

Homework #9 - due Wed., April 28  
MATH 4443/5443

1. (Follow-up from last week.) Let  $E$  be a compact subset of metric space  $(X, d)$  and let  $f, g : E \rightarrow \mathbb{R}^n$  be uniformly continuous on  $E$ . Prove that the function  $f \cdot g$  is uniformly continuous on  $E$ , where  $f \cdot g : E \rightarrow \mathbb{R}$  is the real-valued function which gives the dot product of  $f(x)$  and  $g(x)$ .

2. Suppose that functions  $\{f_k\}_{k=1}^{\infty}$  are continuous real functions that map  $[a, b] \rightarrow [0, \infty)$ . Suppose that

$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$

converges pointwise on  $[a, b]$ . If  $f$  is continuous on  $[a, b]$ , prove that

$$\int_a^b \sum_{k=1}^{\infty} f_k(x) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x) dx.$$

3. Let  $E$  be a closed bounded subset of  $\mathbb{R}^n$ . Suppose the functions  $g, f_k, g_k$  are continuous functions from  $E$  to  $[0, \infty)$  for each  $k$ . Also, suppose  $\{f_k\}$  is pointwise monotone decreasing on  $E$ . If  $g = \sum_{k=1}^{\infty} g_k$  converges pointwise on  $E$ , prove that  $\sum_{k=1}^{\infty} f_k g_k$  converges uniformly on  $E$ .

4. Prove that  $\{(1 - \frac{x}{k})^k\}_{k=1}^{\infty}$  converges uniformly to  $e^{-x}$  on any closed bounded subset in  $\mathbb{R}$ . (Hint: When we first learned this limit for fixed  $x$ , we made use of the monotonicity of the sequence, which arose from the Binomial Theorem. See Sec. 3.4 in Krantz.)

5. For the example shown in class, show that the first partial derivatives of  $f$  exist on all  $\mathbb{R}^2$  and are continuous on  $\mathbb{R}^2$  (especially at  $(0, 0)$ ). Compute the mixed second-order partial derivatives. Explain why they are not equal at  $(0, 0)$ .

$$f(x, y) = \begin{cases} xy \left( \frac{x^2 - y^2}{x^2 + y^2} \right) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

6. Evaluate  $\lim_{y \rightarrow 0} \int_0^1 e^{x^3 y^2 + x} dx$ .