Homework #10 - due Fri. May 7
MATH 4443/5443

1. Prove that

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{\sin\sqrt{x^2+y^2}} & 0 \le \|(x,y)\| \le \pi \\ 0 & (x,y) = (0,0) \end{cases}$$

is not differentiable at $(0,0)$.

2. Let $f_1, f_2, \ldots, f_n$ be continuously differentiable (i.e. C^1) real functions on $(-1,1)$. Define the function $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = f_1(x_1)f_2(x_2)\cdots f_n(x_n)$$

on the cube $I = (-1,1)^n$, where $x = (x_1, x_2, \ldots, x_n)$. Prove that g is differentiable on I.

3. Suppose that $f, g : \mathbb{R} \to \mathbb{R}^m$ are differentiable at a. Use the notation $f'(a)$ to represent the vector derivative of f at a. Suppose that there exists a $\delta > 0$ such that $g(x) \neq 0$ for all $0 < |x - a| < \delta$. If $f(a) = g(a) = 0$ (by which we mean the zero vector $(0,0,\ldots,0)$) and $g'(a) \neq 0$, prove that

$$\lim_{x\to a} \frac{\|f(x)\|}{\|g(x)\|} = \frac{\|f'(a)\|}{\|g'(a)\|}.$$

4. Let T be an $m \times n$ matrix with real entries, representing a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$. Prove that T is differentiable everywhere on $\mathbb{R}^n$ and the total derivative matrix at a is

$$DT(a) = T \qquad \text{for any } a \in \mathbb{R}^n.$$

(This is the analog of the fact that the derivative of the line $f(x) = mx$ is m at every point x!)