

The Cantor Ternary Set:

The Cantor ternary set is a subset of the real interval $S_0 = [0, 1]$. We use a recursive definition to define the Cantor ternary set. To form S_{i+1} from S_i , we remove the open interval which is the middle third of each of the intervals comprising S_i . Explicitly, we have:

$$\begin{aligned}S_0 &= [0, 1] \\S_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\S_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\end{aligned}$$

etc...

Let $S = \bigcap_{i=0}^{\infty} S_i$, which we call the Cantor ternary set.

1. List five elements of the set S .
2. Find the sum of the lengths of the intervals which are removed to construct S . (This demonstrates that S is an example of a set of *measure zero*.)

Another way to view the Cantor ternary set is in terms of ternary expansions. Given $x \in (0, 1)$ a real number, there is a sequence of integers $(a_i)_{i=1}^{\infty}$, $a_i \in \{0, 1, 2\}$ such that the series

$$\sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

converges to x . In other words, we can write x in a ternary (base 3) form:

$$x = 0.a_1a_2a_3\dots$$

This ternary form exists for all $x \in (0, 1)$. Just like the decimal expansion, the ternary expansion is unique except when x is of the form $\frac{q}{3^i}$, $q \in \mathbb{Z}^+$, in which case there are exactly two such sequences converging to x . For example

$$\frac{1}{9} = (0.01000\dots)_3 = (0.002222\dots)_3.$$

3. Prove that the Cantor ternary set is equal to the subset of $[0, 1]$ consisting of all x which have a ternary expansion for which $a_i \in \{0, 2\}$ for all i , i.e. the numbers which have an expansion with no 1's. (Read this carefully in the cases where x has two possible expansions. If x has one ternary expansion which contains no 1's, then it is in the Cantor set.)
4. Show that the Cantor ternary set S has the same cardinality as the real numbers by finding a bijection between S and the interval $[0, 1]$.
5. Prove that if $a, b \in S$ with $a < b$, then there exists a real number $r \notin S$ such that $a < r < b$.
6. (The Cantor ternary function.) Given any $x \in [0, 1]$, we have discussed how x has a ternary expansion

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \quad a_i \in \{0, 1, 2\}.$$

For each x , we first assign an N as follows: if none of the a_i are 1, then $N = \infty$, otherwise let N be the smallest index such that $a_N = 1$. Next, let $b_n = \frac{1}{2}a_n$ for all $n \leq N$ and let $b_N = 1$. Define the function

$$f(x) = \sum_{n=1}^N \frac{b_n}{2^n}.$$

1. Prove that this definition of f is well-defined, i.e. if x has two ternary expansions, they both yield the same value for $f(x)$.
2. Prove that f is continuous and increasing on $[0, 1]$.
3. Prove that f is constant on each interval from the complement of the Cantor ternary set C .
4. Prove that f maps the Cantor ternary set C onto the interval $[0, 1]$. (This is another proof that the Cantor set is uncountable.)

References:

Royden, H.L. *Real Analysis* Third Ed., 1988, Macmillan Publishing Company.

Steen, L.A. and Seebach, Jr., J.A. *Counterexamples in Topology*, Dover Publications, 1978.