Homework #7 - due Wed., April 14
MATH 4443/5443

1. Let f be a continuous, one-to-one function from a compact metric space (X, d) onto metric space (Y, ρ), (i.e. f is invertible). Prove that $f^{-1} : Y \to X$, which is given by $f(f^{-1}(x)) = x$, is continuous. on Y.

2. Let f be a continuous function from metric space (X, d) to metric space (Y, ρ). Prove that if E is compact in X, then $f(E)$ is compact in Y.

3. Let f be a continuous function from metric space (X, d) to metric space (Y, ρ). Given $E \subseteq X$, prove that $f(\overline{E}) \subseteq \overline{f(E)}$. Give an example where the inclusion is strict.

4. Let f be a continuous function from metric space (X, d) to the real line $(\mathbb{R}, |\cdot|)$. Define

$$Z(f) = \{x \in X \,:\, f(x) = 0$$

to be the *zero set* of f. Prove that $Z(f)$ is a closed set.

5. Let f be a function from a metric space (X, d) to itself. Let $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of open sets in X.

a) Prove that $f\left(\bigcup_{\alpha \in \mathcal{A}} V_\alpha\right) = \bigcup_{\alpha \in \mathcal{A}} f(V_\alpha)$.

b) Prove that $f\left(\bigcap_{\alpha \in \mathcal{A}} V_\alpha\right) \subseteq \bigcap_{\alpha \in \mathcal{A}} f(V_\alpha)$. Give an example where the inclusion is proper.

c) Prove that $f^{-1}\left(\bigcup_{\alpha \in \mathcal{A}} V_\alpha\right) = \bigcup_{\alpha \in \mathcal{A}} f^{-1}(V_\alpha)$.

d) Prove that $f^{-1}\left(\bigcap_{\alpha \in \mathcal{A}} V_\alpha\right) = \bigcap_{\alpha \in \mathcal{A}} f^{-1}(V_\alpha)$.

6. In this problem, we prove that every metric space has a completion. In the same sense that the rationals $\mathbb{Q}$ are contained in a complete metric space $\mathbb{R}$, we can construct a complete metric space which contains (a copy of) a given metric.

Let (X, d) be a metric space and let M be the set of all Cauchy sequences $\{x_j\}$ of elements from X. We define an equivalence relation on M by saying $\{x_j\}$ and $\{y_j\}$ are equivalent if $\lim_{j\to\infty} d(x_j, y_j) = 0$. Let $\widehat{M}$ be the set of all equivalence classes $\hat{x}$ under this relation.

a) Prove that the relation defined above is an equivalence relation.

b) Define $\hat{d}(\hat{x}, \hat{y}) = \lim_{j\to\infty} d(x_j, y_j)$, where $\{x_j\}$ is a representative in the equivalence class $\hat{x}$ and $\{y_j\}$ is a representative in the equivalence class $\hat{y}$. Prove that this limit exists and is well-defined, i.e. does not depend on the choice of representatives $\{x_j\}$ and $\{y_j\}$.

c) Show that $\hat{d}$ is a metric on $\widehat{M}$.

d) Define the map $i : X \to \widehat{M}$ which maps element $x \in X$ to the equivalence class which contains the constant sequence $\{x, x, x, \ldots\}$. Prove that $d(x, y) = \hat{d}(i(x), i(y))$ for all $x, y \in X$. In this sense, we can think of $i(X)$ as a copy of X contained in $\widehat{M}$.

e) Prove that $i(X)$ is dense in $\widehat{M}$, i.e. every $\hat{x} \in \widehat{M}$ is either an element of $i(X)$ or a limit point of $i(X)$.

f) Prove that $(\widehat{M}, \hat{d})$ is complete. We call this the completion of (X, d).