Homework #5
MATH 4443/5443

1. Find all values of p such that the following functions $f(x)$ are improperly integrable on the interval I.

a) $f(x) = \dfrac{1}{x^p}$, $I = (1, \infty)$.

b) $f(x) = \dfrac{1}{x^p}$, $I = (0, 1)$.

c) $f(x) = \dfrac{1}{x \log^p x}$, $I = (e, \infty)$.

2. Define the function $L : (0, \infty) \to \mathbb{R}$ by

$$L(x) = \int_1^x \frac{1}{t}\, \mathrm{d}t.$$

a) Prove that L is differentiable and strictly increasing on its domain. Show that $L(1) = 0$ and that $L'(x) = \frac{1}{x}$.

b) Prove that $\lim_{x \to \infty} L(x) = \infty$. Hint: It may help to compute $L(2^n)$ by way of

$$L(2^n) = \sum_{k=1}^{n} \int_{2^{k-1}}^{2^k} \frac{1}{t}\, \mathrm{d}t,$$

and then use a comparison to find the limit as $n \to \infty$.

c) Prove that $L(x^r) = rL(x)$ for all $r \in \mathbb{R}$ using the power rule for derivatives.

d) Prove that $L(xy) = L(x) + L(y)$ for all $x, y \in (0, \infty)$.

3. Use known power series to find the function $f(x)$ with power series $\displaystyle\sum_{k=0}^{\infty} \frac{x^{3k}}{k+1}$. What is the interval of convergence for this power series?