

Handout on e
MATH 4433 Introduction to Analysis

You have proved in class that the sequence

$$a_j = \left(1 + \frac{1}{j}\right)^j$$

is increasing and bounded above, so it converges by the Monotone Convergence Theorem (Prop. 3.3). We denote its limit, Euler's constant, by the symbol e .

You also proved that the sequence

$$b_j = \left(1 + \frac{2}{j}\right)^j$$

converges, and in fact, it converges to e^2 .

The goal is now to prove (Homework exercise # 14) that for any real number x , the sequence

$$c_j = \left(1 + \frac{x}{j}\right)^j$$

converges to e^x . The following is an outline of the steps you will need.

1. Prove that if a sequence $\{d_j\}$ has $d_j > 0$ for all j and converges to a limit L , then $\{d_j^n\}$ converges to L^n , where n is any natural number. (Hint: product rule of limits.)
2. Prove that if a sequence $\{d_j\}$ has $d_j > 0$ for all j and converges to a limit L , then $\{d_j^{\frac{1}{n}}\}$ converges to $L^{\frac{1}{n}}$, where n is any natural number. (You have done this for $n = 2$ already.)
3. Given r a positive rational number, prove that

$$\lim_{j \rightarrow \infty} \left(1 + \frac{r}{j}\right)^j = e^r.$$

(Hint: given $r = \frac{p}{q}$, show that the subsequence indexed by multiples of p converges to e^r . What else do you need to show that the sequence itself converges to e^r ?)

4. Next, given any $x > 0$, show that the limit of

$$\left(1 + \frac{x}{j}\right)^j$$

exists by showing it is monotone increasing and bounded above. (This will be similar to what you did to show the original sequence $\{a_j\}$ above converged to e .)

5. Using the definition of e^x from the book (p. 89):

$$e^x = \sup\{e^r : r \in \mathbb{Q}, r < x\},$$

show that

$$\lim_{j \rightarrow \infty} \left(1 + \frac{x}{j}\right)^j = e^x.$$