Ternary Expansions and the Cantor Set

Geometric Series: Recall a geometric series, which converges for all $|a| < 1$:

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

This is a result of the telescoping formula

$$(1-a)(1+a+a^2+\cdots+a^{N-1}) = 1-a^N$$

When $|a| < 1$, taking the limit as N goes to infinity gives the geometric series formula.

We consider the special case where $a = \dfrac{1}{p}$ for p a natural number greater than or equal to 2. When $p = 2$ we just have the familiar series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

This series also has the cool property that every tail of the series sums to exactly the previous term before the tail:

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\ \frac{1}{2} &= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \\ \frac{1}{2^N} &= \frac{1}{2^{N+1}} + \frac{1}{2^{N+2}} + \frac{1}{2^{N+3}} \cdots \end{aligned}$$

This property is not unique, however. Let $a = \frac{1}{p}$. Prove that

$$\sum_{k=N+1}^{\infty} \frac{p-1}{p^k} = \frac{1}{p^N} \tag{1}$$

for any $p \in \mathbb{N}, p \geq 2$.

Note that, in particular, $\displaystyle\sum_{k=1}^{\infty} \frac{p-1}{p^k} = 1$, regardless of the choice of p.

Expansions of real numbers

Given $p \geq 2$, take any series of the form

$$\sum_{k=1}^{\infty} \frac{a_k}{p^k}, \qquad a_k \in \{0, 1, \ldots, p-1\}.$$

Prove that this series converges to a real number in $[0, 1]$.

Consider the space of all sequences $\{a_k\}_{k=1}^{\infty}$ where $a_k \in \{0, 1, \ldots, p-1\}$. The map

$$F : \{a_k\}_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} \frac{a_k}{p^k} \tag{2}$$

is therefore a well-defined map from the space of such sequences to $[0, 1]$. We observe that F is not quite injective. If $x \in [0, 1]$ is of the form $\frac{q}{p^N}$ where $q \in \mathbb{N}, q < P^N$, show that there are two different sequences $\{a_k\}_{k=1}^{\infty}$ that F maps to x.

Next, we want to prove that F is surjective. Given any $x \in [0, 1]$, prove (constructively) that there exists a sequence $\{a_k\}_{k=1}^{\infty}$ such that $a_k \in \{0, 1, \ldots, p-1\}$ for all k and

$$\sum_{k=1}^{\infty} \frac{a_k}{p^k} = x.$$

Hint: observe that since the series consists of positive terms, the partial sums form a strictly increasing sequence. Also, use Equation (1) above.

You can now describe a bijection between the interval $[0, 1]$ and the space of sequences $\{a_k\}_{k=1}^{\infty}$, $a_k \in \{0, 1, \ldots, p-1\}$. When $p = 10$, the sequence is just exactly the decimal expansion of the number x, for example,

$$x = 0.141592... = \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \cdots;$$

when $p = 2$ the sequence is called the *binary* expansion; when $p = 3$, it is the *ternary* expansion.

The Cantor Set

One way to view the Cantor ternary set is in terms of ternary expansions. Given $x \in [0, 1]$, there is a sequence of integers $\{a_k\}_{k=1}^{\infty}$, $a_k \in \{0, 1, 2\}$ such that the series

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

converges to x. In other words, we can associate x to the ternary sequence

$$\{a_1, a_2, a_3 \ldots\}, a_k \in \{0, 1, 2\}.$$

Prove that the Cantor ternary set is equal to the subset of $[0, 1]$ consisting of all x which have a ternary expansion for which $a_k \in \{0, 2\}$ for all k, i.e. the numbers which have an expansion with no 1's. (Read this carefully in the cases where x has two possible expansions. If x has one ternary expansion which contains no 1's, then it is in the Cantor set.)

Prove that the map we defined in class:

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad b_k = \frac{a_k}{2}$$

maps the Cantor ternary set C onto $[0, 1]$, hence proving C is uncountable.

Prove that if $a, b \in C$, the Cantor ternary set, with $a < b$, then there exists a real number $r \notin C$ such that $a < r < b$. In other words, the Cantor set contains no intervals.