1-0| = 0$$

e.g. Find the power series representation for

$$y = \int x \arctan x \, dx$$

$$y' = x \arctan x$$

$$\frac{y'}{x} = \arctan x$$

$\arctan x \cdot 1$

$dv = 1 \quad u = \arctan x$

$v = x \quad du = \frac{1}{1+x^2}$

$$z = \arctan x$$

$$z' = \frac{1}{1+x^2}$$

$$\frac{1}{(1-x)^2} = \frac{1}{1-2x+x^2}$$

$$= \frac{1}{1-(-x^2)}$$

$$= 1 + (-x^2) + (-x^2)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\int z' dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$$

$$\arctan x \quad = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$0 = \arctan 0 = C$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$z = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$y' = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1}$$

$$\int y' dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+3}}{(2n+3)(2n+1)} + C$$

Power series Representation for $f(x)$

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots$$

$$f(0) = c_0$$

$$f'(x) = c_1 + 2c_2 x + 3c_2 x^2 + \dots$$

$$f'(0) = c_1$$

$$\frac{f''(0)}{2} = c_2 \qquad \frac{f^{(3)}(0)}{6} = c_3$$

$$\frac{f^{(3)}(0)}{3!}$$

$$f''(x) = 2c_2 + 3\cdot 2\cdot c_2 x + \cdots$$

$$f^{(3)}(x) = 3\cdot 2\cdot c_2 x + \cdots$$

$$\boxed{c_n = \frac{f^{(n)}(0)}{n!}}$$

6/22 Fri

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$ - T

Taylor Series expansion at $x=0$
(Maclaurin Series)

$$c_n = \frac{f^{(n)}(0)}{n!}$$

$$c_0 = \frac{f(0)}{0!}$$

$1! = 1$

$0!$

$2! \cdot 3 = 3!$

$0! \cdot 1 = 1!$

In general $$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Taylor Series expansion at $x=a$

e.g. $f(x) = 1+x$

$c_0 = f(0) = 1$

$f'(x) = 1$

$c_1 = \frac{f'(0)}{1} = 1$

$f''(0) = 0$

$c_2 = 0$

$\vdots$

$* + *x + *x^2 + \cdots$

$f(x) = 1+x$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

e.g. $f(x) = \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \dots$

$C_0 = f(0) = \sin 0 = 0$

$f'(x) = \cos x \qquad f'(0) = \cos 0 = 1 \qquad C_1 = 1$

$f''(x) = -\sin x \qquad f''(0) = -\sin 0 = 0 \qquad C_2 = 0$

$f'''(x) = -\cos x \qquad f^{(3)}(0) = -\cos 0 = -1 \qquad C_3 = \frac{-1}{3!}$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$\cos x$

n	0	1	2	3
$2n+1$	1	3	5	7
$(-1)^n$	+1	-	+	-

Hw. $\cos x$

e^x

$f'(x) = e^x \qquad f^{(n)}(0) = 1$

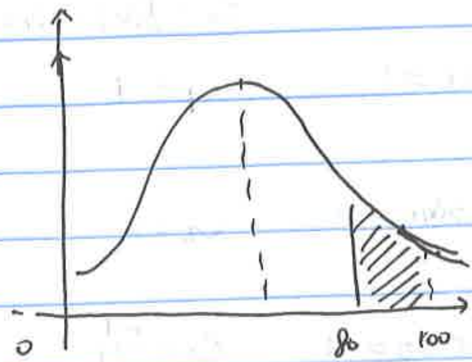
$f''(x) = e^x \qquad C_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$

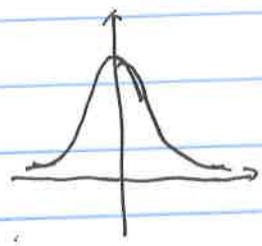
$\vdots$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$f^{(n)}(x) = e^x$

e.g. $f(x) = e^{-x^2}$





$$y = \int e^{-x^2}\,dx$$

(Find the MacLaurin Series expansion)

$$y' = e^{-x^2}$$

$$= e^{(-x^2)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n$$

$$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n$$

$$= \sum_{n=0}^{\infty} \int \frac{1}{n!}(-x^2)^n$$

$$= \sum_{n=0}^{\infty} \int \frac{1}{n!}(-1)^n \cdot x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\, n!}$$

$$\int_{0.2}^{1} e^{-x^2} dx$$

cdf (0.2, ,)

e.g. $(1+x)^c$. (Maclaurin Series)

$$f'(x) = c(1+x)^{c-1}$$

$$f''(x) = c(c-1)(1+x)^{c-2}$$

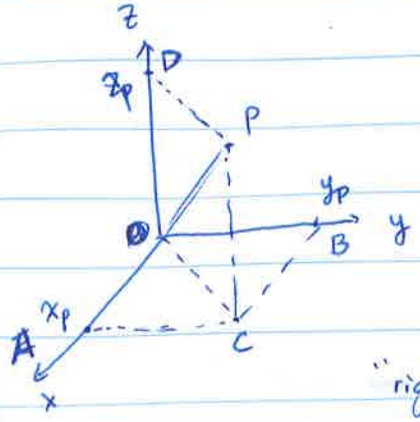
$$\vdots$$

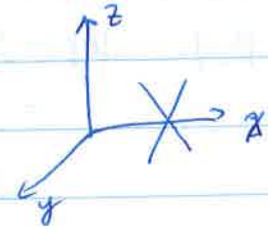
$$f^{(n)}(x) = \underbrace{c(c-1)\cdots(c-n+1)}_{n} (1+x)^{c-n}$$

$$C_n = \frac{f^{(n)}(0)}{n!} = \frac{c(c-1)\cdots(c-n+1)}{n!}$$

e.g. Find the Maclaurin series expansion for the following series

6/26 Mon





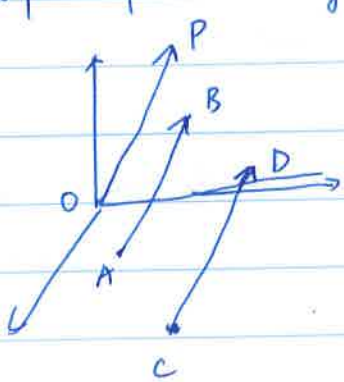
"right hand rule"

key facts: parellel lines stay parellel in 2-D drawing.

$\overrightarrow{AB}$

Def A vector is a directed line segment from A to B

up to parallel shifts



$\overrightarrow{AB} \neq \overrightarrow{BA}$

$\vec{v} = \overrightarrow{OP}$

the coordinates of $\vec{v}$ are the coordinates (x_p, y_p, z_p) of P

$\vec{v} = (x_p, y_p, z_p)$ $\sqrt{x_p^2 + y_p^2}$

A vector is uniquely determined by its direction and length $\|v\|$

Th $\|\vec{v}\| = \sqrt{x_p^2 + y_p^2 + z_p^2}$

Proof. $|OP| = \sqrt{|OC|^2 + |CP|^2}$

$= \sqrt{|OA|^2 + |AC|^2 + |CP|^2}$

$= \sqrt{x_p^2 + y_p^2 + z_p^2}$

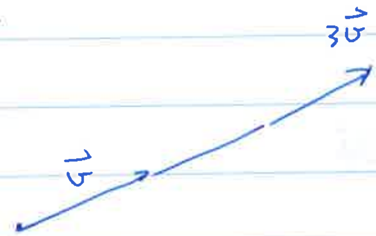
Scalar multiplication of vectors.

Scalar x vector = vector

α real number $\alpha > 0$

$\alpha\vec{v}$: the vector in the same direction as $\vec{v}$ with length $\alpha\|\vec{v}\|$

(opposite if $\alpha < 0$)



$\vec{v} = (x_p, y_p, z_p)$

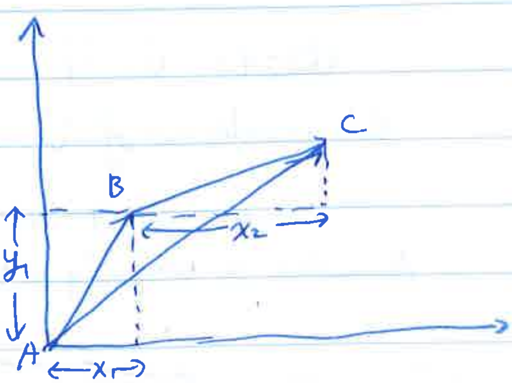
$\alpha\vec{v} = (\alpha v_p, \alpha y_p, \alpha z_p)$

Addition of vectors

$\vec{v} = \overrightarrow{AB} = (x_1, y_1, z_1)$

$\vec{u} = \overrightarrow{BC} = (x_2, y_2, z_2)$

$\vec{v} + \vec{u} = \overrightarrow{AC}$



$\vec{v} + \vec{u} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

A linear combination of $\vec{u}$ and $\vec{v}$ is a vector (in the form of) that can be written as

$$\alpha \vec{u} + \beta \vec{v}$$

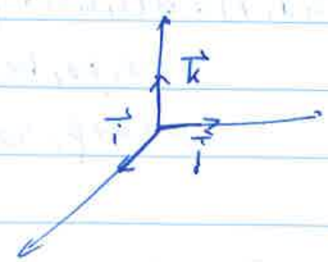
for some real numbers α, β

e.g. $\vec{i} = (1, 0, 0)$

$\vec{j} = (0, 1, 0)$

$\vec{k} = (0, 0, 1)$

" unit vectors "

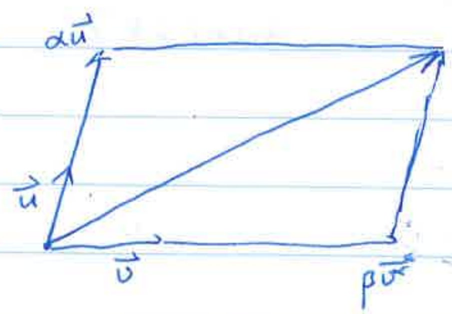


$\vec{v} = (x_p, y_p, z_p)$

$= x_p \vec{i} + y_p \vec{j} + z_p \vec{k}$

~~e.g. Show that ( ) is NOT a lin~~

remark



~~All linear~~ if $\vec{u}, \vec{v}$ are not in the same line

all linear combinations of $\vec{u}$ & $\vec{v}$ form a plane spanned by $\vec{u}$ & $\vec{v}$

e.g. Show that $(3,8,7)$ is NOT a linear combination of $(1,2,3)$
$(1,3,5)$

$$
\begin{aligned}
(3,8,7) &= \alpha(1,2,3) + \beta(1,3,5) \\
&= (\alpha, 2\alpha, 3\alpha) + (\beta, 3\beta, 5\beta) \\
&= (\alpha+\beta,\ 2\alpha+3\beta,\ 3\alpha+5\beta)
\end{aligned}
$$

$$
\begin{cases}
3 = \alpha+\beta \\
8 = 2\alpha+3\beta \\
7 = 3\alpha+5\beta
\end{cases}
$$

$\alpha = 3-\beta$

$8 = 2(3-\beta)+3\beta$

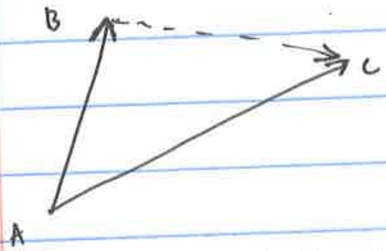
$= 6+\beta$

$\beta = 2$

$\alpha = 1$

$3\cdot 1 + 5\cdot 2 \neq 7$

6/27 Tue



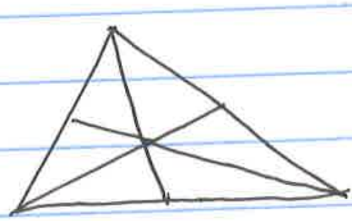
$\vec{AC} - \vec{AB}$

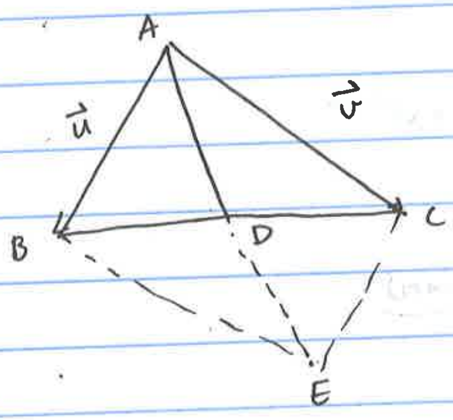
$= \vec{AC} + \vec{BA}$

$= \vec{BC}$

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$





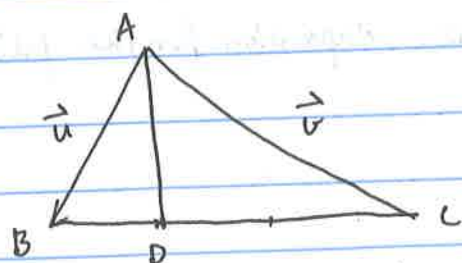
Write $\vec{AD}$ as a linear combination of $\vec{u}$ and $\vec{v}$

$\vec{AD} = ?\,\vec{u} + ?\,\vec{v}$

$= \frac{1}{2}(\vec{u} + \vec{v}) \quad = \frac{1}{2}\vec{AE}$

$= \frac{1}{2}(\vec{AB} + \vec{BE})$

$= \frac{1}{2}(\vec{u} + \vec{v})$



Write $\vec{AD}$ as a linear combination of $\vec{u}$ and $\vec{v}$

① $\vec{AD} = \vec{AB} + \vec{BD} \quad = \vec{u} + \vec{BD}$

$\vec{AD} = \vec{AC} + \vec{CD} \quad = \vec{v} + \vec{CD}$

$\vec{BD} = \vec{BC} - \vec{DC}$

$|DC| = 2|BD|$ $\qquad$ $BC = 3BD$

$$\boxed{\overrightarrow{DC} = 2\overrightarrow{BD}}$$

$$-\overrightarrow{CD} = 2\overrightarrow{BD}$$

① $\times 2$ $\qquad$ $2\overrightarrow{AD} = 2\vec{u} + 2\overrightarrow{BD} = 2\vec{u} - \overrightarrow{CD}$

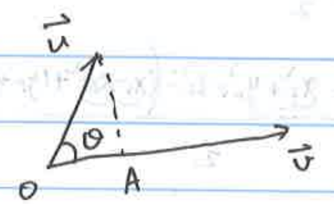
$$\overrightarrow{AD} = \vec{v} + \overrightarrow{CD}$$

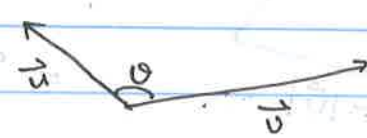
$$3\overrightarrow{AD} = 2\vec{u} + \vec{v}$$

$$\overrightarrow{AD} = \frac{2}{3}\vec{u} + \frac{1}{3}\vec{v}$$

Dot Product

Vector $\cdot$ Vector $=$ Scalar





$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta}$$

$$= |\vec{u}| \cos\theta \cdot |\vec{v}|$$

$$= |OA| \cdot |\vec{v}|$$

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$(\alpha\vec{u}) \cdot \vec{v} = \vec{u} \cdot (\alpha\vec{v})$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\vec{u} \cdot \vec{u} = |\vec{u}|^2$$

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

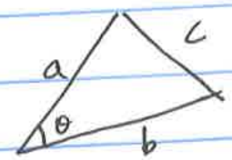
Th. $\vec{u} = (x_1, y_1, z_1)$

$\vec{v} = (x_2, y_2, z_2)$

$\vec{u} \cdot \vec{v} = ?$

Law of Cosines.

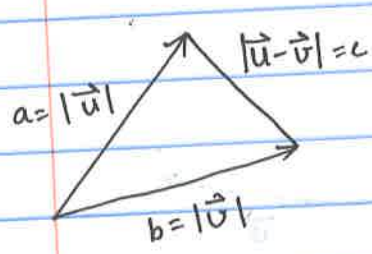
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



$$\cos\theta = \frac{a^2+b^2-c^2}{2ab}$$

$$\vec{u} - \vec{v} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$$

Proof $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cdot \frac{|\vec{u}|^2 + |\vec{v}|^2 - |\vec{u}-\vec{v}|^2}{2|\vec{u}|\,|\vec{v}|}$



$$= \frac{\sqrt{x_1^2+y_1^2+z_1^2}^{\,2} + \sqrt{x_2^2+y_2^2+z_2^2}^{\,2} - \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}^{\,2}}{2}$$

$$= \frac{x_1^2+y_1^2+z_1^2 + x_2^2+y_2^2+z_2^2 - \left((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\right)}{2}$$

$$= \frac{x_1^2 + x_2^2 - (x_1^2 - 2x_1x_2 + x_2^2) + \cdots}{2}$$

$$= \frac{2x_1x_2 + \cdots}{2}$$

$$= x_1x_2 + y_1y_2 + z_1z_2$$

e.g. $|u|\,|v|\cos\theta = x_1x_2 + y_1y_2 + z_1z_2$

$$\cos\theta = \frac{x_1x_2+y_1y_2+z_1z_2}{|u|\cdot|v|}$$

e.g. $\vec{u} = (2, 1, 7)$

$\vec{v} = (1, 0, -3)$

$$\vec{u} \cdot \vec{v} = 2 - 21 = -19$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{-19}{\sqrt{2^2+1^2+7^2} \cdot \sqrt{1^2+0^2+(-3)^2}}$$

$$= \frac{-19}{\sqrt{54} \cdot \sqrt{10}}$$

Wed 6/28

$x_1x_2 + y_1y_2 + z_1z_2$

$\cancel{\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cos}$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cdot|\vec{w}|}$$

$$\boxed{\begin{aligned} &\vec{w}\cdot\vec{v} = \vec{v}\cdot\vec{w} = |\vec{v}|\cdot|\vec{w}|\cos\theta \\ &\quad \alpha\,\vec{v}\cdot\vec{w} \\ &\vec{v}\cdot(\alpha\vec{w}) = (\alpha\vec{v})\cdot\vec{w} = \alpha(\vec{v}\cdot\vec{w}) \\ &\quad \vec{v}\cdot(\vec{s}+\vec{t}) = \vec{v}\cdot\vec{s} + \vec{v}\cdot\vec{t} \end{aligned}}$$

$\theta = 0$ $\quad \vec{u}\cdot\vec{u} = |\vec{u}|^2$

$\sqrt{\vec{u}\cdot\vec{u}} = |\vec{u}|$

$\theta = \frac{\pi}{2}$ $\quad \vec{v}\cdot\vec{w} = 0$

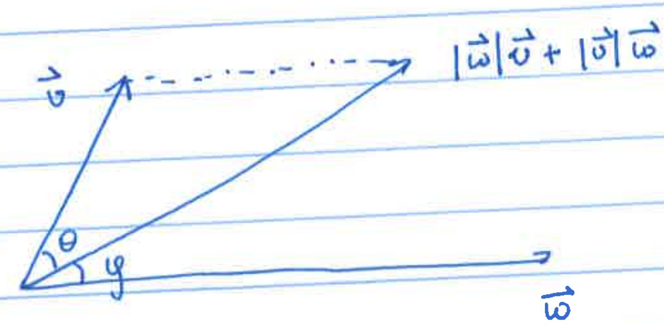
e.g. $|\vec{v}| = 10$

$|\vec{w}| = 15$

$\vec{v}\cdot\vec{w} = 70$

$$|2\vec{v}+3\vec{w}| = \sqrt{(2\vec{v}+3\vec{w})(2\vec{v}+3\vec{w})}$$
$$= \sqrt{2\vec{v}\cdot 2\vec{v} + 3\vec{w}\cdot 2\vec{v} + 2\vec{v}\cdot 3\vec{w} + 3\vec{w}\cdot 3\vec{w}}$$
$$= \sqrt{4|\vec{v}|^2 + 6\,\vec{w}\cdot\vec{v} + 6\,\vec{v}\cdot\vec{w} + 9|\vec{w}|^2}$$
$$= \sqrt{4\cdot 10^2 + 6\cdot 70 + 6\cdot 70 + 9\cdot 15^2}$$

e.g.



$\cos\theta = ?$

$$\cos\theta = \frac{\vec{v}\cdot(|\vec{w}|\vec{v} + |\vec{v}|\cdot\vec{w})}{|\vec{v}|\cdot\left||\vec{w}|\vec{v} + |\vec{v}|\cdot\vec{w}\right|}$$

numerator: $\vec{v}\cdot|\vec{w}|\cdot\vec{v} + \vec{v}\,|\vec{v}|\cdot\vec{w}$

$= |\vec{w}|\cdot|\vec{v}|^2 + |\vec{v}|\cdot\vec{v}\cdot\vec{w}$

$$\left||\vec{w}|\vec{v} + |\vec{v}|\cdot\vec{w}\right| = \sqrt{(|\vec{w}|\vec{v} + |\vec{v}|\cdot\vec{w})(|\vec{w}|\vec{v} + |\vec{v}|\cdot\vec{w})}$$

$$= \sqrt{|\vec{w}|^2\,\vec{v}\cdot\vec{v} + |\vec{v}|\,|\vec{w}|\,\vec{w}\cdot\vec{v} + |\vec{w}|\cdot|\vec{v}|\,\vec{v}\cdot\vec{w} + |\vec{v}|^2\,\vec{w}\cdot\vec{w}}$$

$$= \sqrt{|\vec{w}|^2|\vec{v}|^2 + 2|\vec{v}|\,|\vec{w}|\,\vec{w}\cdot\vec{v} + |\vec{v}|^2|\vec{w}|^2}$$

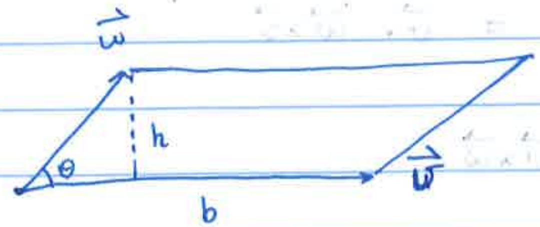
$\vec{v}\cdot\vec{s}\cdot\vec{t}$

$(\vec{v}\cdot\vec{s})\cdot\vec{t}$

$\vec{v}\cdot(\vec{s}\cdot\vec{t})$

Vector x Vector = Vector

Coming outward



$|\vec{v}\times\vec{w}| = b\cdot h$

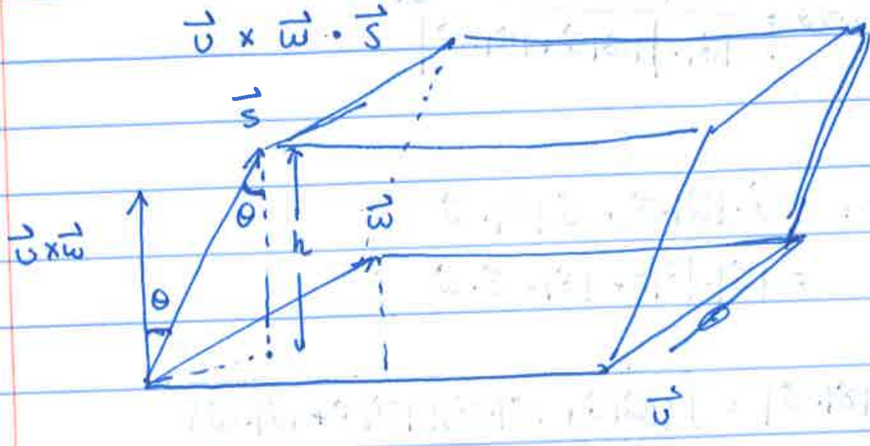
$= |\vec{v}|\cdot|\vec{w}|\sin\theta$

$\sin\theta = \dfrac{|\vec{v}\times\vec{w}|}{|\vec{v}|\cdot|\vec{w}|}$

$\vec{v}\times\vec{w}$:

direction: perpendicular to both $\vec{w}$ and $\vec{v}$ that obeys the right hand rule

magnitude: the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$



$(\vec{v} \times \vec{w}) \cdot \vec{s}$

$= |\vec{v} \times \vec{w}| \cdot |\vec{s}| \cdot \cos\theta$ $\quad\quad$ θ: angle between $\vec{v} \times \vec{w}$ and $\vec{s}$

$= |\vec{v} \times \vec{w}| \cdot h$

$=$ Area $\cdot\, h$

$=$ Volume

$=$ Volume of the parallelepiped formed by $\vec{v}, \vec{w}, \vec{s}$.

$$\vec{v} \times \vec{w} \cdot \vec{s} = \vec{v} \cdot \vec{w} \times \vec{s}$$

$$\alpha\, \vec{v} \times \vec{w}$$

$$\vec{v} \times (\vec{s} + \vec{t}) \overset{?}{=}$$

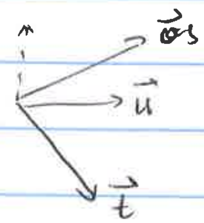
distribution property of the cross product

Thur 6/29

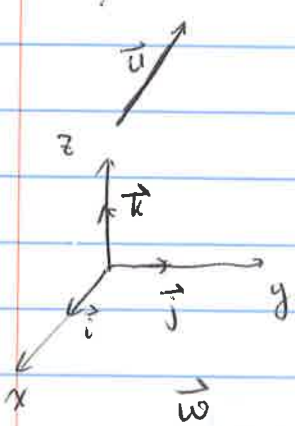
$\vec{v}, \vec{s}, \vec{t}$

$\vec{v} \times \vec{s} \cdot \vec{t} = \vec{v} \cdot \vec{s} \times \vec{t}$

Volume of $= |\vec{v} \times \vec{s} \cdot \vec{t}|$



$\vec{u} \times \vec{u} = \vec{0}$



$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$

$\vec{u} \times \vec{w} = -\vec{w} \times \vec{u}$

$\vec{v} \cdot (\vec{s} + \vec{t}) = \vec{v} \cdot \vec{s} + \vec{v} \cdot \vec{t}$

Th $\quad \vec{v} \times (\vec{s} + \vec{t}) \stackrel{?}{=} \vec{v} \times \vec{s} + \vec{v} \times \vec{t}$

$\vec{d} = \vec{v} \times (\vec{s} + \vec{t}) - (\vec{v} \times \vec{s} + \vec{v} \times \vec{t})$

$$|\vec{d}| = \sqrt{\vec{d}\cdot\vec{d}}$$

$$\vec{d}\cdot\vec{d} = \vec{d}\cdot\left(\vec{u}\times(\vec{s}+\vec{t}) - (\vec{u}\times\vec{s} + \vec{u}\times\vec{t})\right)$$
$$= \vec{d}\cdot\vec{u}\times(\vec{s}+\vec{t}) - \vec{d}\cdot(\vec{u}\times\vec{s}) - \vec{d}\cdot\vec{u}\times\vec{t}$$
$$= (\vec{d}\times\vec{u})\cdot(\vec{s}+\vec{t}) - (\vec{d}\times\vec{u})\cdot\vec{s} - (\vec{d}\times\vec{u})\cdot\vec{t}$$
$$= (\vec{d}\times\vec{u})\cdot(\vec{s}+\vec{t}-\vec{s}-\vec{t})$$
$$= (\vec{d}\times\vec{u})\cdot\vec{0} = 0$$

e.g. $\vec{u} = (2,1,7) = 2\vec{i}+\vec{j}+7\vec{k}$

$\vec{w} = (0,-1,3) = -\vec{j}+3\vec{k}$

$$\vec{u}\times\vec{w} = (2\vec{i}+\vec{j}+7\vec{k})\times(-\vec{j}\times 3\vec{k})$$
$$= 2\vec{i}\times(-\vec{j}) + 2\vec{i}\times(3\vec{k}) + \cdots$$

$\vec{u} = (x_1, y_1, z_1)$

$\vec{w} = (x_2, y_2, z_2)$

$$\vec{u}\times\vec{w} = (x_1\vec{i}+y_1\vec{j}+z_1\vec{k})\times(x_2\vec{i}+y_2\vec{j}+z_2\vec{k})$$
$$= x_1\vec{i}\times x_2\vec{i} + \cancel{y_1\vec{j}\times x_2}\ x_1\vec{i}\times y_2\vec{j} + x_1\vec{i}\times z_2\vec{k}$$
$$+ y_1\vec{j}\times x_2\vec{i} + y_1\vec{j}\times y_2\vec{j} + y_1\vec{j}\times z_2\vec{k}$$
$$+ z_1\vec{k}\times x_2\vec{i} + z_1\vec{k}\times y_2\vec{j} + z_1\vec{k}\times z_2\vec{k}$$

$$= x_1y_2\vec{k} + x_1z_2(-\vec{j}) + y_1x_2(-\vec{k}) + y_1z_2\vec{i}$$
$$+ z_1x_2\vec{j} + z_1y_2(-\vec{i})$$
$$= (y_1z_2 - z_1y_2)\vec{i} + (z_1x_2 - x_1z_2)\vec{j} + (x_1y_2 - y_1x_2)\vec{k}$$

e.g. $-(-z_1x_2 + x_1z_2)\vec{j}$

Determinant of a Matrix

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = AD - BC$$

$$\parallel$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

$$\det\left(\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\right) = A\det\left(\begin{bmatrix} E & F \\ H & I \end{bmatrix}\right) - B\det\left(\begin{bmatrix} D & F \\ G & I \end{bmatrix}\right) + C\det\left(\begin{bmatrix} D & E \\ G & H \end{bmatrix}\right)$$

$$\vec{u}\times\vec{w} = \det\left(\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix}\right)\vec{i} - \vec{j}\det\left(\begin{bmatrix} x_1 & z_1 \\ x_2 & z_2 \end{bmatrix}\right) + \vec{k}\det\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}\right)$$

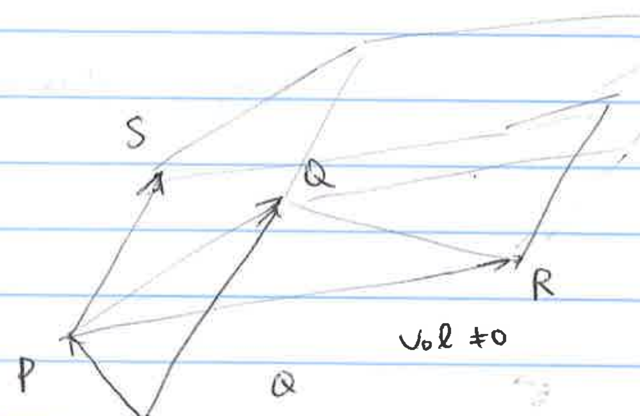
e.g. $P = (1,-1,0) = \vec{OP}$

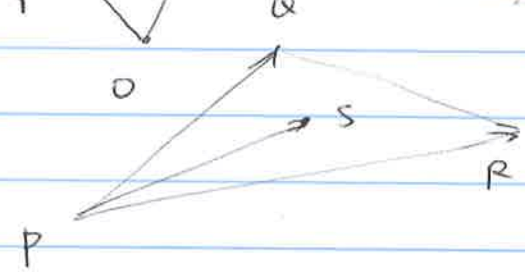
$Q = (3,-2,7) = \vec{OQ}$

$R = (5,-2,23) = \vec{OR}$

$S = (2,-1,8) = \vec{OS}$

Determine if they lie on the Same plane





Vol = 0

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

P, Q, R, S are coplane if and only if

Vol $= 0$

$\overrightarrow{PQ} \times \overrightarrow{PR} \cdot \overrightarrow{PS}$

$= (3-1, -2-(-1), 7-0) \times (4,-1,23) \cdot (1,0,8)$

$= \det\left(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 7 \\ 4 & -1 & 23 \end{bmatrix}\right) \cdot (1,0,8)$

$= \left(\vec{i}(-23+7) - \vec{j}(2\cdot 23 - 4\cdot 7) + \vec{k}(2\cdot(-1) - 4\cdot(-1))\right) \cdot (1,0,8)$

$= (-16, 16, 2) \cdot (1,0,8) = -16+16 = 0$

therefore P, Q, R, S are coplane

7/6 Thur

$\vec{v} \times \vec{w}$ the vector with length : area of the parallelogram formed by $\vec{v}$ and $\vec{w}$

direction : perpendicular to $\vec{v}$ & $\vec{w}$ that obeys the right hand rule

$\vec{v} = (x_1, y_1, z_1)$ $\quad |\vec{v}|$: invariant under the choice of coord. sys.

$\vec{w} = (x_2, y_2, z_2)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$A = (1, 3, 7)$

$B = (2, -1, 5)$ $\quad$ Find the area of the triangle ABC

$C = (-1, 7, 3)$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= (1, -4, -2)$

$\vec{AC} = (2, -4, 4)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -4 & -2 \\ 2 & -4 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} -4 & -2 \\ -4 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -4 \\ 2 & -4 \end{vmatrix}$$

$= -24\vec{i} - 8\vec{j} + 4\vec{k}$

Area of ABC $= \frac{1}{2}$ Area of $\square$

$\frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$= \frac{1}{2} |-24\vec{i} - 8\vec{j} + 4\vec{k}|$

$= \frac{1}{2} \sqrt{24^2 + 8^2 + 4^2}$

lines

Def A directional vector $\vec{v}$ ~~on~~ of a line ℓ is a vector on the line ℓ

if $\vec{w}$ is another directional vector

$$\vec{w} = \lambda \vec{v}$$

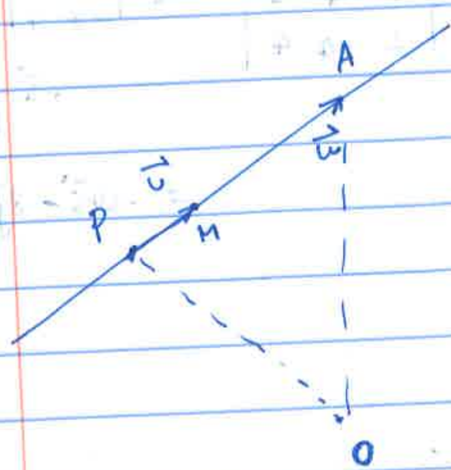
$$|\vec{w}| = |\lambda \vec{v}| = \lambda |\vec{v}|$$

$$\lambda = \frac{|\vec{w}|}{|\vec{v}|}$$

e.g. Find a vector $\vec{w}$ with the same direction as $\vec{v}\,(2,1,4)$ with length 10

$$\vec{w} = \lambda \vec{v} \qquad \lambda = \frac{|\vec{w}|}{|\vec{v}|} = \frac{10}{\sqrt{2^2+1^2+4^2}} = \frac{10}{\sqrt{21}}$$

Find the equations of the line ℓ passing through $P(1,2,3)$ with directional vector $\vec{v} = (7,8,9)$



$$A = (x, y, z)$$

$$\overrightarrow{PA} = \vec{w} = \lambda \vec{v} = (7\lambda, 8\lambda, 9\lambda)$$

$$\overrightarrow{PM} = \vec{v}$$

$\vec{AO} = (0,0,0) - (x, y, z) = (-x, -y, -z)$

$\vec{OP} = (1,2,3)$

$\vec{OP} + \vec{PA} = \vec{OA}$

$(1,2,3) + (7\lambda, 8\lambda, 9\lambda) = (x, y, z)$

$$\begin{cases} x = 1 + 7\lambda \\ y = 2 + 8\lambda \qquad \lambda = \frac{y-2}{8} \\ z = 3 + 9\lambda \end{cases}$$

The parametric equations of l with parameter λ.

$$\lambda = \frac{x-1}{7}$$

$$y = 2 + 8 \cdot \frac{x-1}{7}$$

$$y - 2 = 8 \cdot \frac{x-1}{7}$$

$$\frac{y-2}{8} = \frac{x-1}{7} = \frac{z-3}{9}$$

The symmetric equations of l

the line passing through point $P(x_0, y_0, z_0)$ with directional vector (a, b, c)

has parametric equations

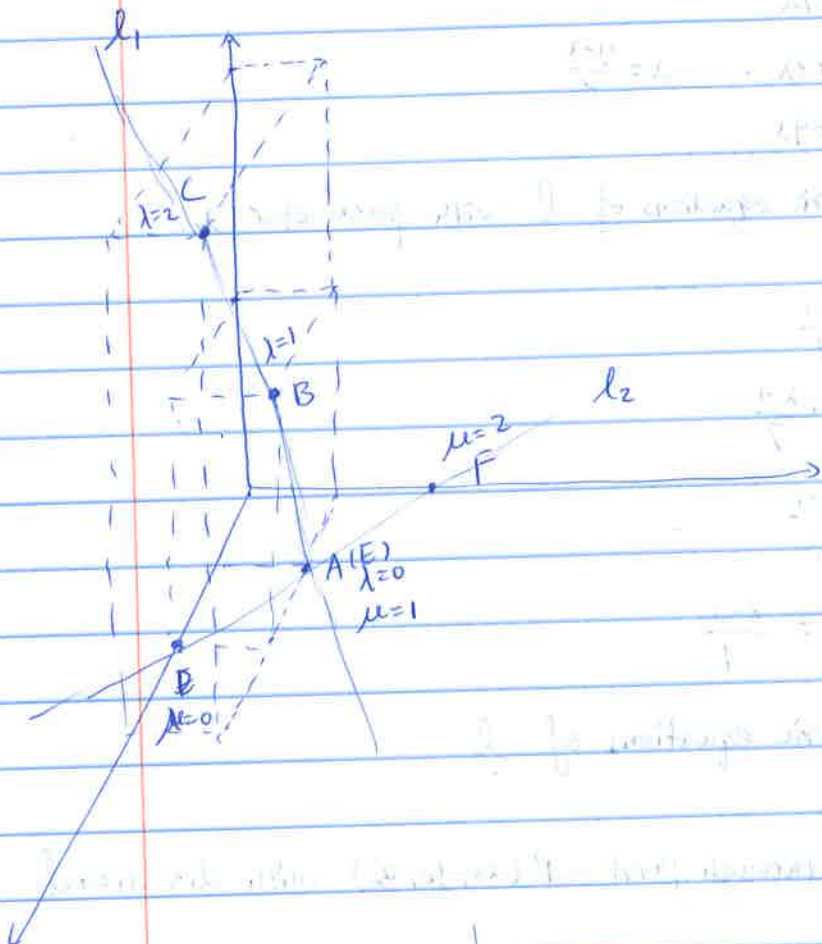
$$\begin{cases} x = x_0 + a\lambda \\ y = y_0 + b\lambda \\ z = z_0 + c\lambda \end{cases}$$

Symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Fri 7/7

l_1	l_2
$x = 1+\lambda$	$x = 2-\lambda$
$y = 1$	$y = \lambda$
$z = 2\lambda$	$z = 0.$



$\lambda=0$	$A(1,1,0)$	$\lambda=0$	$D(2,0,0)$
$\lambda=1$	$B(2,1,2)$	$\lambda=1$	$E(1,1,0)$
$\lambda=2$	$C(3,1,4)$	$\lambda=2$	$F(0,2,0)$

Determine if

$$l_3: \quad \frac{x-3}{2} = \frac{y-1}{4} = \frac{z+2}{7} = \lambda$$

$y = f_1(x) = x+1$

$y = f_2(x) = x^2$

$$l_4: \quad \frac{x-8}{3} = \frac{y-3}{-2} = \frac{z-6}{1} = \mu$$

$x+1 = x^2$

intersect

two lines are skew if they don't intersect

$(5,5,5)$

$l_3 \quad x_0 = 2\lambda+3 \quad y = 4\lambda+1 \quad z = 7\lambda-2$

$l_4 \quad x_0 = 3\mu+8 \quad y_0 = -2\mu+3 \quad z = \mu+6$

$(5,5,5)$

$2\lambda+3 = 3\mu+8$

$\lambda = \frac{3\mu+5}{2}$

$\lambda = \frac{-3+5}{2} = 1$

$4\lambda+1 = -2\mu+3$

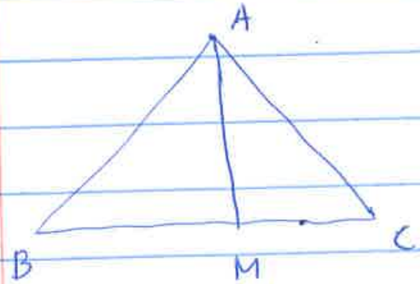
$\cancel{4}^{2} \cdot \frac{3\mu+5}{\cancel{2}} + 1 = -2\mu+3$

$6\mu+10+1 = -2\mu+3$

$8\mu = 3-11$

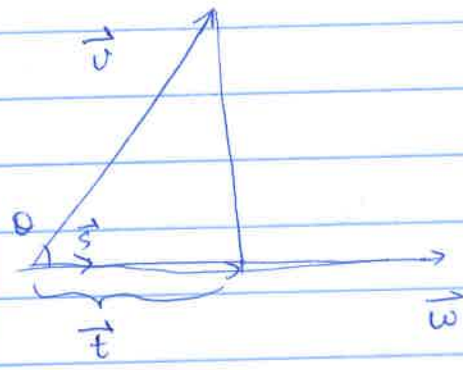
$= -8$

$\mu = -1$

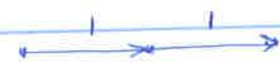


$$\vec{BM} = \vec{AM} - \vec{AB}$$

$$\vec{AM} = \vec{AB} + \vec{BM}$$



$$|\vec{t}| = |\vec{v}| \cos\theta = |\vec{v}| \cdot \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$$



$$\vec{s} = \frac{\vec{w}}{|\vec{w}|}$$

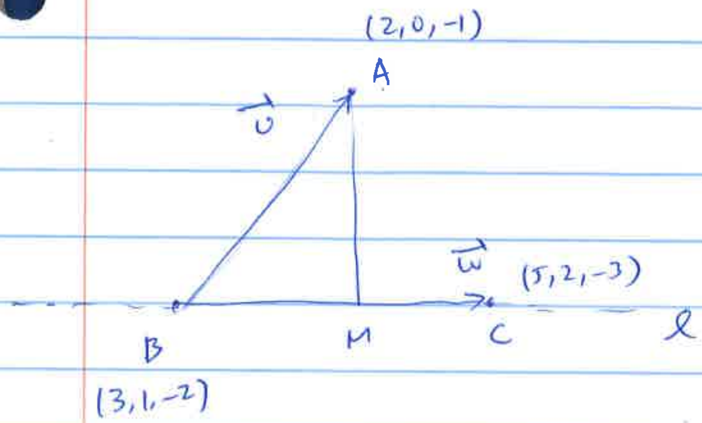
$$\vec{s} = \frac{\vec{w}}{2}$$

$$\vec{t} = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} \cdot \vec{s}$$

$$= \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} \cdot \frac{\vec{w}}{|\vec{w}|}$$

$$|\vec{w}|^2 = \vec{w} \cdot \vec{w}$$

$$= \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \cdot \vec{w}$$



ℓ:

$x = 3 + 2\lambda$

$y = 1 + \lambda$

$z = -2 - \lambda$

A: $(2, 0, -1)$

find $\vec{AM}$

$\lambda = 0: \quad (3,1,-2)$

$\lambda = 1: \quad (5,2,-3)$

$\vec{v} = \vec{BA} = (-1,-1,1)$

$\vec{w} = \vec{BC} = (2,1,-1)$

$\vec{v} \cdot \vec{w} = -4$

$|\vec{w}|^2 = 6$

$\vec{BM} = \frac{-4}{6} \cdot (2,1,-1)$

$\vec{AM} = \vec{AB} + \vec{BM} = (1,1,-1) + \frac{-4}{6}(2,1,-1)$

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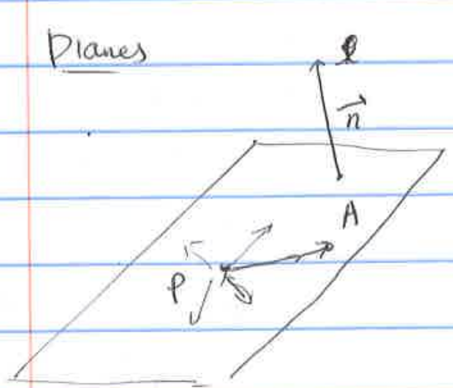
Mon

Planes



$\ell \perp p$

p a normal vector $\vec{n}$ of a plane is a vector perpendicular to the plane p

Normal vectors are unique up to rescaling

$\vec{n_1}, \vec{n_2}$: normal vectors of plane p

$\vec{n_1} = \lambda \vec{n_2}$

e.g. Find the equation of a plane that goes through $P(3,2,1)$ (x_p, y_p, z_p) with normal vector $\vec{n}(7,8,9)$ (a,b,c)

$\vec{PA} \perp \vec{n}$

$\vec{PA} \cdot \vec{n} = 0$

$P(3,2,1)$

$A(x,y,z)$

$\vec{PA} = (x-3, y-2, z-1)$

$(x-3, y-2, z-1)(7,8,9) = 0$

$\boxed{7(x-3) + 8(y-2) + 9(z-1) = 0}$

$7x - 21 + 8y - 16 + 9z - 9 = 0$

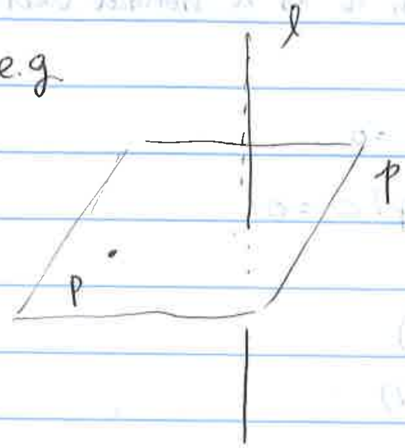
$\boxed{7x + 8y + 9z = 46}$

$a(x-x_p) + b(y-y_p) + c(z-z_p) = 0$

~~Ab Eq~~. A plane $2x + 3y - 5z = 10$

has normal vector $(2, 3, -5)$

e.g.



$l: \frac{x-1}{2} = \frac{y+2}{3} = \frac{z-7}{7}$

$p: (-1, 2, 7)$

Find eq. of p

$x = 2\lambda + 1$ $\quad \lambda = 0$ $\quad$ ~~A(2,3,7)~~

$y = 3\lambda - 2$ $\quad$ $A(1, -2, 7)$

$z = 7\lambda + 7$ $\quad \lambda = 1$ $\quad B(3, 1, 14)$

$\lambda = 2$ $\quad C(5, 4, 21)$

$\vec{AB} = (2, 3, 7)$

$(x-(-1))2 + (y-2)3 + (z-7)7 = 0$

$\lambda = 2$ $\quad \vec{AC} = (4, 6, 14)$

$(x-(-1))4 + (y-2)6 + (z-7)14 = 0$

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$P(x_p, y_p, z_p)$

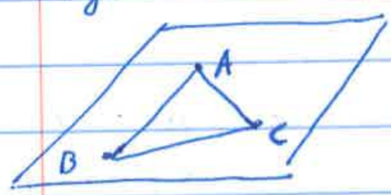
$\vec{n} = (a, b, c)$

the plane going through point P with $\vec{n}$ as a normal vector ~~as~~ has equation

$$(x-x_p,\ y-y_p,\ z-z_p)\cdot(a,b,c)=0$$

$$(x-x_p)a+(y-y_p)b+(z-z_p)c=0$$

e.g.



A (3,2,1)

B (1,-1,7)

C (2,0,-3)

$\overrightarrow{BA} = (2,3,-6)$

$\overrightarrow{AC} = (-1,-2,-4)$

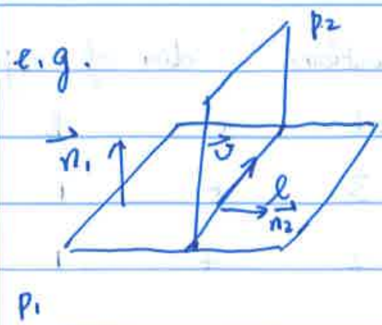
$\overrightarrow{BC} = \cancel{(1,1,-10)}\ \cancel{(1,1,}$

$(1,1,-10)$

$$\overrightarrow{BA}\times\overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -6 \\ -1 & -2 & -4 \end{vmatrix} = (-12-12)\vec{i} - (-8+6)j + (-4+3)\vec{k} = (-24, 2, -1)$$

$$(x-3)(-24)+(y-2)\cdot 2+(z-1)(-1)=0$$

e.g.



$$\ell: \frac{x-2}{2} = \frac{y+3}{7} = \frac{z-1}{1} = \lambda$$

$$p_1: \quad 3x - y + z = 10$$

eq of p_2 ?

$$\vec{n}_1 = (3, -1, 1)$$
$$\vec{v} = (2, 7, 1)$$

$\vec{n}_2 \perp \vec{n}_1$

since $\vec{n}_1$ is normal to p_1 and $\vec{n}_2$ is within p_1.

$\vec{v} \perp \vec{n}_2$

$$\vec{n}_2 = \vec{n}_1 \times \vec{v}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 2 & 7 & 1 \end{vmatrix} = -8\vec{i} - \vec{j} + 23\vec{k}$$

$\lambda = 0 \qquad x = 2,\ y = -3,\ z = 1$

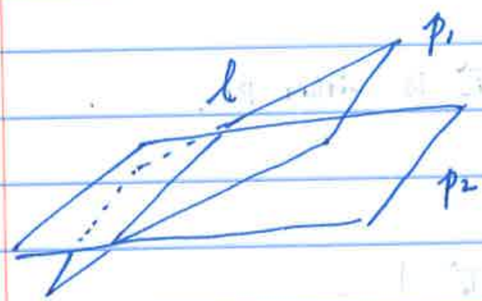
$$(x-2)(-8) + (y+3)(-1) + (z-1)\,23 = 0.$$

	# of variables		# of equations		dim of obj
plane	3	=	1	+	2
parametric eq.	4	=	3	+	1
sym.	3	=	2	+	1

$x = * + * \cdot z$

$y = * + * \cdot z$

$z = z$



$p_1 \quad 2x + y + 7z = 2 \quad ①$

$p_2 \quad 3x - y - 2z = 1 \quad ②$

① $\times (-3)$ $\quad \cancel{-3-2y}$

② $\times 2$

$-3y - 2y - 21z - 4z = -6 + 2$

$-5y - 25z = -4$

$y = \frac{-4 + 25z}{-5}$

① $\times 2$ + ② $\times 7$ $\quad 4x + 21x + 2y - 7y = 4 + 7$

$25x - 5y = 11$

$x = \frac{11 + 5y}{25} = \frac{11 + 5 \cdot \frac{-4 + 25z}{-5}}{25}$

Thur 7/13

$x = 2 + \lambda$

$y = -3 + 2\lambda$

$z = 1 - 2\lambda$

10.1 Parametric Curves

<u>Def</u>. A parametric curve is a curve given by

$x = f(t) \quad y = g(t)$

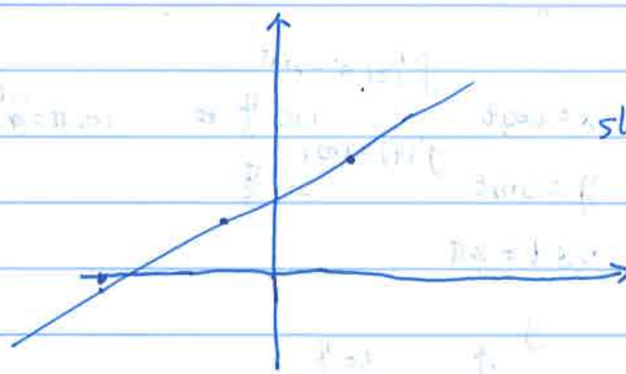
f, g : functions of t (real valued)

for real values of t on a closed interval.

e.g. $x = -2t + 1 = f(t) \qquad f'(t) = -2$

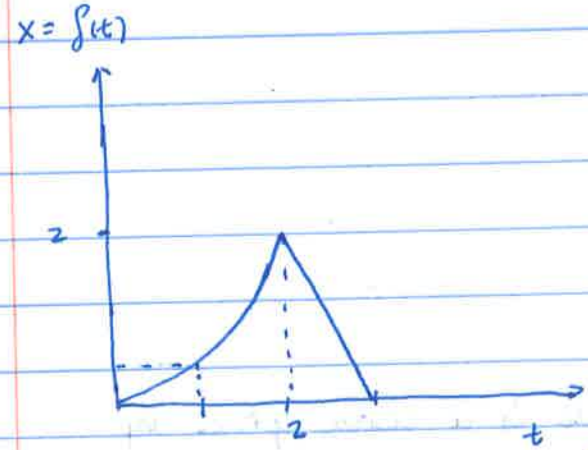
$y = -t + 2 = g(t) \qquad g'(t) = -1$

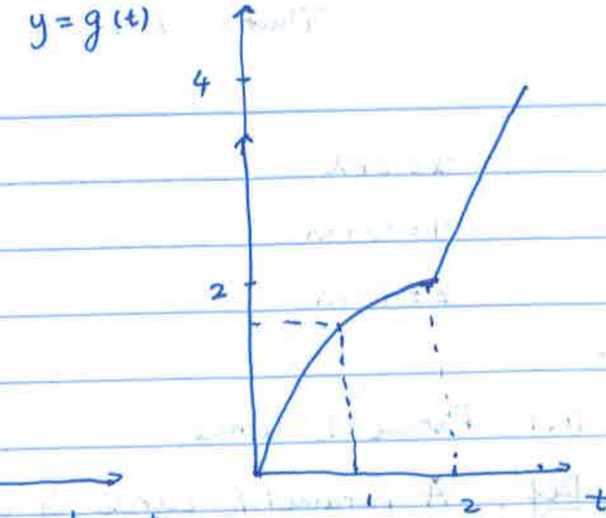
t	x	y
0	1	2
1	-1	1
2	-3	0

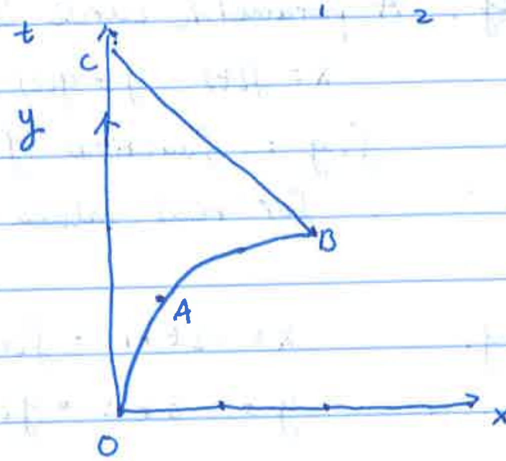


slope $= \frac{1}{2} = \frac{g'(t)}{f'(t)}$

e.g.







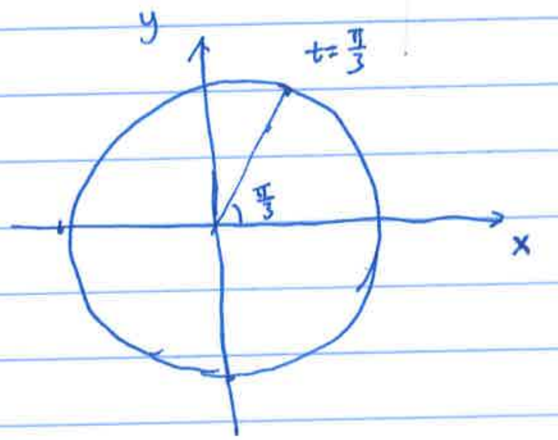
	t	x	y
O	0	0	0
A	1	0.5	1.5
B	2	2	2
C	3	0	4

e.g. $x = \cos t$ $\quad f'(t) = -\sin t$ $\quad \cos\frac{\pi}{3}$ $\quad \cos\pi = -1, \ \sin\pi = 0$

$g'(t) = \cos t$

$y = \sin t$ $\quad \sin\frac{\pi}{3}$

$0 \le t \le 2\pi$



$\cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$

$$y^2 = 1 - x^2$$

$$y = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \cdot (1-x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \cdot (-2x)$$

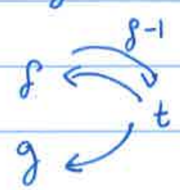
$$= \frac{-x}{\sqrt{1-x^2}} = \frac{-x}{y} = \frac{-\cos t}{\sin t} = \frac{g'(t)}{f'(t)}$$

$$\frac{dy}{dx} \overset{?}{=} \frac{g'(t)}{f'(t)}$$

① $\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot g'(t)$

② $\frac{d}{dt} h^{-1}(t) = \frac{1}{h'(h^{-1}(t))}$

$$f \leftarrow g \leftarrow t$$



$$x = f(t)$$

$$f^{-1}(x) = t$$

$$y = g(t) = g(f^{-1}(x))$$

$$\frac{dy}{dx} = g'(f^{-1}(x)) \cdot \boxed{\frac{1}{f'(f^{-1}(x))}} = \frac{g'(t)}{f'(t)}$$

$$(f^{-1}(x))'$$

e.g. $x = \sin t \cos t = f(t)$

$y = \cos^2 t = g(t)$

$f'(t) = \cos t \cos t + \sin t(-\sin t)$

$g'(t) = 2\cos t(-\sin t)$

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{-2\sin t \cos t}{\cos^2 t - \sin^2 t}$$

$$\frac{x}{y} = \frac{\sin t \cancel{\cos t}}{\cos^2 t} = \tan t$$

$$\tan^2 t + 1 = \sec^2 t = \frac{1}{\cos^2 t}$$

$$\left(\frac{x}{y}\right)^2 + 1 = \frac{1}{y^2}$$

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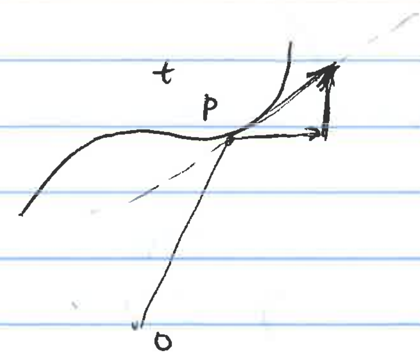
$$x = \cancel{g(t)}\ f(t) \qquad y = g(t)$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dt} \qquad \frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{\text{rise}}{\text{run}}$$



$$\vec{v} = f'(t)\,\vec{i} + g'(t)\,\vec{j}$$
$$= (f'(t),\ g'(t))$$
$$\overrightarrow{op} = (f(t),\ g(t))$$

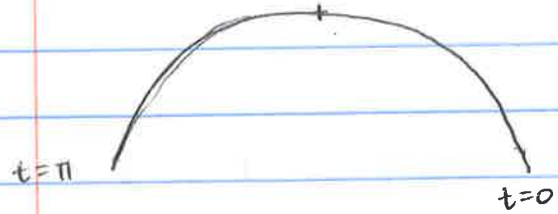
e.g. $f:\ x = e^t \cos t$

$g:\ y = e^t \sin t$

$$f'(t) = e^t \cos t + e^t(-\sin t)$$
$$g'(t) = e^t \sin t + e^t \cos t$$
$$\frac{dy}{dx} = \frac{e^t \sin t + e^t \cos t}{e^t \cos t - e^t \sin t}$$

$$\frac{d^2}{dx^2}y = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{g''(t)f'(t) - g'(t)f''(t)}{(e^t \cos t - e^t \sin t)^2}}{(e^t \cos t - e^t \sin t)} = h(t)$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{\left(\frac{g'(t)}{f'(t)}\right)'}{f'(t)}$$



$x = a\cos t$

$y = b\sin t$

$y = b\sqrt{1-\left(\frac{x}{a}\right)^2}$

$\int y\,dx = \int$

$x = a\cos t$

$dx = -a\sin t$

$= \int y \; \underline{-a\sin t} \; dt$

$= \int b\sin t\,(-a\sin t)\,dt$

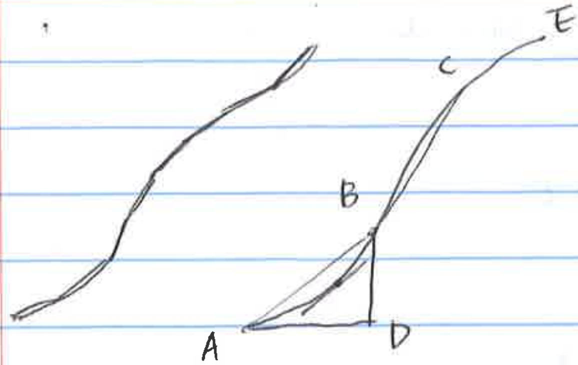
$= -\int_{\pi}^{0} ab\sin^2 t\,dt = -ab\int_{\pi}^{0}\sin^2 t\,dt$

$= -ab\int_{\pi}^{0}(1-\cos^2 t)\,dt$

$\int y$.

Area from $t=a$ to $t=b$

$$\int y\,dx = \int_a^b g(t)\,f'(t)\,dt$$



$$\widehat{AE} \approx AB + BC + CE$$

$$AB = \sqrt{AD^2 + BD^2}$$

$$= \sqrt{AD^2\left(1 + \left(\frac{BD}{AD}\right)^2\right)} = AD \cdot \sqrt{1 + \left(\frac{BD}{AD}\right)^2}$$

$$= dx \cdot \sqrt{1 + (f'(x_A))^2}$$

$$\widetilde{AE} = dx \cdot \sqrt{1 + (f'(x_A))^2} + dx \cdot \sqrt{1 + (f'(x_B))^2} + \cdots$$

$$= \int \sqrt{1 + (f'(x))^2}\, dx$$

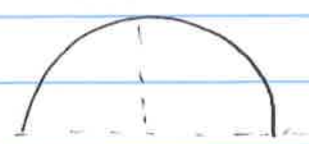
e.g. $x = \cos t$

$y = \sin t$

$2\pi r$

2π

π



$$f'(x) = \frac{dy}{dx} = \frac{\cos t}{-\sin t}$$

$dx = -\sin t$

$$\text{arc length} = \int \sqrt{1 + \left(\frac{\cos t}{-\sin t}\right)^2}\, dx$$

$$= \int_{\pi}^{0} \sqrt{1 + \frac{\cos^2 t}{\sin^2 t}} \cdot (-\sin t)\, dt$$

$$= -\int_{\pi}^{0} \sqrt{\frac{\sin^2 t + \cos^2 t}{\sin^2 t}} \sin t \, dt$$

$$= -\int_{\pi}^{0} \sqrt{\frac{1}{\sin^2 t}} \sin t \, dt$$

$$= -\int_{\pi}^{0} \sqrt{\csc^2 t} \sin t \, dt \qquad \sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$$

$$= -\int_{\pi}^{0} |\csc t| \sin t \, dt$$

$$= -\int_{\pi}^{0} \frac{1}{\sin t} \sin t \, dt$$

$$= -\int_{\pi}^{0} 1 \, dt$$

$$= -t \Big|_{\pi}^{0} = \pi$$

Arc length from $t=a$ to $t=b$:

$$\int \sqrt{1+(f'(x))^2}\, dx = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

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#2

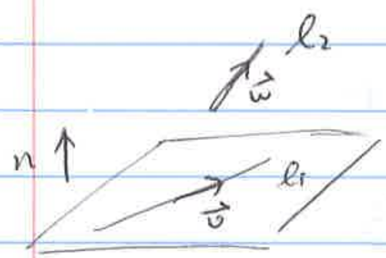
$$4(2\lambda+3)+(3\lambda-1)-11(\lambda+4)=-33$$
$$8\lambda+12+3\lambda-1-11\lambda-44=-33$$
$$8\lambda+3\lambda-11\lambda=0$$
$$0=0$$

#3

$$x=3+2\lambda$$
$$y=2+\lambda$$
$$z=1+7\lambda$$

$$3(3+2\lambda)+(2+\lambda)-(1+7\lambda)=10$$
$$6\lambda+\lambda-7\lambda=0$$

#6



$$\ell_1: \frac{x-2}{3}=\frac{y+5}{2}=\frac{z-7}{-1}$$

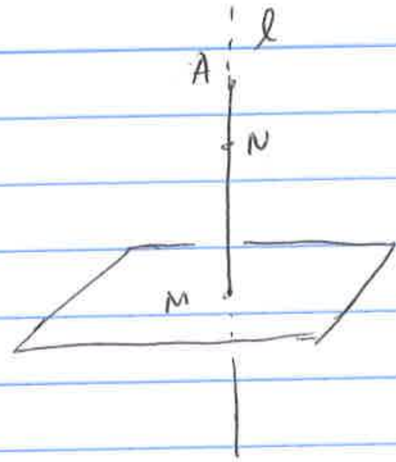
$$\ell_2: \frac{x-1}{1}=\frac{y}{-1}=\frac{z-8}{3}$$

$$\vec{n}=\vec{v}\times\vec{w}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 2 & -1\\ 1 & -1 & 3\end{vmatrix}=5\vec{i}-10\vec{j}-5\vec{k}$$

$(2,-5,7)$

$$5(x-2)-10(y+5)-5(z-7)=0$$

\# 8.



$8x+9y+10z=10$

$MN = (8, 9, 10)$

$l: \dfrac{x-1}{8} = \dfrac{y-2}{9} = \dfrac{z-3}{10}$

$x = 8\lambda+1$ $\quad\quad$ $x = 8 \cdot \dfrac{46}{245} - 1$

$y = 9\lambda+2$ $\quad\quad$ $y = \dots$

$z = 10\lambda+3$ $\quad\quad$ $z = \dots$

$$(8\lambda+1)8 + 9(9\lambda+2) + 10(10\lambda+3) = 10$$

$$345\lambda + 56 = 10$$

$$\lambda = \frac{46}{245}$$

#12

$$\frac{x}{a} = \cos t$$

$$\frac{y}{b} = \sin t$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{b\cos t}{-a\sin t} \qquad \frac{d}{dt}\left(\frac{dy}{dx}\right) =$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

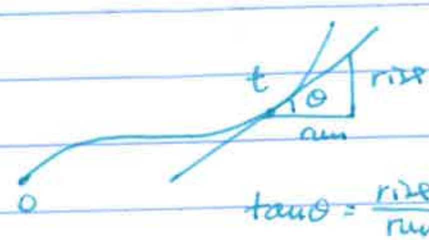
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}} = \frac{\text{quotient rule}}{-a\sin t}.$$

Tue 7/18

$x = f(t)$

$y = g(t)$

$$\tan\theta = \frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$



$$\tan\theta = \frac{\text{rise}}{\text{run}} = \text{slope}$$

$$\frac{d\theta}{ds} = \frac{\frac{d\theta}{dt}}{\frac{ds}{dt}}$$

Arc length from $t=a$ to $t=b$

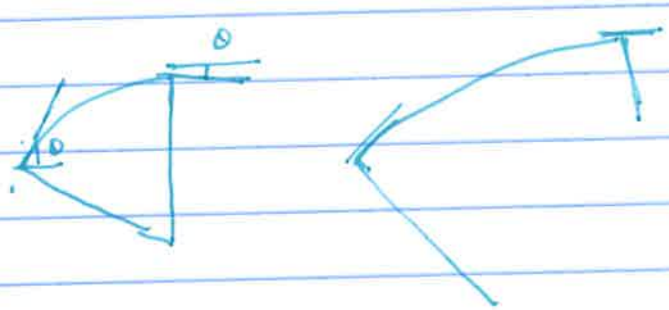
$$\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

$$\int_a^b \sqrt{(f'(\lambda))^2 + (g'(\lambda))^2}\, d\lambda$$

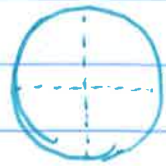
Arc length from 0 to t

$$\int_0^t \sqrt{(f'(\lambda))^2 + (g'(\lambda))^2}\, d\lambda$$

Curvature



$$\frac{d\theta}{ds}$$



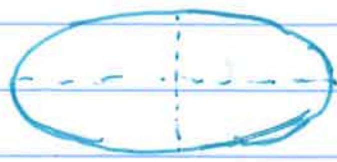
$x = \cos t$

$y = \sin t$

$2\pi \sim 2\pi R$

$t \sim tR$

$$\frac{d\theta}{ds} = \frac{t}{tR} = \frac{1}{R}.$$



$x = a\cos t$

$y = b\sin t$

$$\tan\theta = \frac{b\cos t}{-a\sin t}$$

$$\theta = \tan^{-1}\left(\frac{b}{-a}\cot t\right) \qquad (\tan^{-1}x)' = \frac{1}{1+x^2}$$

$$s(t) = \int_0^t \sqrt{b^2\cos^2\lambda + a^2\sin^2\lambda}\, d\lambda$$

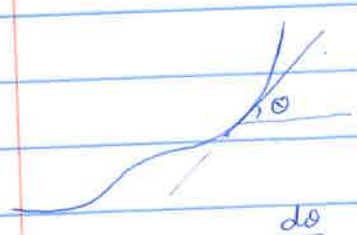
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{b}{-a}\cot t\right)^2} \cdot \frac{b}{-a}\csc^2 t$$

$$\frac{ds}{dt} = b^2\cos^2 t + a^2\sin^2 t$$

$t=0$

$$\frac{\frac{d\theta}{dt}}{\frac{ds}{dt}} = \frac{1}{1+\left(\frac{b}{-a}\right)^2} \cdot \frac{b}{-a} \qquad \csc 0 = \frac{1}{\sin 0}$$

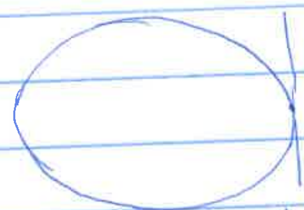
Thur 7/20



curvature $\frac{d\theta}{ds} = \dfrac{\frac{d\theta}{dt}}{\frac{ds}{dt}}$

$$S(t) = \int_0^t \sqrt{(f'(\lambda))^2 + (g'(\lambda))^2}\, d\lambda$$

e.g.



$t=0 \quad \theta = \frac{\pi}{2}$

$x = a\cos t$

$y = b\sin t$

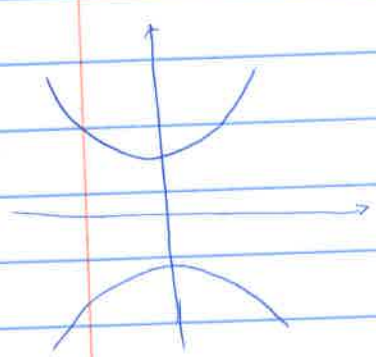
$\tan\theta = \frac{dy}{dx} = \frac{b}{-a}\cot t$

$$\frac{d\theta}{ds} = \frac{\frac{1}{1+\left(\frac{b}{-a}\cot t\right)^2}\cdot\left(\frac{b}{-a}\right)\csc^2 t}{\sqrt{b^2\cos^2 t + a^2\sin^2 t}}$$

$$= \frac{\frac{b}{-a}\cdot\frac{1}{\sin^2 t + \left(\frac{b}{a^2}\right)\cos^2 t}}{\sqrt{b^2\cos^2 t + a^2\sin^2 t}}$$

$$t=0. \quad \frac{d\theta}{ds} = \frac{\frac{b}{-a}\cdot\frac{1}{\left(\frac{b}{a}\right)^2}}{\sqrt{b^2}} = \frac{b}{-a}\cdot\frac{a^2}{b^2}\cdot\frac{1}{b} = -\frac{a}{b^2}$$

e.g.



$x^2 - y^2 = 1$

$x = \sec t$

$y = \tan t$

$$\tan\theta = \frac{dy}{dx} = \frac{\cancel{\tan t}\sec^2 t}{\cancel{\sec t}\tan t} = \frac{\frac{1}{\cos t}}{\frac{\sin t}{\cos t}} = \frac{1}{\sin t} = \csc t$$

$$\theta = \tan^{-1}(\csc t) \qquad (\tan^{-1} x)' = \frac{1}{1+x^2}$$

$$\kappa = \frac{\dfrac{d\theta}{dt}}{\dfrac{ds}{dt}} = \frac{\dfrac{1}{1+\csc^2 t} \cdot (-\csc t \cot t)}{\sqrt{(\tan \sec t)^2 + (\sec^2 t)^2}}$$

$$\frac{d\theta}{dt} = \frac{1}{1+\csc^2 t} \cdot (-\csc t \cot t)$$

$$\frac{ds}{dt} = \sqrt{(f'(t))^2 + (g'(t))^2} = \sqrt{(\tan \sec t)^2 + (\sec^2 t)^2}$$

$$x = f(t)$$

$$y = g(t)$$

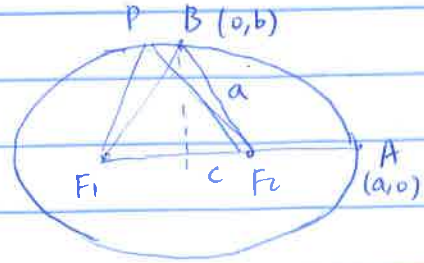
$$\tan\theta = \frac{g'(t)}{f'(t)}$$

$$\theta = \tan^{-1}\left(\frac{g'(t)}{f'(t)}\right)$$

$$\kappa = \frac{\dfrac{d\theta}{dt}}{\dfrac{ds}{dt}} = \frac{\dfrac{1}{1+\left(\dfrac{g'(t)}{f'(t)}\right)^2} \cdot \dfrac{g''(t) f'(t) - f''(t) g'(t)}{(f'(t))^2}}{\sqrt{(f'(t))^2 + (g'(t))^2}}$$

$$= \frac{g''(t) f'(t) - f''(t) g'(t)}{\left((f'(t))^2 + (g'(t))^2\right) \cdot \sqrt{(f'(t))^2 + (g'(t))^2}}$$

$$= \frac{g''(t) f'(t) - f''(t) g'(t)}{\left((f'(t))^2 + (g'(t))^2\right)^{3/2}}$$



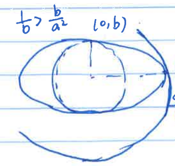
$c = \sqrt{a^2 - b^2}$

$|PF_1| + |PF_2|$: fixed total lengths

$|BF_1| + |BF_2|$

$|F_2A| + |F_1A| = a - c + 2c + (a - c) = 2a$

Fri 7/21

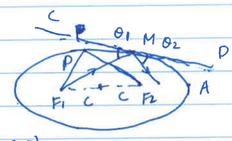


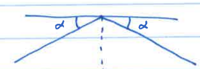
$$\frac{d\theta}{dS} = \frac{1}{R}$$

R, radius

$x = a\cos t$

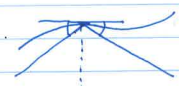
$y = b\sin t$

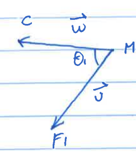




$F_1(-c, 0)$

$|PF_1| + |PF_2|$: fixed constant



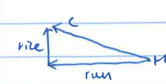


$(a\cos t, b\sin t)$

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos\theta$$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|}$$

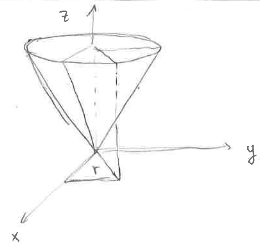
$\vec{v} = ($ $\overrightarrow{MF_1} = (-c - a\cos t, -b\sin t)$



$$\frac{\text{rise}}{\text{run}} = \frac{dy}{dx} = \frac{b\cos t}{-a\sin t}$$

$$\overrightarrow{MC} = (b\cos t, -a\sin t)$$

$$\cos\theta = \frac{(-c - a\cos t) \cdot b\cos t + (-b\sin t)(-a\sin t)}{\sqrt{(-c - a\cos t)^2 + (-b\sin t)^2}\sqrt{(b\cos t)^2 + (-a\sin t)^2}}$$



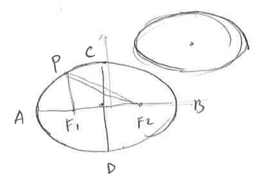
$z = y$

$z = x$

$z = r = \sqrt{x^2+y^2}$

$$\begin{cases} z^2 = x^2+y^2. \\ 2x+y-z=10 \end{cases}$$

$3x^2+y^2-\underline{2xy}+4x+y-7=0$



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Shifted to an ellipse Centered at $(2,3)$

AB: major axis $=2a$

CD: minor

e.g. Find the equation of the ellipse centered at $(2,3)$ and $(8,11)$ with major axis of length 20

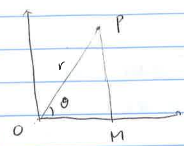
$2a = 20$

$|PF_1| + |PF_2| = 2a$

$\sqrt{(x-2)^2+(y-3)^2} + \sqrt{(x-8)^2+(y-11)^2} = 20$

7/24 Mon

Polar Coordinates



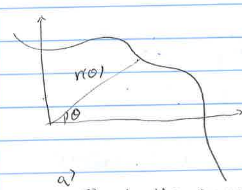
The polar coordinates for P

(r, θ)

$r = |\overrightarrow{OP}|$

$OM = x_p = r\cos\theta$

$PM = y_p = r\sin\theta$

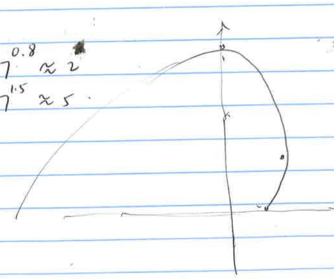


A curve given by ~~coor~~ polar coordinates is a function $r = h(\theta)$

e.g. a) Sketch the curve given by
$r = e^{\theta}$

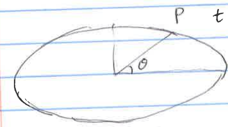
b) find a parametric equation.

θ	$r = e^{\theta}$
0	$e^{0} = 1$
$\frac{\pi}{4}$	$e^{\frac{\pi}{4}} \approx 2.7^{0.8} \approx 2$
$\frac{\pi}{2}$	$e^{\frac{\pi}{2}} \approx 2.7^{1.5} \approx 5$



b) $x = e^{\theta}\cos\theta$

$y = e^{\theta}\sin\theta$



$x = 2\cos t$

$y = \sin t$

$\theta = \frac{\pi}{4}$

$x_p = y_p$

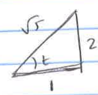
$2\cos t = \sin t$

$2 = \tan t$

$t = \tan^{-1} 2$

$r = \sqrt{x_p^2 + y_p^2}$

$= \sqrt{(2\cos(\tan^{-1} 2))^2 + (\sin(\tan^{-1} 2))^2}$

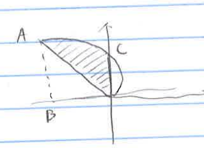


$= \sqrt{\left(2 \cdot \frac{1}{\sqrt{5}}\right)^2 + \dots}$

e.g a) Sketch $r = \theta$

b) Find the enclosed area from $\theta = \frac{\pi}{2}$ to $\theta = \frac{3\pi}{4}$

θ	r
0	0
$\frac{\pi}{4}$	$\frac{\pi}{4}$



$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \theta\, d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} r\, d\theta$$

enclosed Area = Area under the arc $\hat{AC}$ − Area of ABO

Area under $\hat{AC}$. $x = \theta\cos\theta$ $dx = (\theta(-\sin\theta) + \cos\theta)\, d\theta$

$y = \theta\sin\theta$

$$\int y\, dx$$

$$= \int \theta\sin\theta(-\theta\sin\theta + \cos\theta)\, d\theta$$

$$= \int -\theta^2\sin^2\theta \overset{d\theta}{} + \int \theta\sin\theta\cos\theta\, d\theta$$

$$= \int -\theta^2 \cdot \frac{1-\cos 2\theta}{2}\, d\theta + \int \theta \cdot \frac{1}{2}\sin 2\theta\, d\theta$$

$$= \int -\frac{1}{2}\theta^2 d\theta + \ldots *$$

$\int \theta^2 \cos 2\theta$ $u = \theta^2$ $dv = \cos 2\theta\, d\theta$

$du = 2\theta\, d\theta$ $v = \frac{1}{2}\sin 2\theta$

$= \frac{1}{2}\theta^2\sin 2\theta - \int \theta\sin 2\theta\, d\theta$

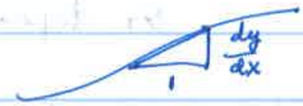
$= \frac{1}{2}\theta^2\sin 2\theta$ $u = \theta$ $dv = \sin 2\theta\, d\theta$

$+ \frac{1}{2}\theta\cos 2\theta - \int \frac{1}{2}\cos\theta\, d\theta$ $du = d\theta$ $v = -\frac{1}{2}\cos 2\theta$

$= \frac{1}{2}\theta^2 \sin$

Wed 7/26

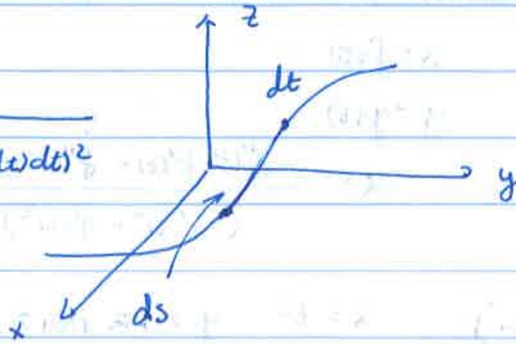
Arc length $s = \int \sqrt{f'(t)^2 + g'(t)^2}\, dt$



$$s = \int \sqrt{dx^2 + dy^2 + dz^2}$$

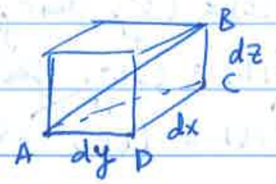
$$= \int \sqrt{(f'(t)dt)^2 + (g'(t)dt)^2 + (h'(t)dt)^2}$$

$$= \int \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt$$



Arc length from 0 to t

$$S(t) = \int_0^t \sqrt{f'(u)^2 + g'(u)^2 + h'(u)^2}\, du$$

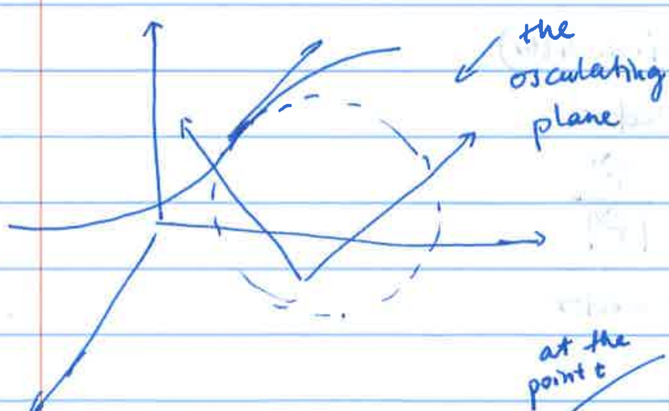


$$AB = \sqrt{AC^2 + BC^2}$$
$$= \sqrt{dy^2 + dx^2 + dz^2}$$

$$\kappa = \frac{d\theta}{ds}$$

$$\frac{dx}{dt} = f'(t)$$

$$dx = f'(t)\,dt$$



Def the osculating plane at the point t is the plane that contains the small arc segment near t

θ: depends on the choice of the coordinate system

Goal: find a formula for κ independent of the coordinate system

$x = f(t)$

$y = g(t)$

$$\kappa = \frac{g''(t) f'(t) - f''(t) g'(t)}{(f'(t)^2 + g'(t)^2)^{3/2}}$$

e.g. $x = t^3$ $y = \cancel{\sec}\ \tan t$

$$\vec{r}(t) = (t^3, \tan t, 0)$$

$$\vec{r}'(t) = (\overset{f'}{3t^2}, \overset{g'}{\sec^2 t}, 0)$$

$$\vec{r}''(t) = (\overset{f''}{6t}, \overset{g''}{2\sec^2 t \tan t}, 0)$$

independent of the coordinate system

$$\kappa = \frac{2\sec^2 t \tan t \cdot 3t^2 - \sec^2 t \cdot 6t}{\left(\sqrt{(3t^2)^2 + (\sec^2 t)^2}\right)^{3}} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

$$\vec{r}' \times \vec{r}'' = (0, 0, 2\sec^2 t \tan t \cdot 3t^2 - \sec^2 t \cdot 6t)$$

$\vec{r}' = (f'(t), g'(t), h'(t))$

: a tangent vector

$\dfrac{d\vec{r}'}{ds}$ $\vec{T} = \dfrac{\vec{r}'}{|\vec{r}'|}$

unit tangent vector

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

Thur 7/27

Arc length from $t=0$ to t

$$S(t) = \int_0^t \sqrt{f'(\lambda)^2 + g'(\lambda)^2 + h'(\lambda)^2}\, d\lambda \qquad \frac{ds}{dt} = \sqrt{f'^2 + g'^2 + h'^2} = |\vec{r}'|$$

$$K = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$K = \left|\frac{d\vec{T}}{ds}\right|$$

$$\vec{r}(t) = (\ln(\cos t), \sin t, \cos t)$$

$$\vec{r}'(t) = \left(\frac{1}{\cos t}(-\sin t), \cos t, -\sin t\right)$$

$$= (-\tan t, \cos t, -\sin t)$$

$$\vec{r}''(t) = (-\sec^2 t, -\sin t, -\cos t)$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\tan t & \cos t & -\sin t \\ -\sec^2 t & -\sin t & -\cos t \end{vmatrix} = +\left(\cos t(-\cos t) - (-\sin t)(-\sin t)\right)\vec{i} - \left((-\tan t)(-\cos t) - (-\sin t)(-\sec^2 t)\right)\vec{j} + \left(-\tan t(-\sin t) - \cos t(-\sec^2 t)\right)\vec{k}$$

$$= \left(-\cos^2 t - \sin^2 t, \; -\frac{\sin t}{\cos t}\cos t + \sin t \cdot \sec^2 t, \; \sin t \cdot \frac{\sin t}{\cos t} + \cos t \sec^2 t\right)$$

$$= \left(-1, -\sin t + \sin t \sec^2 t, \frac{\sin^2 t}{\cos t} + \sec t\right)$$

$$= \left(-1, \sin t \tan^2 t, \frac{\sin^2 t}{\cos t} + \frac{1}{\cos t}\right)$$

$$\left(\frac{\sin^2 t}{\cos t} + \frac{1}{\cos t}\right)\left(\frac{\sin^2 t}{\cos t} + \frac{1}{\cos t}\right)$$

$$|\vec{r}'|^3 = \sqrt{\tan^2 t + \cos^2 t + \sin^2 t}^{\,3} = \sqrt{\tan^2 t + 1}^{\,3} = \sqrt{\sec^2 t}^{\,3} = \begin{cases} \sec^3 t \\ (-\sec t)^3 \end{cases}$$

$$\kappa = \cos^3 t \cdot \sqrt{1 + \frac{\sin^6 t}{\cos^4 t} + \frac{\sin^4 t}{\cos^2 t} + \frac{1}{\cos^2 t} + 2 \cdot \frac{\sin^2 t}{\cos^2 t}}$$

$(A+B)(A+B)$

$$\vec{T} = \frac{1}{\sec t}(-\tan t, \cos t, -\sin t)$$
$$= \left(-\cos t \cdot \frac{\sin t}{\cos t}, \cos^2 t, -\sin t \cos t\right)$$
$$= (-\sin t, \cos^2 t, -\sin t \cos t)$$

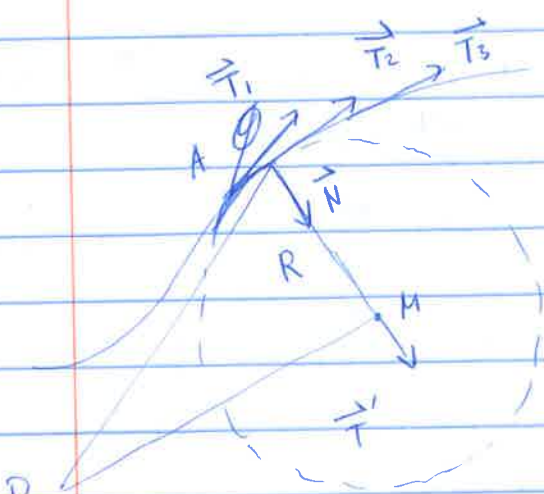
$$\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = \left(\frac{-\cos t}{\cancel{\sec t}\frac{ds}{dt}}, \frac{-2\cos t \sin t}{\frac{ds}{dt}}, \frac{-\left(\cos^2 t + \sin t(-\sin t)\right)}{\frac{ds}{dt}}\right)$$

$$= \frac{1}{\sec t}\left(-\cos t, -2\cos t \sin t, -\cos^2 t + \sin^2 t\right)$$

$$\frac{d\vec{T}}{ds} = \left(\frac{d(-\sin t)}{ds}, \frac{d(\cos^2 t)}{ds}, \frac{d(-\sin t \cos t)}{ds}\right)$$

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d(-\sin t)}{ds} = \frac{\frac{d(-\sin t)}{dt}}{\frac{ds}{dt}}$$



$$\kappa = \frac{1}{R}$$

$\vec{T}$ stays in the same plane

$|\vec{T}|=1$

$\vec{T}' \perp \vec{T}$

$\vec{N} = \dfrac{\vec{T}'}{|\vec{T}'|}$: normal vector

$\vec{B} = \vec{T} \times \vec{N}$ is normal to the osculating plane

the binormal vector (is normal to the osculating plane)

Fri 7/28

$$\theta = \tan^{-1}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \tan^{-1}\left(\frac{e^t\cos t + e^t\sin t}{-e^t\sin t + e^t\cos t}\right)$$

$$\theta'(t) = \frac{1}{1+(\quad)^2}\cdot(\text{quotient rule})$$

$$\tan^{-1}(x)' = \frac{1}{1+x^2}$$

$$S(t) = \int_0^t \sqrt{y'^2 + x'^2}\, dt$$

$$= \int_0^t \sqrt{2e^{2t}(\cos^2 t + \sin^2 t)}\, dt$$

$$= \int_0^t \sqrt{2e^{2t}}\, dt = \sqrt{2}e^t\Big|_0^t = \sqrt{2}(e^t - 1)$$

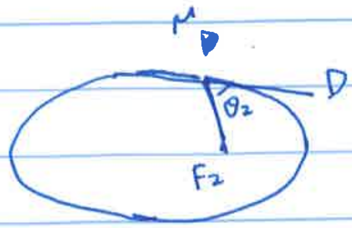
$$s'(t) = \sqrt{2}e^t$$

$$\theta = \tan^{-1}(h'(x))$$

$$\frac{d\theta}{dx} = \frac{1}{1+h'(x)^2}\cdot h''(x)$$

$$\frac{ds}{dx} = \sqrt{1+h'(x)^2}$$

$$\frac{d\theta}{ds} = \frac{\frac{d\theta}{dx}}{\frac{ds}{dx}} = \frac{\frac{1}{1+h'(x)^2}h''(x)}{\sqrt{1+h'(x)^2}}$$



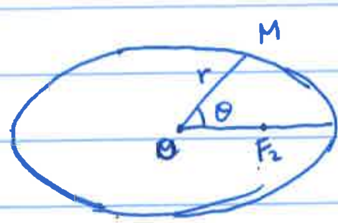
$$\cos\theta_2 = \frac{\vec{MD}\cdot\vec{MF_2}}{|\vec{MD}|\cdot|\vec{MF_2}|}$$

$$\vec{MD} = (+2\sin\theta, -\cos\theta)$$

$$\vec{MF_2} = (c-2\cos t, -\sin t)$$

$F_2: (c, 0)$

$M: (2\cos t, \sin t)$



$$\cos\theta = \frac{\vec{OM}\cdot\vec{OF_2}}{|\vec{OM}|\cdot|\vec{OF_2}|}$$

$$\vec{OM} = (a\cos t, b\sin t)$$

$$\vec{OF_2} = (c, 0)$$

$$= \frac{a\cos t}{\sqrt{a^2\cos^2 t + b^2\sin^2 t}\cdot c}$$