Rotations in $\mathbb{R}^2$

Proposition 1. *Counterclockwise rotation about the origin by θ (in radians) is a linear transformation, and its standard matrix is*

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Proof. First, consider the linear transformation T which acts as rotation by θ CCW on the standard basis $e_1 = (1, 0)$, $e_2 = (0, 1)$. Note e_1 and e_2 lie on the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$. Drawing triangles (see video lecture, or Example 1 in Section 8.1), we see

$$\begin{aligned} Te_1 &= (\cos\theta, \sin\theta) = \cos\theta \cdot e_1 + \sin\theta \cdot e_2 \\ Te_2 &= (\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})) = (-\sin\theta, \cos\theta) = -\sin\theta \cdot e_1 + \cos\theta \cdot e_2. \end{aligned}$$

Thus T has standard matrix $[T] = A$.

To check that our given rotation in the plane actually equals T on the whole plane, it suffices to understand what T does to an arbitrary point $(x, y) \in \mathbb{R}^2$. We compute

$$T(x, y) = x \cdot Te_1 + y \cdot Te_2 = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Again, drawing circles and triangles (see video lecture, or Example 1 in Section 8.1) shows this rotates (x, y) by θ radians CCW in $\mathbb{R}^2$. $\square$

Corollary 1. *Clockwise rotation about the origin by θ (in radians) is a linear transformation, and its standard matrix is*

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Proof. Simply apply the above proposition to compute CCW rotation by $-\theta$, and use the identities $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$. $\square$

Its easy to forget which matrix is clockwise rotation and which matrix is counterclockwise rotation. What I usually do to double check is to test the case $\theta = \frac{\pi}{2}$. E.g., if you plug this in the matrix in Proposition 1, then you get $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Now we compute $Ae_1 = e_2$, which tells you that the matrix in Proposition 1 must be rotation in the CCW direction.

Rotations in $\mathbb{C}$

I've mentioned that sometimes we want to consider vector spaces and matrices over the set of complex numbers $\mathbb{C}$. One reason this is useful is that multiplication of complex number has a nice geometric interpretation.

Recall $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$, and we can view $\mathbb{C}$ either as a 1-dimensional complex vector space or a 2-dimensional real vector space. Here we will view $\mathbb{C}$ as a 2-dimensional real vector space, with "standard basis" $\{1, i\}$. Consider any $z \in \mathbb{C}$. We can write

$$z = x + iy, \qquad (x, y) \in \mathbb{R}^2.$$

The ordered pair (x, y) is called the **Cartesian coordinates** for z.

Proposition 2. *The map $T : \mathbb{C} \to \mathbb{R}^2$ given by $T(x + iy) = (x, y)$ is an isomorphism of real vector spaces.*

Proof. The map T is a linear map sends the standard basis $1, i$ of $\mathbb{C}$ to the standard basis e_1, e_2 of $\mathbb{R}^2$. It is clear that $\ker T = \{0\}$ and the image of T is $\mathbb{R}^2$. $\square$

There is another way to represent complex numbers that is useful for our purposes. Any point z on the complex plane lies on some circle of radius $r \geq 0$ centered at 0. Let θ be such that CCW rotation by θ maps r (i.e., $(r, 0)$ in Cartesian coordinates) to z. (Draw yourself a picture, or see video lecture.) Thus the Cartesian coordinates (x, y) of z are given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix},$$

i.e.

$$z = r(\cos\theta + i\sin\theta) =: re^{i\theta},$$

where we define $e^{i\theta} = \cos\theta + i\sin\theta$ to be the point on the unit circle that is θ radians CCW from $1 \in \mathbb{C}$. The representation $z = re^{i\theta}$ is called the **polar representation** of z, and the parameters r, θ are unique if $r > 0$ and $0 \leq \theta < 2\pi$.

Now multiplication of two complex numbers $z = re^{i\theta}$, $z' = r'e^{i\theta'}$ is simply given by

$$zz' = rr'e^{i(\theta+\theta')}. \tag{1}$$

Since multiplication is commutative, the only thing to check here is that $e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}$. This can be done algebraically using the formulas for $e^{i\theta}, e^{i\theta'}$ together with appropriate trigonometric identities.

Proposition 3. *Let $z = re^{i\theta} \in \mathbb{C}$. Then the map $T : \mathbb{C} \to \mathbb{C}$ given by $T(w) = zw = re^{i\theta}w$ is the linear transformation corresponding to rotation in $\mathbb{C}$ about 0 by θ radians CCW and then radial scaling (outward from 0 in every direction) by r.*

Proof. It is clear that multiplication in $\mathbb{C}$ by a real number $r \geq 0$ acts as radial scaling by r. From (1) we see that multiplication by $e^{i\theta}$ simply acts as rotation by θ CCW. $\square$

In other words, multiplication in $\mathbb{C}$ provides and algebraic way to describe the operations of rotation about 0 and radial scaling in the plane. This perspective on planar rotations—specifically the fact that multiplication by $e^{i\theta}$ acts as rotation by θ—will be helpful when we use eigenvalues to study the geometry of linear transformations in Chapter 14.

Rotations in $\mathbb{R}^3$

Historical remarks:

Before linear algebra was invented, William Rowan Hamilton spend many years trying to develop an algebraic system to describe rotations in 3-space. He initially tried to develop a 3-dimensional system for this, and realized it's impossible, but eventually developed a 4-dimensional system for this called the *quaternions*. This is a 4-dimensional number system $\mathbb{H}$ containing $\mathbb{R}$ and $\mathbb{C}$ with infinitely many square roots of -1 where multiplication is not commutative. It is similar to the space $M_2(\mathbb{R})$ of 2×2 matrices in that both are

4-dimensional real vector spaces, both have well defined multiplication laws for arbitrary elements in the space, and in both cases the multiplication is not in general commutative. One advantage $\mathbb{H}$ has over $M_2(\mathbb{R})$ is that every non-zero element of $\mathbb{H}$ has a multiplicative inverse!

The quaternions were a precursor to the modern perspective of linear algebra, where the standard approach to rotations in $\mathbb{R}^3$ uses 3×3 matrices. We will not cover quaternions in this course, but they are interesting enough that they merit mention (they are briefly discussed in Exercise 6 in Appendix B of the text). The modern approach to linear algebra with vector spaces, linear transformations and matrices has the advantage that the theory applies to $\mathbb{R}^n$, and is applicable to more general linear transformations than just rotations and radial scaling. However, Hamilton's quaternions $\mathbb{H}$ does still have its advantages, and is still used in pure mathematics, engineering and computer science. One reason the quaternions are useful in practic is that they provide a more efficient way to encode rotations (4 real parameters) than 3×3 matrices (9 real parameters).

Now let us consider the problem of representing rotations in $\mathbb{R}^3$. One approach is to directly follow the approach in Proposition 1: use trigonometry to write down what a given rotation does to basis.

Example 1. The matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents rotation in $\mathbb{R}^3$ about the z-axis by θ. Note that CW and CCW don't quite make sense in $\mathbb{R}^3$. We can specify the direction of rotation by saying the positive x-axis rotates toward the positive y-axis. This is easy to see from the $\mathbb{R}^2$ case because this rotation simply sends (x, y, z) to

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z. \end{aligned}$$

Similarly, we can represent rotations about the y-axis and the x-axis respectively by

$$B = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

A basic problem in certain aspects of engineering, robotics and computer graphis is: given two rotations (or more general transformations) how to understand they interact with each other—specifically, what happens if you do one and then the other? This reduces to a question about composition of linear transformations, which you can answer at least in terms of formulas by doing matrix multiplication, provided you know how to write down matrix representations for your transformations.

In $\mathbb{R}^2$ it is not hard to see that rotations commute: doing rotation by θ then rotation by ϕ is the same as doing rotation by ϕ and rotation by θ. This is true because there is only one "axis" to rotate about in $\mathbb{R}^2$ (it is not true if you consider rotations about different points in $\mathbb{R}^2$, which will not be linear transformations). However, in $\mathbb{R}^3$ if you rotate about two different axes then in general the order in which you do the rotations matters.

Example 2. Let us consider the rotation matrices A and B about the z- and y-axes from the previous example, both with $\theta = \frac{\pi}{2}$, and see what happens when we do one rotation and then the other.

First, consider rotation about the y-axis and then rotation about the z-axis. This transformation has matrix

$$AB = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Next, consider doing first rotation about the z-axis and then about they y-axis. This transformation has matrix

$$BA = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

So the order in which you do these rotations matters.

To get a sense of the geometry of the compositions from Example 2, note that

$$AB \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -z \\ x \end{pmatrix}, \qquad BA \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ x \\ -y \end{pmatrix}. \tag{2}$$

From these formulas it's not clear if the resulting transformations are rotations, reflections or something else.

It should at least be clear that doing rotations preserves distances and angles, so the composition must also preserve distances and angles. Such transformations are called **isometries** (defined in Chapter 15, which we will not have time for), and it turns out that every (linear) isometry in $\mathbb{R}^2$ or $\mathbb{R}^3$ is either a rotation or a reflection. There is a notion of orientation in $\mathbb{R}^3$, and one can use this to show that the composition of rotations is never a reflection. This leads to the following theorem, which we will not prove (but the result should be obvious for $\mathbb{R}^2$). To relate the statement of the theorem to linear transformations, we first give a lemma.

Lemma 1. *A rotation in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a linear transformation if and only if it fixes the origin.*

Proof. It is clear that a rotation must fix the origin to be a linear transformation. Now suppose T is a rotation which fixes the origin. If T is a rotation of $\mathbb{R}^2$, then it is a linear transformation by Proposition 1.

So suppose T is a rotation of $\mathbb{R}^3$. Then it is rotation by θ about some axis W, which is a line in $\mathbb{R}^3$. Assume T is a nontrivial rotation (i.e., $\theta \neq 0$—otherwise T is simply the identity transformation, which we know is linear). Then only fixes points on the axis W, so W must be line through the origin. Let V be the plane through the origin which is perpendicular to W. Then T restricted to V is a planar rotation R about the origin in the plane $V \subset \mathbb{R}^3$, and one can deduce R is linear from Proposition 1. Then one can check that T must be the unique linear transformation which acts as the identity on W and acts as R on V. (I have suppressed some details in the last two sentences, but you can work everything out explicitly with trigonometric formulas.) $\square$

Theorem 1. *The composition of two rotations fixing the origin in $\mathbb{R}^2$ or $\mathbb{R}^3$, is again a rotation fixing the origin.*

Proof. Omitted. $\square$

This implies that the compositions AB and BA above are indeed rotations in $\mathbb{R}^3$. Now using the formulas in (2), we can figure out what the axes of these rotations are. The key is that the axis of rotation ℓ will be a line through an origin, which is spanned by some vector $v_0 = (x_0, y_0, z_0)$, and since these are nontrivial rotations only vectors on the axis will be fixed by this rotation.

Hence to determine the axes of rotation for AB, say, it suffices to find a vector v_0 such that $ABv_0 = v_0$, i.e.,

$$AB \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} -y_0 \\ -z_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

This implies $y_0 = -x_0$ and $z_0 = x_0$. Taking $x_0 = 1$, we see that AB is a rotation about the line spanned by $(1, -1, 1)$. Similarly, one can check that BA is a rotation about the line spanned by $(1, 1, -1)$.

With more work, one an also determine the angle and direction of rotation. Explicitly, what you could do is let V be a plane through the origin orthogonal to ℓ. For $\ell = \mathrm{Span}(1, -1, 1)$ this is simply the plane $x - y + z = 0$. Then you can choose a nonzero vector $v \in V$ and see what your rotation does to the vector v. From this, with a little trigonometry, one can determine the angle of rotation. We will not do this, but see below for a cleaner approach.

This idea of looking for vectors or lines which are fixed by linear transformations is a special case of the study of **eigenvectors** and **eigenvalues**, which we will take up in Chapter 14. This theory provides a general method to understanding the geometry of linear transformations, and has many applications.

We have explained how to write down rotations about the x-, y- and z-axes, and their compositions. While, as stated above, you can use some trigonometry to directly write down the formula for any rotation in $\mathbb{R}^3$, it can get a bit technical in general. A cleaner approach using compositions is as follows.

Let us say you want to construct a rotation about some line ℓ through the origin. First, you can find rotations R_x and R_z about the x-axis and z-axis such that $S = R_z R_x$ transforms the z-axis to ℓ. (If you have seen spherical coordinates before, the angles in the spherical coordinates for a non-zero point on ℓ tell you the angles to choose for R_x and R_z.) Let T be a rotation the z-axis by θ. Then a rotation about ℓ by θ is given by:

$$STS^{-1} = R_z R_x T R_x^{-1} R_z^{-1}. \tag{3}$$

This is a special case of change of bases: with S^{-1}, you are first changing coordinates so ℓ becomes the z-axis, then you apply the rotation T about the z-axis in your new coordinate system, then you apply S to go back to your original coordinate system.

Note that $[T]$ is given by the matrix of the form of A in Example 1. If R_x is rotation about the x-axis by ϕ, then R_x^{-1} is simply rotation about the x-axis by $-\phi$, so R_x and R_x^{-1} are both given by matrices of the form of C in Example 1. Similarly, R_z and R_z^{-1} are both given by matrices of the form of A in Example 1. Hence we can use these 3 basic types of rotation matrices from Example 1 to write down an arbitrary rotation in $\mathbb{R}^3$. (Actually, we have only used rotations about the x- and z-axes, but sometimes it is convenient to use the rotations about the y-axis as well.)

Example 3. Let ℓ be the line spanned by $v_0 = (0,1,1)$. Find a matrix that represents rotation by θ about ℓ.

Solution. Note that we can transform the z-axis to ℓ by rotating by $\frac{\pi}{4}$ about the x-axis (rotating in the direction sending the z-axis to the y-axis). Let R_x be this rotation. It has standard matrix

$$C = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} \end{pmatrix}.$$

Note: you should check that indeed

$$C\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix}$$

is on the z-axis to make sure that you are rotating in the right direction, i.e., that $\theta = \frac{\pi}{4}$ rather than $\theta = -\frac{\pi}{4}$ is the right choice when using formula for C in Example 1.

Let A be as in Example 1. Then from (3) we see that rotation by theta about ℓ is given by the matrix

$$\begin{aligned} CAC^{-1} &= \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & -\sqrt{2} & \sqrt{2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2\cos\theta & -\sqrt{2}\sin\theta & -\sqrt{2}\sin\theta \\ \sqrt{2}\sin\theta & \cos\theta + 1 & \cos\theta - 1 \\ \sqrt{2}\sin\theta & \cos\theta - 1 & \cos\theta + 1 \end{pmatrix}. \end{aligned}$$

While we didn't specify the direction of the rotation here, one can do this if desired.

We remark that this idea to relating general rotations to "standard ones" via change of basis can be used to study the problem mentioned earlier: given some rotation T about a line ℓ through the origin in $\mathbb{R}^3$, how do we determine the angle of the rotation? We can use some rotations R_x and R_z to rotation ℓ to the z-axis, and then linear transformation

$$T' = S^{-1}TS, \qquad S = R_zR_x$$

will be a rotation about the z-axis, and hence its matrix will be of the form A from Example 1 for some θ. This θ will be the angle of rotation of T about ℓ.

Takeaway summary

- We can use linear algebra to completely understand rotations and do any necessary calculations (though the precise calculations may be somewhat complicated).
- The ideas of using compositions of linear transformations and change of bases are very powerful tools that are useful for simple geometric questions like: how do I compute rotation about some line in $\mathbb{R}^3$.
- Looking for vectors or lines fixed by a rotation will tell us the axis of that rotation. This idea extends to general linear transformations with the notion of **eigenvectors** and **eigenvalues** that we will study in Chapter 14.