Homework #12 Problems
MATH 4433 Introduction to Analysis

1. Let f be an increasing **bounded** function on an open bounded nonempty interval $I = (a, b)$.

a) Prove that the one-sided limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ both exist as finite numbers.

b) Prove that for such a function, f is continuous on (a, b) if and only if f is uniformly continuous on (a, b).

2. Suppose that f is continuous on $[0, 1]$. For a fixed natural number n, define the intervals $I_k = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ for each $k = 1, 2, \ldots, 2^n$. Prove that given any $\epsilon > 0$, there is a natural number N such that for all $n > N$ we have

$$\sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x) < \epsilon \qquad \text{for each } k = 1, 2, \ldots 2^n.$$

3. Use the definition of the derivative to determine where the derivative exists and to compute $f'(x)$ for the functions: $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$.

4. Use the Mean Value Theorem to prove that $\sqrt{1 + 2x} < 1 + x$ for all $x > 0$.