Here are some supplementary practice problems for the final exam. **Warning: they are not meant to be a comprehensive set of review questions.** In particular, I tried to avoid too much repetition from Exams 1 and 2, the review problems for Exam 2, and the final homework (HW 7). You should definitely make sure you are comfortable with those problems as well.

1. Give an example of a sentence which is not a statement.

   Discussed in class (e.g., "This sentence is false.")

2. Prove or disprove: every function from $A = \{1, 2\}$ to $B = \{1, 2, 3\}$ is injective.

   Discussed in class (false, give counterexample)

3. Prove or disprove: no function from $A = \{1, 2\}$ to $B = \{1, 2, 3\}$ is surjective.

   Discussed in class (true, pigeonhole principle)

4. Show $x^2 = 1 - x^4$ has no solutions in $\mathbb{Z}$.

   Discussed in class (think contradiction—suggestion, use positivity of x^2)

5. Show that if x is irrational, so is $\sqrt{x}$.

   Discussed in class (think contrapositive)

6. Prove that an integer is divisible by 2 if and only if its last digit is.

   Discussed in class (remember, for if and only if proofs, prove one direction then the other)

7. How many relations are there on $\{a, b, c, d\}$?

   Discussed in class—the number of relations on a set A is the size of the power set of $A \times A$

8. How many reflexive relations are there on $\{a, b, c, d\}$?

   Discussed in class—count the subsets of $A \times A$ which contain $\{(a, a), (b, b), (c, c), (d, d)\}$—you should get 2^{12}

9. How many equivalence relations are there on $\{a, b, c, d\}$? (For in class, just do this for $\{a, b, c\}$.)

   Discussed in class—count partitions, for $\{a, b, c\}$ you should get $1 + 3 + 1 = 5$

10. What is the coefficient of $x^{97}y^3$ in $(x + y)^{100}$?

    Use binomial theorem—$\binom{97}{3} = (97 \cdot 96 \cdot 95)/6$

11. Negate the statement: if $x^2 > 1$, then $x > 1$. Is this statement or its negative true?

    One way to state the negation: There exists $x \leq 1$ such that $x^2 > 1$. The negation is true—e.g., take $x = -2$.

12. Let A, B be sets in a universal set X. Prove or disprove: $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

False: e.g., take $X = \mathbb{R}$, $A = (-\infty, 0)$, $B = (0, \infty)$. Then $\overline{A} \cup \overline{B} = \mathbb{R}$, but $\overline{A \cup B} = \{0\}$. However something close is true: $\overline{A} \cup \overline{B} = \overline{A \cap B}$. This is like one of DeMorgan's laws, but for sets.

13. Show $a \in \mathbb{Z}$ is odd if and only if $a^2 + 2a + 3$ is even.

Similar to Ch7, Ex 1–5. Again, prove one direction then the other. For $\implies$, assume $a = 2k + 1$. May want to use contrapositive for other direction and suppose $a = 2k$ is even. Note: you can realize $a^2 + 2a + 3 = (a + 1)^2 + 2$ to make your arguments easier, but you don't have to.

14. Prove $n^2 \leq n^3$ for all $n \in \mathbb{N}$.

You could do induction, but direct proof is easiest. This is equivalent to $n^3 - n^2 = n^2(n - 1) \geq 0$ for all $n \in \mathbb{N}$. Since $n^2 > 0$ and $n \geq 1$, the left hand side is ≥ 0. Alternatively, you could divide by n^2, which you should note is always > 0 (both for division to be defined, and not to flip the inequality).

15. Prove $3^n \geq 2^n + 1$ for all $n \in \mathbb{N}$.

Use induction. See Ch 10, Ex 16.

16. Prove or disprove: if A and B are sets, then $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

See Ch 10, Ex 9.

17. Consider a 5-card hand dealt from a standard 52-card deck. How many hands are there such that:

(a) there are at least 2 cards from 1 suit?

(b) there are at least 2 cards which are clubs?

(c) all cards are clubs?

(d) all cards are clubs but non-consecutive? (a flush in clubs, but not a straight flush—recall if your cards are 2 3 ... 10 J Q K A, then you can think of J as 11, Q as 12, K as 13 and A can be either 1 or 14)

Note that the order in which the cards appear in your hand do not matter.

(a) By the pigeonhole principle, this is all 5-card hands, so there are $\binom{52}{5} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48/120$ of them.

(b) To get the number of hands with exactly k clubs, you choose k out of the 13 clubs, then $5 - k$ out of the 39 non-clubs, which gives $\binom{13}{k} \cdot \binom{39}{5-k}$ possibilities. You could either add this up from $k = 2$ to $k = 5$, or just subtract the $k = 0$ and $k = 1$ values from the total number of hands.

(c) As in (b), this is $\binom{13}{5} = 13 \cdot 11 \cdot 9 = 1287$.

(d) There are 10 possibilities for the lowest numerical value of a card in a straight, and this number together with a suit completely determines a hand that is a straight flush. Hence from (c) there are $1287 - 10 = 1277$ possibilities.

18. Prove or disprove: if A, B, C, D are sets, then $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

    See Ch 10, Ex 5

19. Prove $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$.

    See Ch 10, Ex 15

20. Give 2 infinite sets with the same cardinality, and 2 infinite sets with different cardinalities. For the 2 sets with the same cardinality, prove they have the same cardinality.

    For 2 sets with the same cardinality, you could do something like $\mathbb{Z}$ and $\mathbb{N}$, or $\mathbb{N} \cup \{0\}$ and $\mathbb{N}$. You prove they have the same cardinality by exhibiting a bijection—for the latter pair the bijection is just given by $f(x) = x + 1$. In general, you should check your function is a bijection.

    For 2 sets with different cardinalities, you could take $\mathbb{Z}$ and $\mathbb{R}$, or $\mathbb{Q}$ and $\mathbb{R}$.

21. Explain Russell's paradox. What does it mean for set theory?

    See Sec 1.10. What it means is you need to be careful about the definition of sets in formal logic. (Just like not every declarative sentence can be made true/false in a consistent way, not every collection of objects you can give a description for can be resolved in a consistent way. Of course the natural mathematical objects we look at, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\{f : \mathbb{R} \to \mathbb{R}\}$, ... are sets, so we don't really have to worry about this stuff when we're doing "everyday mathematics.")