

ON CENTRAL CRITICAL VALUES OF THE DEGREE FOUR L -FUNCTIONS FOR $\mathrm{GSp}(4)$: THE FUNDAMENTAL LEMMA. II

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ABSTRACT. We propose a new relative trace formula concerning the central critical values of the spinor L -functions for $\mathrm{GSp}(4)$. The main result is a proof of the fundamental lemma for the unit element of the Hecke algebra. Our new relative trace formula has some significant advantages over the previous ones for the subsequent development.

1. INTRODUCTION

Investigating special values of automorphic L -functions is a central theme of modern number theory. In particular, central values are of significant interest because of their relevance to the Birch & Swinnerton-Dyer conjecture and its generalizations. In addition to the theta correspondence, the relative trace formula has become a powerful tool in proving explicit formulas for special values.

In [3, Conjectures 1.8, 1.9], two relative trace formulas, which should ultimately lead to a proof of Böcherer's conjecture [1] and its generalization on the central critical values of the spinor L -functions for $\mathrm{GSp}(4)$, were proposed. We refer to [3, Introduction] for the statement of Böcherer's conjecture and to [3, Conjectures 1.10, 1.11] for its generalizations, respectively. The main results [3, Theorems 1.13, 1.14] were the fundamental lemma for the unit element of the Hecke algebra for the regular double cosets for the two conjectural relative trace formulas.

In this manuscript we propose another relative trace formula to tackle the same problem, which will have several advantages over the previous ones. On one side of the trace formula, we take the global distribution related to the Bessel periods of the automorphic forms as in the previous two relative trace formulas. On the other side we take a new global distribution which is inspired by Novodvorsky's integral representation [9] of the L -function for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ (see also Bump [2, Section 3.4] for unfolding and Soudry [11] for non-archimedean local theory). The main result of the current manuscript is a proof of the fundamental lemma for the unit element for the regular orbital integrals.

This article is organized as follows. In the rest of this section, we introduce the new global distribution (1.1) and will state the conjectural relative trace formula (1.3). Then we briefly discuss the special value formula (1.4) which should follow (1.3) and compare our suggested trace formula to those proposed in [3]. In Section 2,

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the geometric decomposition of the global distribution (1.1) will be computed. In Section 3, preliminary evaluations of the regular geometric *local* terms for the unit element of the Hecke algebra will be performed. In Section 4, the final section, the regular geometric terms will be explicitly evaluated and the main result of this article, the fundamental lemma for the unit element of the Hecke algebra for the regular terms, will be proved as Theorem 1 (the inert case) and Theorem 2 (the split case).

Now let us explain our conjectural relative trace formula in some detail.

Notation. Let F be a number field and let \mathbb{A} be its ring of adeles. Let ψ be a non-trivial character of \mathbb{A}/F . Let E be a quadratic extension of F and let \mathbb{A}_E be its ring of adeles. Let $\kappa = \kappa_{E/F}$ denote the quadratic character of $\mathbb{A}^\times/F^\times$ corresponding to the quadratic extension E/F in the sense of class field theory. Let σ denote the unique non-trivial element in $\text{Gal}(E/F)$ and we take $\eta \in E^\times$ such that $\eta^\sigma = -\eta$. Let Ω be a character of $\mathbb{A}_E^\times/E^\times$ and let ω be its restriction to $\mathbb{A}^\times/F^\times$. We denote by Ω' the character of $\mathbb{A}_E^\times/E^\times$ defined by $\Omega'(x) = \Omega(x^\sigma)$.

1.1. Setup.

1.1.1. *GSp(4) and its subgroups.* Let G be the group $\text{GSp}(4)$, an algebraic group over F defined by

$$G = \{g \in \text{GL}(4) \mid {}^t g J g = \lambda(g) J, \lambda(g) \in \mathbb{G}_m\}, \quad \text{where } J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

Here ${}^t g$ denotes the transpose of g and $\lambda(g)$ is called the *similitude factor* of g .

We consider some subgroups of G . Let

$$H = \{(h_1, h_2) \in \text{GL}(2) \times \text{GL}(2) \mid \det h_1 = \det h_2\}.$$

For $h = (h_1, h_2) \in H$, let

$$\iota(h_1, h_2) = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \quad \text{where } h_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

Then we have $\iota(h_1, h_2) \in G$ with $\lambda(\iota(h_1, h_2)) = \det h_1 = \det h_2$. Thus we regard H as a subgroup of G by identifying h with $\iota(h)$. Let N denote the unipotent radical of the standard Borel subgroup of G , i.e. N consists of elements of G of the form

$$u(x, y, z, w) = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y & z \\ 0 & 1 & z & w \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x, y, z, w \in \mathbb{G}_a.$$

1.1.2. *Quaternion similitude unitary groups and subgroups.* For each $\epsilon \in F^\times$, let D_ϵ denote the quaternion algebra over F defined by

$$D_\epsilon = \left\{ \begin{pmatrix} a & b\epsilon \\ b^\sigma & a^\sigma \end{pmatrix} \mid a, b \in E \right\}.$$

We shall identify $a \in E$ with $\begin{pmatrix} a & 0 \\ 0 & a^\sigma \end{pmatrix} \in D_\epsilon$. We recall that $\{D_\epsilon\}_\epsilon$ gives a set of representatives for the isomorphism classes of central simple quaternion algebras

over F containing E when ϵ runs over a set of representatives for $F^\times/\mathbb{N}_{E/F}(E^\times)$. Let $D_\epsilon \ni \alpha \mapsto \bar{\alpha} \in D_\epsilon$ denote the canonical involution of D_ϵ , i.e.

$$\overline{\begin{pmatrix} a & b\epsilon \\ b^\sigma & a^\sigma \end{pmatrix}} = \begin{pmatrix} a^\sigma & -b\epsilon \\ -b & a \end{pmatrix}.$$

We define the *quaternion similitude unitary group* G_ϵ of degree two over D_ϵ to be

$$G_\epsilon = \left\{ g \in \mathrm{GL}(2, D_\epsilon) \mid g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \mu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mu(g) \in \mathbb{G}_m \right\}$$

where $g^* = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$ for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We recall that the G_ϵ 's are inner forms of $G = \mathrm{GSp}(4)$. When $\epsilon = 1$, we have $D_1 \simeq \mathrm{Mat}_{2 \times 2}(F)$ and $G = \alpha G_1 \alpha^{-1}$ in $\mathrm{GL}_4(E)$ where

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ \eta & -\eta & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \eta & -\eta \end{pmatrix}.$$

We define the *upper* (resp. *lower*) *Bessel subgroup* R_ϵ (resp. \bar{R}_ϵ) of G_ϵ by

$$R_\epsilon = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \mid a \in E^\times, X \in D_\epsilon^- \right\},$$

$$\bar{R}_\epsilon = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \mid a \in E^\times, Y \in D_\epsilon^- \right\},$$

where $D_\epsilon^- = \{X \in D_\epsilon \mid X + \bar{X} = 0\}$.

1.2. The special value side.

1.2.1. *Eisenstein series.* For Φ in $\mathcal{S}(\mathbb{A}^2)$, the space of Schwartz-Bruhat functions on \mathbb{A}^2 , and $s \in \mathbb{C}$, the function

$$f_\Phi(g, s) = |\det g|^{s+\frac{1}{2}} \int_{\mathbb{A}^\times} \Phi[(0, t)g] \kappa^{-1}(t) |t|^{2s+1} d^\times t$$

satisfies

$$f_\Phi \left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g, s \right) = \left| \frac{a}{b} \right|^{s+\frac{1}{2}} \kappa(b) f_\Phi(g, s) \quad \text{for} \quad \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in B_2(\mathbb{A}),$$

where B_2 denotes the standard Borel subgroup of $\mathrm{GL}(2)$. Then the corresponding Eisenstein series

$$E_\Phi(g, s) = \sum_{\gamma \in B_2(F) \backslash \mathrm{GL}_2(F)} f_\Phi(\gamma g, s)$$

converges for $\Re(s) > \frac{1}{2}$ and it can be rewritten as

$$E_\Phi(g, s) = |\det g|^{s+\frac{1}{2}} \int_{\mathbb{A}^\times/F^\times} \sum_{\xi \in F^2 \setminus \{(0,0)\}} \Phi(t\xi g) \kappa^{-1}(t) |t|^{2s+1} d^\times t.$$

The Fourier transform $\hat{\Phi}$ of $\Phi \in \mathcal{S}(\mathbb{A}^2)$ is defined by

$$\hat{\Phi}(x, y) = \int_{\mathbb{A}} \int_{\mathbb{A}} \Phi(u, v) \psi^{-1}(xu + yv) du dv$$

where the Haar measure on \mathbb{A} is normalized so that $\int_{\mathbb{A}/F} dx = 1$. By the Poisson summation formula, $E_{\Phi}(g, s)$ extends to an entire function of s and satisfies a functional equation

$$E_{\Phi}(g, s) = E_{\hat{\Phi}}({}^t g^{-1}, -s).$$

1.2.2. *Theta series.* Let (V, q) denote the quadratic space of dimension two over F where $V = E$ and $q(x) = N_{E/F}(x)$. We have

$$\begin{aligned} \mathrm{GO}(V) &= \{h \in \mathrm{GL}(V) \mid q(hx) = \nu(h) \cdot q(x), \forall x \in V\} = E^{\times} \rtimes \mathrm{Gal}(E/F), \\ \mathrm{GSO}(V) &= \{h \in \mathrm{GO}(V) \mid \nu(h) = \det h\} = E^{\times} \end{aligned}$$

where E^{\times} acts on V by multiplication. Then a character χ of $\mathbb{A}_E^{\times}/E^{\times}$ determines a unique irreducible automorphic representation $\Pi(\chi)$ of $\mathrm{GO}(V, \mathbb{A})$. Here we note that $\Pi(\chi) = \Pi(\chi')$ where $\chi'(x) = \chi(x^{\sigma})$. Let

$$r_{\psi} : \mathrm{O}(V, \mathbb{A}) \times \mathrm{SL}_2(\mathbb{A}) \rightarrow \mathrm{Aut}(\mathcal{S}(V(\mathbb{A})))$$

denote the Weil representation corresponding to the additive character ψ . Let

$$G(\kappa) = \{g \in \mathrm{GL}_2(\mathbb{A}) \mid \kappa(\det g) = 1\}, \quad \text{where } \kappa = \kappa_{E/F},$$

and let

$$R = \{(h, g) \in \mathrm{GO}(V, \mathbb{A}) \times G(\kappa) \mid \nu(h) = \det g\}.$$

Following Harris and Kudla [4, Section 3], we may extend r_{ψ} to a representation $r : R \rightarrow \mathrm{Aut}(\mathcal{S}(V(\mathbb{A})))$ by

$$(r(h, g)f)(x) = |\nu(h)|^{-1} \cdot (r_{\psi}(g_1)f)(h^{-1}x)$$

where

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & \det g^{-1} \end{pmatrix} g \in \mathrm{SL}_2(\mathbb{A}).$$

For an automorphic form $\phi \in \Pi(\chi)$, $\Phi' \in \mathcal{S}(V(\mathbb{A}))$ and $g \in G(\kappa)$, let

$$\theta_{\phi}(g, \Phi') = \int_{\mathrm{O}(V, F) \backslash \mathrm{O}(V, \mathbb{A})} \theta(\tau h, g; \Phi') \phi(\tau) d\tau,$$

where $h \in \mathrm{GO}(V, \mathbb{A})$ such that $\nu(h) = \det g$ and

$$\theta(h, g; \Phi') = \sum_{x \in V(F)} (r(h, g)\Phi')(x) \quad \text{for } (h, g) \in R.$$

Then $\theta_{\phi}(\cdot, \Phi')$ is left $\mathrm{GL}_2(F)$ -invariant and we may extend it to the whole of $\mathrm{GL}_2(\mathbb{A})$ by setting it equal to 0 off $G(\kappa)$. We still denote the extension to $\mathrm{GL}_2(\mathbb{A})$ by $\theta_{\phi}(\cdot, \Phi')$. Let $\Theta(\chi)$ denote the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by $\{\theta_{\phi}(\cdot, \Phi') \mid \phi \in \Pi(\chi), \Phi' \in \mathcal{S}(V(\mathbb{A}))\}$.

1.2.3. *The global distribution for the special value side.* Let f be a smooth function on $G(\mathbb{A})$ with compact support. Then we form the kernel function K_f by

$$K_f(x, y) = \int_{Z(F) \backslash Z(\mathbb{A})} \sum_{\gamma \in G(F)} f(x^{-1}\gamma y z) \omega(z) dz$$

where Z denotes the center of G .

For $\Phi \in \mathcal{S}(\mathbb{A}^2)$ and $\theta \in \Theta(\Omega^{-1})$, we define $E : H(\mathbb{A}) \rightarrow \mathbb{C}$ by

$$E(\iota(h_1, h_2)) = E(\iota(h_1, h_2), \Phi, \theta) = E_{\Phi}^*(h_1)\theta(h_2)$$

where

$$E_{\Phi}^*(g) = E_{\Phi}(g, 0).$$

Here we note that $E(zh) = \omega^{-1}(z) E(h)$ for $z \in Z(\mathbb{A})$ and $h \in H(\mathbb{A})$ since the central character of $\Theta(\Omega^{-1})$ is $\omega^{-1} \cdot \kappa$. By abuse of notation, let ψ denote the non-degenerate character of $N(\mathbb{A})$ defined by

$$\psi \left[\begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y & z \\ 0 & 1 & z & w \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] = \psi(x + w).$$

Then we consider the global distribution defined by

$$(1.1) \quad I(f) = I(f, \Phi, \theta) = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} K_f(h, n) E(h) \psi(n) dh dn.$$

1.3. The period side. Let us define a character τ of $R_{\epsilon}(\mathbb{A})$ and a character ξ of $\bar{R}_{\epsilon}(\mathbb{A})$, respectively, by

$$\begin{aligned} \tau \left[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \right] &= \Omega(a) \cdot \psi[\text{tr}(-\eta X)], \\ \xi \left[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \right] &= \Omega'(a) \cdot \psi[\text{tr}(-\eta^{-1}Y)]. \end{aligned}$$

For a smooth function f_{ϵ} on $G_{\epsilon}(\mathbb{A})$ with compact support, we define a kernel function $K_{f_{\epsilon}}$ by

$$K_{f_{\epsilon}}(x, y) = \int_{Z_{\epsilon}(F)\backslash Z_{\epsilon}(\mathbb{A})} \sum_{\gamma \in G_{\epsilon}(F)} f_{\epsilon}(x^{-1}\gamma y z) \omega(z) dz$$

where Z_{ϵ} denotes the center of G_{ϵ} . Then we define a global distribution J_{ϵ} by

$$(1.2) \quad J_{\epsilon}(f_{\epsilon}) = \int_{Z_{\epsilon}(\mathbb{A})\bar{R}_{\epsilon}(F)\backslash \bar{R}_{\epsilon}(\mathbb{A})} \int_{Z_{\epsilon}(\mathbb{A})R_{\epsilon}(F)\backslash R_{\epsilon}(\mathbb{A})} K_{f_{\epsilon}}(\bar{r}, r) \xi(\bar{r})^{-1} \tau(r) d\bar{r} dr.$$

1.4. The relative trace formula. We have geometric decompositions for the distributions $I(f, \Phi, \theta)$ (see below) and $J_{\epsilon}(f_{\epsilon})$ (by double cosets in the usual way). The “regular” geometric terms on the special value side are the distributions $I(s, a, f)$ in (2.5) below. The double coset decomposition for the period side is done in [3], and on this side the “regular” geometric terms will be

$$J_{\epsilon}(u, \mu, f_{\epsilon}) = \int_{Z_{\epsilon}(\mathbb{A})\backslash \bar{R}_{\epsilon}(\mathbb{A})} \int_{Z_{\epsilon}(\mathbb{A})\backslash R_{\epsilon}(\mathbb{A})} \int_{Z_{\epsilon}(\mathbb{A})} f_{\epsilon}(\bar{r}A_{\epsilon}(u, \mu)rz) \xi(\bar{r}) \tau(r) \omega(z) dz d\bar{r} dr$$

where

$$A_{\epsilon}(u, \mu) = \begin{pmatrix} \begin{pmatrix} 1 & \epsilon u \\ u^{\sigma} & 1 \end{pmatrix} & 0 \\ 0 & \mu \begin{pmatrix} 1 & -\epsilon u \\ -u^{\sigma} & 1 \end{pmatrix}^{-1} \end{pmatrix}$$

for $\mu \in F^{\times}$ and $u \in E^{\times}$ such that $\epsilon u u^{\sigma} \neq 1$. We say that (f, Φ, θ) and $\{f_{\epsilon}\}_{\epsilon}$ match on the regular terms if

$$I(s, a, f) = J_{\epsilon}(u, \mu, f_{\epsilon})$$

whenever

$$s = \frac{1 - \epsilon u u^{\sigma}}{4\mu}, \quad a = \frac{1}{1 - \epsilon u u^{\sigma}}.$$

(Of course, one should also consider matching for the remaining geometric terms. But we leave the precise definition of them to our future work.)

Then we conjecture the following relative trace formula.

Conjecture. *Suppose f_ϵ is identically zero for almost all ϵ . If (f, Φ, θ) and $\{f_\epsilon\}_\epsilon$ are matching, then*

$$(1.3) \quad I(f, \Phi, \theta) = \sum_{\epsilon \in F^\times / N_{E/F}(E^\times)} J_\epsilon(f_\epsilon).$$

Note that both sides of (1.3) converge.

Crucial to this matching is knowing the *fundamental lemma* at almost all places, which is a local version of this matching identity (up to a local factor) when f_v and $f_{\epsilon,v}$ are unit elements of the appropriate Hecke algebras. This is what we establish here.

Here let us explain one of the main consequences of the trace formula (1.3). For an automorphic form ϕ on $R_\epsilon(\mathbb{A})$ satisfying $\phi(zr) = \omega(z)\phi(r)$ for $z \in Z_\epsilon(\mathbb{A})$ and $r \in R_\epsilon(\mathbb{A})$, we define the Bessel period $B_{\epsilon, \Omega^{-1}}(\phi)$ by

$$B_{\epsilon, \Omega^{-1}}(\phi) = \int_{Z_\epsilon(\mathbb{A})R_\epsilon(F) \backslash R_\epsilon(\mathbb{A})} \phi(r) \tau^{-1}(r) dr.$$

Then from (1.3), we expect to obtain a special value formula, *roughly* of the form,

$$(1.4) \quad \frac{|W(\varphi)|^2}{\langle \varphi, \varphi \rangle} \cdot L\left(\frac{1}{2}, \pi \otimes \Theta(\Omega^{-1})\right) = \sum_{(\epsilon, \pi_\epsilon)} \frac{|B_{\epsilon, \Omega^{-1}}(\varphi_\epsilon)|^2}{\langle \varphi_\epsilon, \varphi_\epsilon \rangle}.$$

Here the notation is as follows. Let π be a globally generic cuspidal representation of $G(\mathbb{A}) = \mathrm{GSp}_4(\mathbb{A})$ whose central character is ω . Then $W(\varphi)$ denotes a Whittaker–Fourier coefficient of a *new form* φ in the space of π and $\langle \varphi, \varphi \rangle$ is its Petersson norm. The L -function $L(s, \pi \otimes \Theta(\Omega^{-1}))$ is a twist of the degree four L -function of π by $\Theta(\Omega^{-1})$, the representation of $\mathrm{GL}_2(\mathbb{A})$ corresponding to Ω^{-1} . We note that

$$L(s, \pi \otimes \Theta(\Omega^{-1})) = L(s, \mathrm{BC}_{E/F}(\pi) \otimes \Omega^{-1})$$

where $\mathrm{BC}_{E/F}(\pi)$ denotes the base change of π to E . In the summation, (ϵ, π_ϵ) is a pair where $\epsilon \in F^\times / N_{E/F}(E^\times)$, π_ϵ is a cuspidal automorphic representation of $G_\epsilon(\mathbb{A})$ corresponding to π in the functorial sense, and, φ_ϵ is a *test vector* in the space of π_ϵ with $\langle \varphi_\epsilon, \varphi_\epsilon \rangle$ its Petersson norm.

Furthermore in (1.4), when the left hand side is non-zero, we expect only one term on the right hand side to be non-zero because of the local conditions controlled by the ε -factors as in the $\mathrm{GL}(2)$ case. We refer the reader to the recent work of Prasad and Takloo-Bighash [10] concerning the local conditions. We also refer to [10] for the formulation of the conjectural special value formula in question, in the spirit of the important paper of Ichino and Ikeda [5], which is a guiding light for research in this direction.

As we stated in the beginning of the introduction, two conjectural relative trace formulas concerning the special value in question, which are generalizations to $\mathrm{GSp}(4)$ of the relative trace formulas of Jacquet [6, 7], have been introduced in [3]. Jacquet has given another proof of Waldspurger [12] based on these relative trace formulas. Our motivation came from Böcherer’s conjecture [1] on the central critical values of the quadratic twists of the spinor L -functions for Siegel modular forms of degree two, as explained in [3, Introduction].

Our new relative trace formula (1.3) has some significant advantages over the previous ones. First this works without any restriction on the character Ω of \mathbb{A}_E^\times . In contrast, the first trace formula in [3] works only when Ω is trivial. The second one in [3] works with arbitrary Ω but it yields a formula involving the quadratic base change for $\mathrm{GSp}(4)$. In order to obtain a special value formula of the form (1.4), it seems necessary to prove another relative trace formula concerning the quadratic base change for $\mathrm{GSp}(4)$, which establishes a formula, roughly of the form,

$$\frac{|W(\varphi)|^2}{\langle \varphi, \varphi \rangle} = \frac{\overline{W_E(\varphi_E)} \cdot H(\varphi_E)}{\langle \varphi_E, \varphi_E \rangle}$$

where φ_E is a test vector in the space of $\mathrm{BC}_{E/F}(\pi)$, $W_E(\varphi_E)$ denotes a Whittaker-Fourier coefficient of φ_E , and,

$$H(\varphi_E) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \varphi_E(h) (\omega\kappa)(\lambda(h)) dh.$$

This is an interesting problem itself but it surely appears to be another challenging task. Also, preliminary consideration on the $\mathrm{GL}(2)$ case suggests that analytic issues such as truncation are more tractable. For instance, both sides of (1.3) already converge. Even at this unrefined stage, this is no longer true for the trace formulas proposed in [3]. Finally, as we shall see in this manuscript, the orbital integrals are much simpler to handle compared with the previous ones in [3].

The idea to consider another relative trace formula along the lines of (1.3), whose special value side comes from the Rankin-Selberg integral involving Eisenstein series for the L -function in question, was suggested to the first author by Erez Lapid, who kindly provided his personal note [8] on the analogous relative trace formula for $\mathrm{GL}(2)$. We would like to express our deep gratitude to him for generously sharing his ideas with us. Thanks are also due to the referee for some useful comments.

2. DECOMPOSITION OF THE GLOBAL DISTRIBUTION $I(f)$

2.1. Unfolding. By considering the double coset decomposition

$$(2.1) \quad H(F)\backslash G(F)/Z(F)N(F)$$

in the definition of the kernel function, we may rewrite (1.1) as

$$I(f) = \sum_{\gamma \in H(F)\backslash G(F)/Z(F)N(F)} I_\gamma(f)$$

where

$$I_\gamma(f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \sum_{h_0 \in H_\gamma(F)\backslash H(F)} f(h^{-1}h_0^{-1}\gamma zn) \omega(z) E(h) \psi(n) dh dn dz$$

and $H_\gamma = H \cap \gamma(ZN)\gamma^{-1}$. Since there is a bijection

$$H_\gamma(F)\backslash H(F) \approx Z(\mathbb{A})H_\gamma(F)\backslash Z(\mathbb{A})H(F),$$

we may rewrite $I_\gamma(f)$ as

$$I_\gamma(f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{Z(\mathbb{A})H_\gamma(F)\backslash H(\mathbb{A})} f(h^{-1}\gamma zn) \omega(z) E(h) \psi(n) dh dn dz.$$

Let us denote $H \cap \gamma N \gamma^{-1}$ by H_γ^u . Then because of the bijection

$$Z(\mathbb{A}) H_\gamma(F) \backslash H_\gamma(\mathbb{A}) \approx H_\gamma^u(F) \backslash H_\gamma^u(\mathbb{A}),$$

further we have

$$I_\gamma(f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{H_\gamma(\mathbb{A}) \backslash H(\mathbb{A})} \int_{H_\gamma^u(F) \backslash H_\gamma^u(\mathbb{A})} f(h^{-1} u^{-1} \gamma z n) \omega(z) E(uh) \psi(n) du dh dn dz.$$

By a change of variable $n \mapsto (\gamma^{-1} u \gamma) n$, we have

$$(2.2) \quad I_\gamma(f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{H_\gamma(\mathbb{A}) \backslash H(\mathbb{A})} \left(\int_{H_\gamma^u(F) \backslash H_\gamma^u(\mathbb{A})} \psi(\gamma^{-1} u \gamma) E(uh) du \right) f(h^{-1} \gamma z n) \omega(z) \psi(n) dh dn dz.$$

Let us compute the double coset decomposition (2.1) explicitly. Let Q denote the Klingen parabolic subgroup of G , whose Levi decomposition is $Q = M_Q N_Q$ where

$$M_Q = \left\{ \iota \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, h \right) \mid \det h = ab \right\} \subset H, \quad N_Q = \{u(x, y, z, 0) \mid x, y, z \in \mathbb{G}_a\}.$$

The left action of $G(F)$ on F^4 induces a bijection

$$G(F)/Q(F) \approx \mathbb{P}(F^4)$$

where $\mathbb{P}(F^4)$ is the projective space. By the $H(F)$ -orbit decomposition in $\mathbb{P}(F^4)$, we have

$$G(F) = H(F)Q(F) \cup H(F)w_0Q(F) \cup H(F)\bar{u}_1Q(F)$$

where

$$w_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \bar{u}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 1. (1) We have $H(F)Q(F) = H(F)N(F)$.

(2) We have $H(F)w_0Q(F) = H(F)w_0N(F)$.

(3) For $a \in F^\times$, let

$$\bar{u}_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{n}_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & a & 1 & 0 \\ 1 & 0 & 0 & a^{-1} \end{pmatrix}.$$

Then we have

$$H(F)\bar{u}_1Q(F) = \left(\bigcup_{r \in F^\times} H(F)\bar{u}_rN(F) \right) \cup \left(\bigcup_{s \in F^\times} H(F)\bar{n}_sN(F) \right).$$

Proof. The first case is clear since $H(F) \supset M_Q(F)$. As for the second case, since w_0 normalizes $H(F)$, we have

$$H(F)w_0Q(F) = w_0H(F)Q(F) = w_0H(F)N(F) = H(F)w_0N(F).$$

Suppose that $H(F)\gamma N(F) \subset H(F)\bar{u}_1 Q(F)$. Since

$$M_Q(F) = \left\{ \iota \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) \mid a, b \in F^\times \right\} \times \{ \iota(1, h) \mid h \in \mathrm{SL}_2(F) \}$$

and \bar{u}_1 commutes with $\iota \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right)$, we may assume that $\gamma = \bar{u}_1 \iota(1, h)$ where $h \in \mathrm{SL}_2(F)$. By the Bruhat decomposition for $\mathrm{SL}_2(F)$, we may assume further that $\gamma = \bar{u}_1 t_r$ for some $r \in F^\times$ or $\gamma = \bar{u}_1 t_s w_1$ for some $s \in F^\times$, where

$$t_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r^{-1} \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

since \bar{u}_1 commutes with $u(0, 0, 0, w)$. Then we have $\bar{u}_1 t_r = t_r \bar{u}_{r^{-1}}$ where $t_r \in H(F)$ and $\bar{u}_1 t_s w_1 = w_1 \bar{n}_{s^{-1}}$ where $w_1 \in H(F)$. \square

Let us write the integral (2.2) explicitly in each case.

When $\gamma = 1$. Then we have $H_1^u = H \cap N$ and

$$I_1(f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{H_1(\mathbb{A}) \backslash H(\mathbb{A})} f(h^{-1}zn) \omega(z) \psi(n) \left(\int_{(F \backslash \mathbb{A})^2} \psi(w) E \left[\iota \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \right) h \right] dy dw \right) dh dn dz.$$

When $\gamma = w_0$. Then we have $H_{w_0}^u = H \cap w_0 N w_0^{-1} = H_1^u$ and

$$I_{w_0}(f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{H_1(\mathbb{A}) \backslash H(\mathbb{A})} f(h^{-1}w_0zn) \omega(z) \psi(n) \left(\int_{(F \backslash \mathbb{A})^2} \psi(y) E \left[\iota \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \right) h \right] dy dw \right) dh dn dz.$$

We note that $I_{w_0}(f)$ vanishes when θ is cuspidal.

When $\gamma = \bar{u}_r$. Then we have $H_{\bar{u}_r}^u = H \cap \bar{u}_r N \bar{u}_r^{-1} = H_1^u$ and

$$I_{\bar{u}_r}(f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{H_\gamma(\mathbb{A}) \backslash H(\mathbb{A})} f(h^{-1}\bar{u}_r zn) \omega(z) \psi(n) \left(\int_{(F \backslash \mathbb{A})^2} \psi(r^2y + w) E \left[\iota \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \right) h \right] dy dw \right) dh dn dz.$$

When $\gamma = \bar{n}_s$. Let $H_0 = Z H_0^u$ where

$$H_0^u = \left\{ \iota \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \right) \mid y \in \mathbb{G}_a \right\}.$$

We have $H_0 = H \cap \bar{n}_s N \bar{n}_s^{-1}$ for any $s \in F^\times$. Let us denote $I_\gamma(f)$ by $I(s, f)$. Then

$$(2.3) \quad I(s, f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{H_0(\mathbb{A}) \backslash H(\mathbb{A})} f(h^{-1}\bar{n}_s zn) \omega(z) \psi(n) \left(\int_{F \backslash \mathbb{A}} \psi(sy) E \left[\iota \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \right) h \right] dy \right) dh dn dz.$$

Let us utilize the Fourier expansions of $E_{\mathbb{F}}^*$ and θ . For $a \in F$, let

$$W_a^{(1)}(h) = \int_{F \setminus \mathbb{A}} \psi(ax) E_{\mathbb{F}}^* \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h \right) dx$$

and

$$W_a^{(2)}(h) = \int_{F \setminus \mathbb{A}} \psi(ax) \theta \left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} h \right) dx.$$

Then we have

$$\begin{aligned} \int_{F \setminus \mathbb{A}} \psi(sy) E \left[\iota \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} h_1, \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} h_2 \right) \right] dy &= \sum_{a+b=s} W_a^{(1)}(h_1) W_b^{(2)}(h_2) \\ &= \sum_{a \in F} W_{sa}^{(1)}(h_1) W_{s(1-a)}^{(2)}(h_2). \end{aligned}$$

Hence we may rewrite (2.3) as $I(s, f) = \sum_{a \in F} I(s, a, f)$ where

$$(2.4) \quad I(s, a, f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{H_0(\mathbb{A}) \setminus H(\mathbb{A})} f(h^{-1} \bar{n}_s z n) \omega(z) \psi(n) W_{sa}^{(1)}(h_1) W_{s(1-a)}^{(2)}(h_2) dh dn dz.$$

We note that $I(s, 1, f)$ vanishes when θ is cuspidal. For $a \in F^\times$, we have

$$W_a^{(1)}(h_1) = W_1^{(1)} \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h_1 \right), \quad W_a^{(2)}(h_2) = W_1^{(2)} \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} h_2 \right).$$

Hence for $a \in F \setminus \{0, 1\}$, we may write (2.4) as

$$(2.5) \quad I(s, a, f) = \int_{Z(\mathbb{A})} \int_{N(\mathbb{A})} \int_{H_0(\mathbb{A}) \setminus H(\mathbb{A})} f(h^{-1} \bar{n}_s z n) \omega(z) \psi(n) W_1^{(1)} \left(\begin{pmatrix} sa & 0 \\ 0 & 1 \end{pmatrix} h_1 \right) W_1^{(2)} \left(\begin{pmatrix} 1 & 0 \\ 0 & s(1-a) \end{pmatrix} h_2 \right) dh dn dz.$$

3. EVALUATION OF LOCAL ORBITAL INTEGRALS

In the rest of the manuscript, we remain in the local situation and we shall use the following notation.

Notation. Let F be a non-archimedean local field whose residual characteristic is *not equal to two*. Let \mathcal{O} denote the ring of integers in F and ϖ be a prime element of F . Let q denote the cardinality of the residue field $\mathcal{O}/\varpi\mathcal{O}$ and $|\cdot|$ denote the normalized absolute value on F , so that $|\varpi| = q^{-1}$. For $a \in F^\times$, $\text{ord}(a)$ denotes the order of a . Let ψ be an additive character of F of order zero, i.e. ψ is trivial on \mathcal{O} but not on $\varpi^{-1}\mathcal{O}$.

Let E denote either the unique unramified quadratic extension of F , in the inert case, or $F \oplus F$, in the split case. Let $\kappa = \kappa_{E/F}$, i.e. κ is the unique unramified quadratic character of F^\times in the inert case and κ is the trivial character of F^\times in the split case. Let Ω be an unramified character of E^\times and let $\omega = \Omega|_{F^\times}$. Then we may write $\Omega = \delta \circ N_{E/F}$ where δ is an unramified character of F^\times and we have $\omega = \delta^2$.

For a commutative ring A , $M_2(A)$ denotes the set of two by two matrices with entries in A . Let $\text{Sym}^2(A) = \{S \in M_2(A) \mid {}^t S = S\}$.

For an algebraic group \mathbb{G} defined over F , we also write \mathbb{G} for its group of F -rational points.

Whittaker function. Let W denote the $\mathrm{GL}_2(\mathcal{O})$ fixed vector in the Whittaker model, with respect to the upper unipotent subgroup, of the unramified principal series $\pi(1, \kappa)$ of $\mathrm{GL}_2(F)$ which is normalized so that $W(1) = 1$. Thus

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ab & 0 \\ 0 & b \end{pmatrix} k\right) = \psi(-x) \kappa(b) W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

for $k \in \mathrm{GL}_2(\mathcal{O})$. For $a \in F^\times$, let us simply write $W(a)$ for $W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Then it is well known that we have

$$(3.1) \quad W(a) = \begin{cases} |a|^{\frac{1}{2}}, & \text{when } E \text{ is inert and } \mathrm{ord}(a) \in 2\mathbb{Z}_{\geq 0} \\ |a|^{\frac{1}{2}} (1 + \mathrm{ord}(a)), & \text{when } E \text{ splits and } \mathrm{ord}(a) \in \mathbb{Z}_{\geq 0}, \\ 0, & \text{otherwise.} \end{cases}$$

If W' denotes the normalized $\mathrm{GL}_2(\mathcal{O})$ fixed vector in the Whittaker model of $\pi(\delta^{-1}, \delta^{-1} \cdot \kappa)$ with respect to the lower unipotent subgroup, it is easily seen that

$$W'(g) = \delta^{-1} (\det g) W(w_2 g) \quad \text{where } w_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Kloosterman sum. For $r, s \in F^\times$, let

$$\mathcal{Kl}(r, s) = \int_{\mathcal{O}^\times} \psi(r\varepsilon + s\varepsilon^{-1}) d\varepsilon \quad \text{where } \int_{\mathcal{O}^\times} d\varepsilon = 1 - q^{-1}.$$

It is clear from the definition that $\mathcal{Kl}(r, s) = \mathcal{Kl}(s, r)$ and

$$\mathcal{Kl}(r, s) = \mathcal{Kl}(r\xi, s\xi^{-1}) \quad \text{for } \xi \in \mathcal{O}^\times.$$

We also recall the following [3, Proposition 2.7].

- Lemma 2.** (1) If $|r| \leq 1$ and $|s| \leq 1$, then $\mathcal{Kl}(r, s) = 1 - q^{-1}$.
(2) If $\max\{|r|, |s|\} > q$ and $|r| \neq |s|$, then $\mathcal{Kl}(r, s) = 0$.
(3) If $|r| = q$ and $|s| \leq 1$, then $\mathcal{Kl}(r, s) = -q^{-1}$.

Since the following facts will be used often, we record them here as a corollary.

Corollary 1. (1) When $\mathrm{ord}(rs) = -1$, we have

$$\mathcal{Kl}(r, s) = \begin{cases} -q^{-1}, & \text{when } \mathrm{ord}(r) = -1 \text{ or } \mathrm{ord}(s) = -1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) When $\mathrm{ord}(rs) \leq -2$, $\mathcal{Kl}(r, s)$ vanishes unless $\mathrm{ord}(r) = \mathrm{ord}(s)$.

3.1. Local orbital integral. Let Ξ be the characteristic function of the maximal compact subgroup $K = \mathrm{GSp}_4(\mathcal{O})$ of G . Following (2.5), we define the local orbital integral $I(s, a)$ for $s \in F^\times$ and $a \in F \setminus \{0, 1\}$ by

$$(3.2) \quad I(s, a) = \int_Z \int_N \int_{H_0 \setminus H} \Xi\left(\iota(h_1, h_2)^{-1} \bar{n}_s z n\right) \omega(z) \psi(n) \\ \delta^{-1}(s(1-a) \det h_2) W\left(\begin{pmatrix} sa & 0 \\ 0 & 1 \end{pmatrix} h_1\right) W\left(\begin{pmatrix} s(1-a) & 0 \\ 0 & 1 \end{pmatrix} w_2 h_2\right) dh dn dz,$$

where we recall that $H_0 = ZH_0^u$,

$$H_0^u = \left\{ \iota \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \right) \mid y \in F \right\}, \quad \bar{n}_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & s & 1 & 0 \\ 1 & 0 & 0 & s^{-1} \end{pmatrix}.$$

When we consider the Iwasawa decomposition $H = N_H T_H K_H$ where $K_H = H(\mathcal{O})$,

$$N_H = \left\{ \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y & 0 & 1 \end{pmatrix} \mid x, y \in F \right\}, \quad T_H = \left\{ z' \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & bc & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid z', b, c \in F^\times \right\},$$

a Haar measure on H is given by $|c|^2 dn_H dt_H dk_H$. Thus we may rewrite (3.2) as

$$\begin{aligned} I(s, a) &= \int_Z \int_N \int_F \int_{(F^\times)^2} \Xi \left[\iota \left(\begin{pmatrix} b & cx \\ 0 & c \end{pmatrix}, \begin{pmatrix} bc & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} \bar{n}_s z n \right] \\ &\quad \omega(z) \psi(n) \psi(-sax) \delta^{-1}(s(1-a)bc) \kappa(b) \\ &\quad W(sabc^{-1}) W(s(1-a)b^{-1}c^{-1}) |c|^2 d^\times b d^\times c dx dn dz. \end{aligned}$$

Here the similitude

$$\lambda \left[\iota \left(\begin{pmatrix} b & cx \\ 0 & c \end{pmatrix}, \begin{pmatrix} bc & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} \bar{n}_s z n \right] = z^2 b^{-1} c^{-1} \in \mathcal{O}^\times$$

implies $\text{ord}(c) = \text{ord}(z^2 b^{-1})$ and we have

$$\begin{aligned} I(s, a) &= \delta^{-1}(s(1-a)) \int_{F^\times} \int_N \int_F \int_{F^\times} \psi(n) \psi(-sax) \kappa(b) |z^2 b^{-1}|^2 \\ &\quad \Xi \left[\iota \left(\begin{pmatrix} zb^{-1} & -zb^{-1}x \\ 0 & z^{-1}b \end{pmatrix}, \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \right) \bar{n}_s n \right] \\ &\quad W(saz^{-2}b^2) W(s(1-a)z^{-2}) d^\times b dx dn d^\times z. \end{aligned}$$

By a change of variable $b \mapsto zb^{-1}$, we have

$$\begin{aligned} I(s, a) &= \delta^{-1}(s(1-a)) \int_{F^\times} \int_N \int_F \int_{F^\times} \psi(n) \psi(-sax) \kappa(zb) |zb|^2 \\ &\quad \Xi \left[\iota \left(\begin{pmatrix} b & -bx \\ 0 & b^{-1} \end{pmatrix}, \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \right) \bar{n}_s n \right] W(sab^{-2}) W(s(1-a)z^{-2}) d^\times b dx dn d^\times z. \end{aligned}$$

For $b, z \in F^\times$ and $x, y \in F$, let $A(b, z, x, y)$ denote the matrix

$$\iota \left(\begin{pmatrix} b & -bx \\ 0 & b^{-1} \end{pmatrix}, \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \right) \bar{n}_s \begin{pmatrix} 1 & y & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -y & 1 \end{pmatrix} = \begin{pmatrix} b & b(y-sx) & -bx & 0 \\ 0 & sz^{-1} & 0 & 0 \\ 0 & b^{-1}s & b^{-1} & 0 \\ z & yz & -s^{-1}yz & s^{-1}z \end{pmatrix}$$

and let U denote the unipotent radical of the upper Siegel parabolic subgroup of G , i.e.

$$U = \left\{ \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \mid S \in \text{Sym}^2(F) \right\}.$$

Then we have

$$(3.3) \quad I(s, a) = \delta^{-1}(s(1-a)) \int_{F^\times} \int_F \int_U \int_F \int_{F^\times} \psi(u) \psi(y-sax) \kappa(zb) |zb|^2 \\ \Xi [A(b, z, x, y) u] W(sab^{-2}) W(s(1-a)z^{-2}) d^\times b dx dy du d^\times z.$$

Let us examine the support of the integral (3.3).

Lemma 3. *For the matrix $A(b, z, x, y)$, we have*

$$(3.4) \quad A(b, z, x, y) u \in K \quad \text{for some } u \in U$$

if and only if

$$(3.5) \quad \max\{|b|, |z|\} = 1,$$

$$(3.6) \quad \max\{|b(y-sx)|, |sz^{-1}|, |b^{-1}s|, |yz|\} \leq 1,$$

$$(3.7) \quad \max\{|bsz^{-1}|, |sboxz|, |b^{-1}sz|\} = 1.$$

Proof. By Lemma 4.9 in [3], the condition (3.4) is equivalent to (3.5), (3.6) and

$$(3.8) \quad \max\{|bsz^{-1}|, |s|, |sboxz|, |b^{-1}sz|\} = 1.$$

Here we note that $|s|^2 = |bsz^{-1}| \cdot |b^{-1}sz|$ and then (3.7) is clearly equivalent to (3.8). \square

We record here as a separate lemma the following consequence of the observation in the proof of Lemma 3.

Lemma 4. *The local orbital integral $I(s, a)$ vanishes unless $|s| \leq 1$.*

3.2. Evaluation of $I(s, a)$ when $|s| = 1$.

Proposition 1. *When $|s| = 1$, we have*

$$(3.9) \quad I(s, a) = \delta^{-1}(1-a) W(a) W(1-a).$$

In particular $I(s, a)$ vanishes unless $\text{ord}(a) \geq 0$.

Proof. When $|s| = 1$, the condition (3.7) is equivalent to

$$\max\{|bz^{-1}|, |boxz|, |bz^{-1}|^{-1}\} = 1.$$

Hence we have $|bz^{-1}| = 1$. Thus the condition (3.4) is equivalent to $|b| = |z| = 1$, $|x| \leq 1$, $|y| \leq 1$ and then we have $A(b, z, x, y) u \in K$. Hence $A(b, z, x, y) u \in K$ if and only if $u \in U \cap K$. Thus we have

$$I(s, a) = \delta^{-1}(1-a) W(a) W(1-a) \int_{\mathcal{O}} \psi(-sax) dx$$

where $W(a)$ vanishes unless $|a| \leq 1$. Hence the equality (3.9) holds. \square

3.3. Evaluation of $I(s, a)$ when $|s| < 1$. First we put

$$\text{ord}(s) = h, \quad \text{ord}(1-a) = k, \quad \text{ord}(a) = k'.$$

By the condition (3.5), the integral $I(s, a)$ becomes a sum of three integrals $I_j(s, a)$ ($j = 0, 1, 2$), supported on

$$|b| = |z| = 1; \quad |b| = 1 > |z|; \quad |b| < |z| = 1,$$

respectively.

3.3.1. *Evaluation of $I_0(s, a)$.* When $|b| = |z| = 1$, we may assume that $b = z = 1$. For $A(1, 1, x, y)$, the condition (3.4) holds if and only if $|x| = |s|^{-1}$ and $|y| \leq 1$. Then

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A(1, 1, x, y) = \begin{pmatrix} 1 & y - sx & -x & 0 \\ 1 & y & -s^{-1}y & s^{-1} \\ 0 & s & 1 & 0 \\ 0 & -s & 0 & 0 \end{pmatrix}$$

where $\begin{pmatrix} 1 & y - sx \\ 1 & y \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O})$. Hence for $S \in \mathrm{Sym}^2(F)$, we have

$$A(1, 1, x, y) \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if

$$(3.10) \quad \begin{pmatrix} 1 & y - sx \\ 1 & y \end{pmatrix} S + \begin{pmatrix} -x & 0 \\ -s^{-1}y & s^{-1} \end{pmatrix} \in M_2(\mathcal{O})$$

and

$$(3.11) \quad \begin{pmatrix} 0 & s \\ 0 & -s \end{pmatrix} S + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{O}).$$

The condition (3.10) is equivalent to

$$S \in \begin{pmatrix} 2s^{-1}y - s^{-2}x^{-1}y^2 & -s^{-1} + s^{-2}x^{-1}y \\ -s^{-1} + s^{-2}x^{-1}y & -s^{-2}x^{-1} \end{pmatrix} + \mathrm{Sym}^2(\mathcal{O})$$

and this implies (3.11). Hence

$$I_0(s, a) = \delta^{-1}(s(1-a)) W(sa) W(s(1-a)) \int_{s^{-1}\mathcal{O}^\times} \psi(-sax - s^{-2}x^{-1}) dx.$$

Thus

$$(3.12) \quad I_0(s, a) = \delta^{-1}(s(1-a)) W(sa) W(s(1-a)) |s|^{-1} \mathcal{Kl}(a, s^{-1}).$$

Here we note that

$$(3.13) \quad \delta^{-1}(a) I_0(s, a) = \delta^{-1}(1-a) I_0(-s, 1-a)$$

since

$$\mathcal{Kl}(1-a, -s^{-1}) = \mathcal{Kl}(-a, -s^{-1}) = \mathcal{Kl}(a, s^{-1}).$$

Proposition 2. (1) *The integral $I_0(s, a)$ vanishes unless $k = -h$, or, $h = 1$ and $k \geq 0$.*

(2) *When $k = -h$, we have*

$$I_0(s, a) = q^h \mathcal{Kl}(a, s^{-1}).$$

(3) *When $h = 1$ and $k \geq 0$, we have*

$$I_0(s, a) = -\delta(\varpi)^{-k-1} W(\varpi^{1+k'}) W(\varpi^{1+k}).$$

Proof. In (3.12), $W(s(1-a))$ vanishes unless $h+k \geq 0$. When $h > 1$ and $h+k \geq 1$, $\mathcal{Kl}(a, s^{-1})$ vanishes by Lemma 2. The rest is clear from (3.12). \square

3.3.2. *Evaluation of $I_1(s, a)$.* When $|b| = 1$ and $|z| < 1$, we may assume that $b = 1$. Then, for $A(1, z, x, y)$, the condition (3.4) is equivalent to

$$\max\{|sz^{-1}|, |sxz|\} = 1, \quad |y - sx| \leq 1,$$

since $yz = z(y - sx) + sxz$. Splitting the first condition into two separate cases

$$|sz^{-1}| = 1 \geq |sxz| \quad \text{or} \quad |sz^{-1}| < 1 = |sxz|,$$

we may write $I_1(s, a) = I_{1,1}(s, a) + I_{1,2}(s, a)$. Here

$$(3.14) \quad I_{1,1}(s, a) = \delta^{-1}(s(1-a)) W(sa) W(s^{-1}(1-a)) \kappa(s) |s|^2 \\ \int_{y \in \mathcal{O}} \int_U \int_{x \in s^{-2}\mathcal{O}} \Xi[A(1, s, x, y + sx) u] \psi(u) \psi(s(1-a)x) dx du dy.$$

As for $I_{1,2}(s, a)$, further splitting into separate cases according to $\text{ord}(z)$, we have $I_{1,2}(s, a) = \sum_{j=1}^{h-1} I_{1,2}^{(j)}(s, a)$ where

$$(3.15) \quad I_{1,2}^{(j)}(s, a) = \delta^{-1}(s(1-a)) W(sa) W(s(1-a)\varpi^{-2j}) \kappa(\varpi)^j q^{-2j} \\ \int_{y \in \mathcal{O}} \int_U \int_{x \in s^{-1}\varpi^{-j}\mathcal{O}^\times} \Xi[A(1, \varpi^j, x, y + sx) u] \psi(u) \psi(s(1-a)x) dx du dy.$$

In (3.14), for $S \in \text{Sym}^2(F)$, we have

$$A(1, s, x, y + sx) \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} = \begin{pmatrix} 1 & y & -x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ s & s(y + sx) & -y - sx & 1 \end{pmatrix} \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if $S \in \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + \text{Sym}^2(\mathcal{O})$ since $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathcal{O})$. Hence

$$I_{1,1}(s, a) = \delta^{-1}(s(1-a)) W(sa) W(s^{-1}(1-a)) \kappa(s) \int_{\mathcal{O}} \psi(s^{-1}(1-a)x) dx.$$

Since $W(s^{-1}(1-a))$ vanishes unless $|s^{-1}(1-a)| \leq 1$, we have

$$(3.16) \quad I_{1,1}(s, a) = \delta^{-1}(s(1-a)) W(sa) W(s^{-1}(1-a)) \kappa(s).$$

In (3.15), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A(1, \varpi^j, x, y + sx) \\ = \begin{pmatrix} 1 & y & -x & 0 \\ \varpi^j & \varpi^j(y + sx) & -s^{-1}\varpi^j(y + sx) & s^{-1}\varpi^j \\ 0 & s & 1 & 0 \\ 0 & -s\varpi^{-j} & 0 & 0 \end{pmatrix}$$

where $\begin{pmatrix} 1 & y \\ \varpi^j & \varpi^j(y + sx) \end{pmatrix} \in \text{GL}_2(\mathcal{O})$. Hence for $S \in \text{Sym}^2(F)$, we have

$$A(1, \varpi^j, x, y + sx) \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if

$$(3.17) \quad \begin{pmatrix} 1 & y \\ \varpi^j & \varpi^j(y+sx) \end{pmatrix} S + \begin{pmatrix} -x & 0 \\ -s^{-1}\varpi^j(y+sx) & s^{-1}\varpi^j \end{pmatrix} \in M_2(\mathcal{O})$$

and

$$(3.18) \quad \begin{pmatrix} 0 & s \\ 0 & -s\varpi^{-j} \end{pmatrix} S + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{O}).$$

The condition (3.17) is equivalent to

$$S \in -\frac{1}{\varpi^j s x} \begin{pmatrix} s^{-1}\varpi^j(y+sx)(y-sx) & -s^{-1}\varpi^j y \\ -s^{-1}\varpi^j y & s^{-1}\varpi^j \end{pmatrix} + \text{Sym}^2(\mathcal{O})$$

and this implies (3.18). Hence

$$(3.19) \quad I_{1,2}^{(j)}(s, a) = \delta^{-1}(s(1-a)) W(sa) W(s(1-a)\varpi^{-2j}) \kappa(\varpi)^j |s|^{-1} q^{-j} \mathcal{Kl}(\varpi^{-j}(1-a), -s^{-1}\varpi^j).$$

Thus we have

$$(3.20) \quad I_1(s, a) = \delta^{-1}(s(1-a)) W(sa) W(s^{-1}(1-a)) \kappa(s) + \delta^{-1}(s(1-a)) W(sa) |s|^{-1} \sum_{j=1}^{h-1} W(s(1-a)\varpi^{-2j}) \kappa(\varpi)^j q^{-j} \mathcal{Kl}(\varpi^{-j}(1-a), -s^{-1}\varpi^j).$$

Proposition 3. (1) *The integral $I_1(s, a)$ vanishes unless $h+k \geq 2$.*

(2) *When $h=1$, we have*

$$I_1(s, a) = \begin{cases} \delta(\varpi)^{-k-1} W(\varpi) W(\varpi^{k-1}) \kappa(\varpi), & \text{when } k \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(3) *Suppose that $h \geq 2$.*

(a) *When $-h+2 \leq k \leq h-2$ and $h \equiv k \pmod{2}$, we have*

$$I_1(s, a) = \delta(\varpi)^{-h-k} W(\varpi^{h+k'}) \kappa(\varpi)^{\frac{h+k}{2}} q^{\frac{h-k}{2}} \mathcal{Kl}(\varpi^{-\frac{h+k}{2}}(1-a), -s^{-1}\varpi^{\frac{h+k}{2}}).$$

(b) *When $k = h-1$, we have*

$$I_1(s, a) = -\delta(\varpi)^{-2h+1} W(\varpi^h) W(\varpi) \kappa(\varpi)^{h-1}.$$

(c) *When $k \geq h$, we have*

$$I_1(s, a) = \delta(\varpi)^{-h-k} W(\varpi^h) \kappa(\varpi)^h \{W(\varpi^{k-h}) - \kappa(\varpi) W(\varpi^{k-h+2})\}.$$

(d) *The integral $I_1(s, a)$ vanishes otherwise.*

Proof. From (3.16), $I_{1,1}(s, a)$ vanishes unless $-h+k \geq 0$. Similarly from (3.19), $I_{1,2}^{(j)}(s, a)$ vanishes unless $h+k-2j \geq 0$. Hence $I_1(s, a)$ vanishes unless $h+k \geq 2$.

When $k \geq h > 0$, we have $k' = 0$. Hence

$$(3.21) \quad I_{1,1}(s, a) = \begin{cases} \delta(\varpi)^{-h-k} W(\varpi^h) W(\varpi^{k-h}) \kappa(\varpi)^h, & \text{when } k \geq h, \\ 0, & \text{otherwise.} \end{cases}$$

When $h=1$, it is clear from (3.21) that $I_1(s, a)$ is given as above.

Suppose that $h \geq 2$. If $I_{1,2}^{(j)}(s, a) \neq 0$, then we have $h + k - 2j \geq 0$ since $W(s(1-a)\varpi^{-2j}) \neq 0$. Hence $-j + k \geq -h + j$ and by Lemma 2, $I_{1,2}^{(j)}(s, a)$ vanishes unless

$$(3.22) \quad j = h - 1, \quad -(h - 1) + k \geq -1$$

or

$$(3.23) \quad 1 \leq j \leq h - 2, \quad -j + k = -h + j.$$

The former case (3.22) occurs when $k \geq h - 2$ and then

$$(3.24) \quad I_{1,2}^{(h-1)}(s, a) = \delta(\varpi)^{-h-k} W\left(\varpi^{h+k'}\right) W\left(\varpi^{k-h+2}\right) \kappa(\varpi)^{h-1} q \mathcal{Kl}\left(\varpi^{1-h}(1-a), -s^{-1}\varpi^{h-1}\right).$$

The latter case (3.23) occurs when $h \equiv k \pmod{2}$ and $2 - h \leq k \leq h - 4$. Then

$$(3.25) \quad I_{1,2}^{\left(\frac{h+k}{2}\right)}(s, a) = \delta(\varpi)^{-h-k} W\left(\varpi^{h+k'}\right) \kappa(\varpi)^{\frac{h+k}{2}} q^{\frac{h-k}{2}} \mathcal{Kl}\left(\varpi^{-\frac{h+k}{2}}(1-a), -s^{-1}\varpi^{\frac{h+k}{2}}\right).$$

Now the assertion follows from (3.21), (3.24) and (3.25). \square

3.3.3. Evaluation of $I_2(s, a)$. When $|b| < 1$ and $|z| = 1$, we may assume that $z = 1$ and then for $A(b, 1, x, y)$ the condition (3.4) holds if and only if

$$\max\{|sbx|, |sb^{-1}|\} = 1, \quad |y| \leq 1.$$

Dividing the first condition into two separate cases according to

$$|sb^{-1}| = 1 \geq |sbx| \quad \text{or} \quad |sb^{-1}| < 1 = |sbx|,$$

we write $I_2(s, a) = I_{2,1}(s, a) + I_{2,2}(s, a)$ where

$$(3.26) \quad I_{2,1}(s, a) = \delta^{-1}(s(1-a)) W(s^{-1}a) W(s(1-a)) \kappa(s) |s|^2 \int_{y \in \mathcal{O}} \int_U \int_{x \in s^{-2}\mathcal{O}} \Xi[A(s, 1, x, y)u] \psi(u) \psi(-sax) dx du dy,$$

$I_{2,2}(s, a) = \sum_{j=1}^{h-1} I_{2,2}^{(j)}(s, a)$, and,

$$(3.27) \quad I_{2,2}^{(j)}(s, a) = \delta^{-1}(s(1-a)) W(sa\varpi^{-2j}) W(s(1-a)) \kappa(\varpi)^j q^{-2j} \int_{y \in \mathcal{O}} \int_U \int_{x \in s^{-1}\varpi^{-j}\mathcal{O}^\times} \Xi[A(\varpi^j, 1, x, y)u] \psi(u) \psi(-sax) dx du dy.$$

In (3.26), for $S \in \text{Sym}^2(F)$, we have

$$A(s, 1, x, y) \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} = \begin{pmatrix} s & s(y-sx) & -sx & 0 \\ 0 & s & 0 & 0 \\ 0 & 1 & s^{-1} & 0 \\ 1 & y & -s^{-1}y & s^{-1} \end{pmatrix} \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if

$$(3.28) \quad \begin{pmatrix} s & s(y-sx) \\ 0 & s \end{pmatrix} S + \begin{pmatrix} -sx & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{O})$$

and

$$(3.29) \quad \begin{pmatrix} 0 & 1 \\ 1 & y \end{pmatrix} S + \begin{pmatrix} s^{-1} & 0 \\ -s^{-1}y & s^{-1} \end{pmatrix} \in M_2(\mathcal{O}).$$

Since $\begin{pmatrix} 0 & 1 \\ 1 & y \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O})$, the condition (3.29) is equivalent to

$$(3.30) \quad S \in \begin{pmatrix} 2s^{-1}y & -s^{-1} \\ -s^{-1} & 0 \end{pmatrix} + \mathrm{Sym}^2(\mathcal{O})$$

and (3.30) implies (3.28). Hence

$$I_{2,1}(s, a) = \delta^{-1}(s(1-a)) W(s^{-1}a) W(s(1-a)) \kappa(s) \int_{\mathcal{O}} \psi(-s^{-1}ax) dx.$$

Since $W(s^{-1}a)$ vanishes unless $|s^{-1}a| \leq 1$, we have

$$I_{2,1}(s, a) = \delta^{-1}(s(1-a)) W(s^{-1}a) W(s(1-a)) \kappa(s).$$

In (3.27), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A(\varpi^j, 1, x, y) = \begin{pmatrix} \varpi^j & \varpi^j(y-sx) & -\varpi^j x & 0 \\ 1 & y & -s^{-1}y & s^{-1} \\ 0 & s\varpi^{-j} & \varpi^{-j} & 0 \\ 0 & -s & 0 & 0 \end{pmatrix}$$

where $\begin{pmatrix} \varpi^j & \varpi^j(y-sx) \\ 1 & y \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O})$. Hence for $S \in \mathrm{Sym}^2(F)$, we have

$$A(\varpi^j, 1, x, y) \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if

$$(3.31) \quad \begin{pmatrix} \varpi^j & \varpi^j(y-sx) \\ 1 & y \end{pmatrix} S + \begin{pmatrix} -\varpi^j x & 0 \\ -s^{-1}y & s^{-1} \end{pmatrix} \in M_2(\mathcal{O})$$

and

$$(3.32) \quad \begin{pmatrix} 0 & s\varpi^{-j} \\ 0 & -s \end{pmatrix} S + \begin{pmatrix} \varpi^{-j} & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{O}).$$

The condition (3.31) is equivalent to

$$S \in -\frac{1}{\varpi^j sx} \begin{pmatrix} s^{-1}\varpi^j y(y-2sx) & -s^{-1}\varpi^j(y-sx) \\ -s^{-1}\varpi^j(y-sx) & s^{-1}\varpi^j \end{pmatrix} + \mathrm{Sym}^2(\mathcal{O})$$

and this implies (3.32). Hence

$$I_{2,2}^{(j)}(s, a) = \delta^{-1}(s(1-a)) W(sa\varpi^{-2j}) W(s(1-a)) \kappa(\varpi)^j |s|^{-1} q^{-j} \mathcal{Kl}(\varpi^{-j}a, s^{-1}\varpi^j).$$

Thus we have

$$(3.33) \quad I_2(s, a) = \delta^{-1}(s(1-a)) W(s^{-1}a) W(s(1-a)) \kappa(s) + \delta^{-1}(s(1-a)) W(s(1-a)) |s|^{-1} \sum_{j=1}^{h-1} W(sa\varpi^{-2j}) \kappa(\varpi)^j q^{-j} \mathcal{Kl}(\varpi^{-j}a, s^{-1}\varpi^j).$$

By comparing (3.33) with (3.20), we have

$$(3.34) \quad I_2(s, a) = \delta^{-1}(1-a)\delta(a)I_1(-s, 1-a).$$

Thus $I_2(s, a)$ is explicitly evaluated as follows by Proposition 3.

Proposition 4. (1) *The integral $I_2(s, a)$ vanishes unless $h + k' \geq 2$.*

(2) *When $h = 1$, we have*

$$I_2(s, a) = \begin{cases} \delta(\varpi)^{-1} W(\varpi) W(\varpi^{k'-1}) \kappa(\varpi), & \text{when } k' \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(3) *Suppose that $h \geq 2$.*

(a) *When $-h + 2 \leq k' \leq h - 2$ and $h \equiv k' \pmod{2}$, we have*

$$I_2(s, a) = \delta(\varpi)^{-h-k} W(\varpi^{h+k}) \kappa(\varpi)^{\frac{h+k'}{2}} q^{\frac{h-k'}{2}} \mathcal{K} \ell \left(\varpi^{-\frac{h+k'}{2}} a, s^{-1} \varpi^{\frac{h+k'}{2}} \right).$$

(b) *When $k' = h - 1$, we have*

$$I_2(s, a) = -\delta(\varpi)^{-h} W(\varpi^h) W(\varpi) \kappa(\varpi)^{h-1}.$$

(c) *When $k' \geq h$, we have*

$$I_2(s, a) = \delta(\varpi)^{-h} W(\varpi^h) \kappa(\varpi)^h \left\{ W(\varpi^{k'-h}) - \kappa(\varpi) W(\varpi^{k'-h+2}) \right\}.$$

(d) *The integral $I_2(s, a)$ vanishes otherwise.*

Here we also note the following functional equation for $I(s, a)$.

Proposition 5 (Functional Equation). *For $s \in F^\times$ and $a \in F \setminus \{0, 1\}$, the function $\delta^{-1}(a) \cdot I(s, a)$ is invariant under the transformation $(s, a) \mapsto (-s, 1-a)$, i.e.*

$$(3.35) \quad \delta^{-1}(1-a) \cdot I(-s, 1-a) = \delta^{-1}(a) \cdot I(s, a).$$

Proof. When $|s| = 1$, (3.35) is clear from (3.9). When $|s| < 1$, (3.35) follows from (3.13) and (3.34), since $I(s, a) = \sum_{i=0}^2 I_i(s, a)$. \square

4. MATCHING

Let us introduce a new set of parameters for the matching. We note that there is a bijection

$$(F \setminus \{0, 1\}) \times F^\times \ni (x, \mu) \xrightarrow{\approx} \left(-\frac{1-x}{4\mu}, \frac{1}{1-x} \right) \in F^\times \times (F \setminus \{0, 1\})$$

whose inverse is given by

$$(s, a) \mapsto \left(-\frac{1-a}{a}, -\frac{1}{4sa} \right).$$

For $x \in F \setminus \{0, 1\}$ and $\mu \in F^\times$, let us define $\mathcal{I}(x, \mu)$ by

$$\mathcal{I}(x, \mu) = I(s, a) \quad \text{where } s = -\frac{1-x}{4\mu}, a = \frac{1}{1-x}.$$

Throughout this section we use the following two set of parameters.

$$\begin{cases} m & = \text{ord}(x) \\ m' & = \text{ord}(1-x) \\ n & = -\text{ord}(\mu) \end{cases} \quad \begin{cases} h & = \text{ord}(s) \\ k & = \text{ord}(1-a) \\ k' & = \text{ord}(a) \end{cases}$$

The dictionary between the two sets is given as follows.

$$\begin{cases} m &= k - k' \\ m' &= -k' \\ n &= h + k' \end{cases} \quad \begin{cases} h &= m' + n \\ k &= m - m' \\ k' &= -m' \end{cases}$$

In terms of the the new parameters, the functional equation (3.35) becomes

$$(4.1) \quad \mathcal{I}(x^{-1}, \mu x^{-1}) = \delta(x) \cdot \mathcal{I}(x, \mu).$$

First we note the following vanishing condition.

Lemma 5. *Then integral $\mathcal{I}(x, \mu)$ vanishes unless*

$$(4.2) \quad \text{ord}(\mu) \leq \min\{0, \text{ord}(x)\}.$$

Furthermore when E/F is inert, $\mathcal{I}(x, \mu)$ vanishes unless $\text{ord}(x)$ and $\text{ord}(\mu)$ are both even.

Proof. It is clear from (3.9), (3.12), (3.20) and (3.33) that $\mathcal{I}(x, \mu)$ vanishes unless $W(sa) \neq 0$ or $W(s(1-a)) \neq 0$. Hence $\mathcal{I}(x, \mu)$ vanishes unless $\text{ord}(sa) = n \geq 0$ or $\text{ord}(s(1-a)) = m+n \geq 0$, i.e. the condition (4.2) holds.

When E/F is inert, $W(b)$ vanishes unless $\text{ord}(b)$ is even. It is clear from (3.9), (3.12), (3.20) and (3.33) that every term contributing to $I(s, a)$ has a product of the form $W(sau^2)W(s(1-a)v^2)$, for some $u, v \in F^\times$, as a factor. Hence $\mathcal{I}(x, \mu)$ vanishes unless $\text{ord}(sa)$ and $\text{ord}(s(1-a))$ are both even. \square

4.1. Matching when E/F is inert. In this case, the Bessel orbital integral defined as follows is the local orbital integral to be compared with $\mathcal{I}(x, \mu)$.

Let $\eta \in E$ such that $\eta^2 \in \mathcal{O}^\times$ and $E = F(\eta)$. Then for $u \in E^\times$ such that $uu^\sigma \neq 1$ and $\mu \in F^\times$, the Bessel orbital integral $\mathcal{B}(u, \mu)$ is defined by

$$\mathcal{B}(u, \mu) = \int_{\mathbb{R}_1/Z_1} \int_{\mathbb{R}_1} \Xi_1 \left[\bar{r} \begin{pmatrix} A_u & 0 \\ 0 & \mu \cdot \bar{A}_u^{-1} \end{pmatrix} r \right] \xi(\bar{r}) \tau(r) d\bar{r} dr$$

where Ξ_1 denotes the characteristic function of $G_1 \cap \text{GL}_4(\mathcal{O}_E)$, \mathcal{O}_E is the integer ring of E , and $A_u = \begin{pmatrix} 1 & u \\ u^\sigma & 1 \end{pmatrix}$. We note that this is $\Omega^{-1}(u)$ times the Bessel orbital integral considered in [3]. We may restate Theorem 5.11 of [3] as follows.

Proposition 6. *Let $\mu \in F^\times$ and $u \in E^\times$ such that $uu^\sigma \neq 1$. Let $x = uu^\sigma$.*

(1) *The Bessel orbital integral $\mathcal{B}(u, \mu)$ vanishes unless $\text{ord}(\mu)$ is even and*

$$\text{ord}(\mu) \leq \min\{0, \text{ord}(x)\}.$$

(2) *We have the functional equation*

$$(4.3) \quad \mathcal{B}(u^{-1}, \mu x^{-1}) = \delta(x) \cdot \mathcal{B}(u, \mu).$$

(3) *Suppose that $\mu \in \mathcal{O}^\times$.*

(a) *When $|1-x| = 1$, we have*

$$\mathcal{B}(u, \mu) = 1.$$

(b) *When $|1-x| < 1$, we have*

$$\mathcal{B}(u, \mu) = |1-x|^{-1} \cdot \mathcal{Kl} \left(2(1-x)^{-1}, -2\mu(1-x)^{-1} \right).$$

(4) *Suppose that $n = -\text{ord}(\mu) > 0$, $m = \text{ord}(x) \geq 0$ and n is even.*

(a) When $m \geq n$, we have

$$\mathcal{B}(u, \mu) = \delta(\varpi)^n q^n \left\{ (-1)^{\frac{n}{2}} \cdot \mathcal{Kl}\left(2\varpi^{-\frac{n}{2}}, -2\varpi^{\frac{n}{2}}\mu\right) + 1 + q^{-1} \right\}.$$

(b) When $0 \leq m < n$, we have

$$\begin{aligned} \mathcal{B}(u, \mu) &= \delta(\varpi)^n |1-x|^{-1} q^n (-1)^{\frac{n}{2}} \\ &\quad \left\{ \mathcal{Kl}\left(\frac{2\varpi^{-\frac{n}{2}}}{1-x}, \frac{-2\varpi^{\frac{n}{2}}\mu}{1-x}\right) + (-1)^{\frac{m}{2}} \cdot \mathcal{Kl}\left(\frac{2\varpi^{\frac{m-n}{2}}}{1-x}, \frac{-2\varpi^{\frac{n-m}{2}}\mu x}{1-x}\right) \right\}. \end{aligned}$$

Note that in the statement of [3, Theorem 5.11] the factor of q^n is missing in Case (4)(b).

Theorem 1 (Matching when E/F is inert). *For $x \in F \setminus \{0, 1\}$ and $\mu \in F^\times$, $\mathcal{I}(x, \mu)$ vanishes unless $\text{ord}(x)$ is even. When $x = uu^\sigma$ for $u \in E^\times$, we have*

$$(4.4) \quad \mathcal{I}(x, \mu) = \delta^{-1} \left(\frac{x}{\mu^2} \right) \left| \frac{x}{\mu^2} \right|^{\frac{1}{2}} \mathcal{B}(u, \mu).$$

Proof. By Lemma 5, the vanishing conditions for $\mathcal{I}(x, \mu)$ and $\mathcal{B}(u, \mu)$ match. Also the functional equations (4.1) and (4.3) are compatible with the matching (4.4). Hence it is enough for us to show (4.4) when

$$m = \text{ord}(x) \geq 0, n = -\text{ord}(\mu) \geq 0 \text{ and } m, n \text{ are both even.}$$

Here we note that then we also have $m' = \text{ord}(1-x) \geq 0$.

When $m' = n = 0$. Then $h = k' = 0$ and $k = m$. Hence by (3.9) we have

$$\mathcal{I}(x, \mu) = \delta(\varpi)^{-m} q^{-\frac{m}{2}}.$$

When $m' > 0$ and $n = 0$. Then $m = 0$ and $k = k' = -h = -m'$. Hence we have $h + k = h + k' = 0$ and both $I_1(s, a)$ and $I_2(s, a)$ vanish. Thus by (3.12),

$$\mathcal{I}(x, \mu) = q^{m'} \mathcal{Kl}\left(2(1-x)^{-1}, -2\mu(1-x)^{-1}\right).$$

When $m' = 0$ and $n > 0$. Then we have $h = n \geq 2$, $k = m$ and $k' = 0$. Hence $I_0(s, a)$ vanishes since $\mathcal{Kl}(a, s^{-1}) = 0$. By Proposition 3 and Proposition 4,

$$\begin{aligned} I_1(s, a) &= \delta(\varpi)^{-m-n} q^{-\frac{m}{2}} \\ &\quad \cdot \begin{cases} (-1)^n (1 + q^{-1}), & \text{when } m \geq n, \\ (-1)^{\frac{m+n}{2}} \mathcal{Kl}\left(\varpi^{-\frac{m+n}{2}}(1-a), -s^{-1}\varpi^{\frac{m+n}{2}}\right), & \text{when } m < n \end{cases} \end{aligned}$$

and

$$I_2(s, a) = \delta(\varpi)^{-m-n} q^{-\frac{m}{2}} (-1)^{\frac{n}{2}} \mathcal{Kl}\left(\varpi^{-\frac{n}{2}}a, s^{-1}\varpi^{\frac{n}{2}}\right).$$

Hence

$$\begin{aligned} \mathcal{I}(x, \mu) &= \delta(\varpi)^{-m-n} q^{-\frac{m}{2}} (-1)^{\frac{n}{2}} \\ &\quad \cdot \begin{cases} \left\{ \mathcal{Kl}\left(2\varpi^{-\frac{n}{2}}, -2\varpi^{\frac{n}{2}}\mu\right) + (-1)^{\frac{n}{2}} (1 + q^{-1}) \right\}, & \text{when } m \geq n, \\ \left\{ \mathcal{Kl}\left(\frac{2\varpi^{-\frac{n}{2}}}{1-x}, \frac{-2\varpi^{\frac{n}{2}}\mu}{1-x}\right) + (-1)^{\frac{m}{2}} \mathcal{Kl}\left(\frac{2\varpi^{\frac{m-n}{2}}}{1-x}, \frac{-2\varpi^{\frac{n-m}{2}}\mu x}{1-x}\right) \right\}, & \text{when } m < n. \end{cases} \end{aligned}$$

When $m' > 0$ and $n > 0$. Then $m = 0$ and $h = m' + n \geq 3$, $k = k' = -m'$. Hence $I_0(s, a)$ vanishes since $\mathcal{Kl}(a, s^{-1}) = 0$. By Proposition 3 and Proposition 4,

$$I_1(s, a) = \delta(\varpi)^{-n} q^{m'} (-1)^{\frac{n}{2}} \mathcal{Kl}\left(\varpi^{-\frac{n}{2}}(1-a), -s^{-1}\varpi^{\frac{n}{2}}\right)$$

and

$$I_2(s, a) = \delta(\varpi)^{-n} q^{m'} (-1)^{\frac{n}{2}} \mathcal{Kl}\left(\varpi^{-\frac{n}{2}}a, s^{-1}\varpi^{\frac{n}{2}}\right).$$

Hence

$$\begin{aligned} \mathcal{I}(x, \mu) &= \delta(\varpi)^{-n} q^{m'} (-1)^{\frac{n}{2}} \\ &\quad \cdot \left\{ \mathcal{Kl}\left(\frac{2\varpi^{-\frac{n}{2}}}{1-x}, \frac{-2\varpi^{\frac{n}{2}}\mu x}{1-x}\right) + \mathcal{Kl}\left(\frac{2\varpi^{-\frac{n}{2}}}{1-x}, \frac{-2\varpi^{\frac{n}{2}}\mu}{1-x}\right) \right\}. \end{aligned}$$

By comparing with Proposition 6, we have

$$\mathcal{I}(x, \mu) = \delta(\varpi)^{-m-2n} q^{-\frac{m}{2}-n} \mathcal{B}(u, \mu)$$

in every case, i.e. the equality (4.4) holds. \square

4.2. Matching when E/F is split. In this case, the Novodvorsky orbital integral defined as follows is the local orbital integral to be compared with $\mathcal{I}(x, \mu)$.

For $x \in F \setminus \{0, 1\}$ and $\mu \in F^\times$, let

$$\begin{aligned} (4.5) \quad \mathcal{N}(x, \mu) &= \int_{(F^\times)^3} \int_U \int_{\bar{U}} \\ &\quad \Xi \left[\begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} h_x & 0 \\ 0 & \mu \cdot {}^t h_x^{-1} \end{pmatrix} \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \right] \\ &\quad \delta(abc) \psi \left[\text{tr} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (X + Y) \right\} \right] d^\times a d^\times b d^\times c dX dY \end{aligned}$$

where $h_x = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}$. The difference between (4.5) and the Novodvorsky integral considered in [3, (4.20)] is the characters on the diagonal torus.

Let us evaluate $\mathcal{N}(x, \mu)$. First we note the following lemma which is proved by an argument identical to the one given in [3, Section 4.3].

Lemma 6. (1) *We have the functional equation*

$$(4.6) \quad \mathcal{N}(x^{-1}, \mu x^{-1}) = \delta(x) \mathcal{N}(x, \mu).$$

(2) *The integral $\mathcal{N}(x, \mu)$ vanishes unless*

$$\text{ord}(\mu) \leq \min\{0, \text{ord}(x)\}.$$

Thus it is enough for us to evaluate $\mathcal{N}(x, \mu)$ when $|\mu| \geq 1$ and $|x| \leq 1$.

Lemma 7. *Suppose that $m = \text{ord}(x) \geq 0$ and $n = -\text{ord}(\mu) \geq 0$. Put $m' = \text{ord}(1-x)$.*

(1) *When $n = 0$, we have*

$$\mathcal{N}(x, \mu) = \begin{cases} 1 + m, & \text{when } m' = 0, \\ q^{m'} \mathcal{Kl}\left(2(1-x)^{-1}, -2\mu(1-x)^{-1}\right), & \text{when } m' > 0. \end{cases}$$

(2) When $n > 0$, we have

$$(4.7) \quad \mathcal{N}(x, \mu) = \mathcal{N}_0(x, \mu) + \mathcal{N}_1(x, \mu) + \mathcal{N}_2(x, \mu)$$

where

$$\mathcal{N}_0(x, \mu) = \begin{cases} -4\delta(\varpi), & \text{when } n = 1 \text{ and } m' = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{N}_1(x, \mu) = \delta(\varpi)^n q^{m'+n} (n+1) \sum_{i=1}^{m+n-1} \mathcal{K}\ell\left(\frac{2\varpi^{-i}x}{1-x}, \frac{-2\varpi^i\mu}{1-x}\right)$$

and

$$\mathcal{N}_2(x, \mu) = \delta(\varpi)^n q^{m'+n} (m+n+1) \sum_{j=1}^{n-1} \mathcal{K}\ell\left(\frac{2\varpi^{-j}}{1-x}, \frac{-2\varpi^j\mu}{1-x}\right).$$

Proof. As in [3, Section 4.3], we have

$$\mathcal{N}(x, \mu) = \delta(\mu)^{-1} \sum_{\substack{0 \leq i \leq m+n \\ 0 \leq j \leq n}} \mathcal{I}(x, \mu, \varpi^{n-i-j}, \varpi^i, \varpi^j)$$

where

$$\begin{aligned} \mathcal{I}(x, \mu, a, b, c) &= \int_U \int_{\bar{U}} \psi \left[\text{tr} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (X + Y) \right\} \right] \\ &\equiv \left[\begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} h_x & 0 \\ 0 & \mu \cdot {}^t h_x^{-1} \end{pmatrix} \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \right] dX dY. \end{aligned}$$

When $n = 0$. Then

$$\mathcal{N}(x, \mu) = \sum_{0 \leq i \leq m} \mathcal{I}(x, \mu, \varpi^{-i}, \varpi^i, 1).$$

If $m' = 0$, then we have $\mathcal{I}(x, \mu, \varpi^{-i}, \varpi^i, 1) = 1$ ($0 \leq i \leq m$) as computed in [3, p.97] and we have $\mathcal{N}(x, \mu) = 1 + m$. If $m' > 0$, then we have $m = 0$ and

$$\mathcal{N}(x, \mu) = q^{m'} \mathcal{K}\ell\left(2(1-x)^{-1}, -2\mu(1-x)^{-1}\right)$$

as computed in [3, Proposition 4.15].

When $n > 0$. Let us write $\mathcal{I}_{i,j}$ for $\mathcal{I}(x, \mu, \varpi^{n-i-j}, \varpi^i, \varpi^j)$. Thus

$$(4.8) \quad \mathcal{N}(x, \mu) = \delta(\varpi)^n \sum_{\substack{0 \leq i \leq m+n \\ 0 \leq j \leq n}} \mathcal{I}_{i,j}.$$

In (4.8), let us call $\mathcal{I}_{i,j}$ an interior term when $0 < i < m+n$ and $0 < j < n$ and let us call $\mathcal{I}_{i,j}$ a boundary term otherwise.

For the boundary terms, by [3, Proposition 4.15], we have

$$\mathcal{I}_{i,0} = \mathcal{I}_{i,n} = q^{m'+n} \mathcal{K}\ell\left(\frac{2\varpi^{-i}x}{1-x}, \frac{-2\varpi^i\mu}{1-x}\right)$$

and

$$\mathcal{I}_{0,j} = \mathcal{I}_{m+n,j} = q^{m'+n} \mathcal{K}\ell\left(\frac{2\varpi^{-j}}{1-x}, \frac{-2\varpi^j\mu}{1-x}\right).$$

As for the interior terms, by [3, Proposition 4.17], we have

$$\mathcal{I}_{i,j} = q^{m'+n} \left\{ \mathcal{Kl} \left(\frac{2\varpi^{-j}}{1-x}, \frac{-2\varpi^j x}{1-x} \right) + \mathcal{Kl} \left(\frac{2\varpi^{-i} x}{1-x}, \frac{-2\varpi^i \mu}{1-x} \right) \right\}.$$

Thus we have

$$\mathcal{N}(x, \mu) = \mathcal{N}_1(x, \mu) + \mathcal{N}_2(x, \mu) + \delta(\mu)^{-1} (\mathcal{I}_{0,0} + \mathcal{I}_{0,n} + \mathcal{I}_{m+n,0} + \mathcal{I}_{m+n,n})$$

where

$$\begin{aligned} & \mathcal{I}_{0,0} + \mathcal{I}_{0,n} + \mathcal{I}_{m+n,0} + \mathcal{I}_{m+n,n} \\ &= 2q^{m'+n} \left\{ \mathcal{Kl} \left(\frac{2x}{1-x}, \frac{-2\mu}{1-x} \right) + \mathcal{Kl} \left(\frac{2\varpi^{-n}}{1-x}, \frac{-2\varpi^n \mu}{1-x} \right) \right\}. \end{aligned}$$

Since

$$\text{ord} \left(\frac{2x}{1-x} \right) = m - m', \quad \text{ord} \left(\frac{-2\mu}{1-x} \right) = -n - m' < 0$$

and they can never be equal, we have

$$\mathcal{Kl} \left(\frac{2x}{1-x}, \frac{-2\mu}{1-x} \right) = \begin{cases} -q^{-1}, & \text{when } n = 1 \text{ and } m' = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly we have

$$\mathcal{Kl} \left(\frac{2\varpi^{-n}}{1-x}, \frac{-2\varpi^n \mu}{1-x} \right) = \begin{cases} -q^{-1}, & \text{when } n = 1 \text{ and } m' = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have (4.7). \square

Theorem 2 (Matching when E/F is split). *For $x \in F \setminus \{0, 1\}$ and $\mu \in F^\times$, we have*

$$(4.9) \quad \mathcal{I}(x, \mu) = \delta^{-1} \left(\frac{x}{\mu^2} \right) \left| \frac{x}{\mu^2} \right|^{\frac{1}{2}} \mathcal{N}(x, \mu).$$

Proof. As in the case when E/F is inert, it is enough for us to show (4.9) when $m \geq 0$ and $n \geq 0$.

When $n = 0$. If $m' = 0$, by (3.9), we have

$$\mathcal{I}(x, \mu) = \delta(\varpi)^{-m} q^{-\frac{m}{2}} (m+1).$$

If $m' > 0$, then $m = 0$ and $k = k' = -h$. Hence both $I_1(s, a)$ and $I_2(s, a)$ vanish and by (3.12) we have

$$\mathcal{I}(x, \mu) = q^{m'} \mathcal{Kl} \left(2(1-x)^{-1}, -2\mu(1-x)^{-1} \right).$$

When $n = 1$. Then we have $\mathcal{N}(x, \mu) = \mathcal{N}_0(x, \mu) + \mathcal{N}_1(x, \mu)$. If $m' > 0$, then $m = 0$ and we have $\mathcal{N}_0(x, \mu) = \mathcal{N}_1(x, \mu) = 0$. When $m' = 0$, we have

$$\mathcal{N}(x, \mu) = -4\delta(\varpi) + 2\delta(\varpi)q \sum_{i=1}^m \mathcal{Kl}\left(\frac{2\varpi^{-i}x}{1-x}, \frac{-2\varpi^i\mu}{1-x}\right).$$

Here the arguments in the Kloosterman sums are all integers when $m > 0$. Thus when $n = 1$, we have

$$\mathcal{N}(x, \mu) = \begin{cases} 2\delta(\varpi)q \{m - (m+2)q^{-1}\}, & \text{when } m \geq 1, \\ -4\delta(\varpi), & \text{when } m = m' = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us compute $\mathcal{I}(x, \mu)$. If $m' > 0$, then $m = 0$ and $I_1(s, a) = I_2(s, a) = 0$ since $h + k = h + k' = 1$. We also have $I_0(s, a) = 0$ since $-h < k' < 0$ and hence $\mathcal{Kl}(a, s^{-1}) = 0$. Suppose that $m' = 0$. Then we have $I_2(s, a) = 0$ since $h + k' = 1$,

$$I_0(s, a) = -2(m+2)\delta(\varpi)^{-m-1}q^{-\frac{m+2}{2}}$$

and

$$I_1(s, a) = \begin{cases} 2m\delta(\varpi)^{-m-1}q^{-\frac{m}{2}}, & \text{when } m \geq 1, \\ 0, & \text{when } m = 0. \end{cases}$$

Hence when $n = 1$, we have

$$\mathcal{I}(x, \mu) = \begin{cases} 2\delta(\varpi)^{-m-1}q^{-\frac{m}{2}} \{m - (m+2)q^{-1}\}, & \text{when } m \geq 1, \\ -4\delta(\varpi)^{-1}q^{-1}, & \text{when } m = m' = 0, \\ 0, & \text{otherwise.} \end{cases}$$

When $n \geq 2$. Then we have

$$\mathcal{N}(x, \mu) = \mathcal{N}_1(x, \mu) + \mathcal{N}_2(x, \mu).$$

On the other hand, we have $I_0(s, a) = 0$ by Proposition 2 since $k + h \geq 2$ and $h \neq 1$. Thus we have

$$\mathcal{I}(x, \mu) = I_1(s, a) + I_2(s, a).$$

We shall show the matching between $\mathcal{N}_i(x, \mu)$ and $I_i(s, a)$ for $i = 1, 2$.

Let us first consider the matching between $I(s, a)$ and $\mathcal{N}_2(x, \mu)$. By Proposition 4, $I_2(s, a)$ vanishes unless n is even. When n is even, we have

$$I_2(s, a) = \delta(\varpi)^{-m-n}(m+n+1)q^{-\frac{m}{2}+m'} \mathcal{Kl}\left(\frac{2\varpi^{-\frac{n}{2}}}{1-x}, \frac{2\varpi^{\frac{n}{2}}\mu}{1-x}\right).$$

On the other hand

$$\mathcal{N}_2(x, \mu) = \delta(\varpi)^n q^{m'+n}(m+n+1) \sum_{j=1}^{n-1} \mathcal{Kl}\left(\frac{2\varpi^{-j}}{1-x}, \frac{-2\varpi^j\mu}{1-x}\right).$$

Since

$$\text{ord}\left(\frac{2\varpi^{-j}}{1-x}\right) + \text{ord}\left(\frac{-2\varpi^j\mu}{1-x}\right) = -2m' - n \leq -2,$$

we have $\mathcal{Kl}\left(\frac{2\varpi^{-j}}{1-x}, \frac{-2\varpi^j\mu}{1-x}\right) = 0$ unless n is even and $j = \frac{n}{2}$. Thus $\mathcal{N}_2(x, \mu)$ vanishes unless n is even and when n is even we have

$$\mathcal{N}_2(x, \mu) = \delta(\varpi)^n q^{m'+n}(m+n+1) \mathcal{Kl}\left(\frac{2\varpi^{-\frac{n}{2}}}{1-x}, \frac{-2\varpi^{\frac{n}{2}}\mu}{1-x}\right).$$

Let us compare $I_1(s, a)$ with $\mathcal{N}_1(x, \mu)$. By Proposition 3, $I_1(s, a)$ is given as follows. We have

$$I_1(s, a) = \delta(\varpi)^{-m-n} (n+1) q^{-\frac{m}{2}} \{(m-n+1) - (m-n+3)q^{-1}\}$$

when $m \geq n$,

$$I_1(s, a) = -2\delta(\varpi)^{-2n+1} (n+1) q^{-\frac{n+1}{2}}$$

when $m = n-1$,

$$I_1(s, a) = \delta(\varpi)^{-m-n} (n+1) q^{-\frac{m}{2}+m'} \mathcal{Kl} \left(\frac{2\varpi^{-\frac{m+n}{2}}x}{1-x}, \frac{-2\varpi^{\frac{m+n}{2}}\mu}{1-x} \right)$$

when $m \leq n-2$ and $m \equiv n \pmod{2}$, and $I_1(s, a)$ vanishes otherwise.

Let us compute $\mathcal{N}_1(x, \mu)$. We recall that

$$(4.10) \quad \mathcal{N}_1(x, \mu) = \delta(\varpi)^n (n+1) q^{m'+n} \sum_{i=1}^{m+n-1} \mathcal{Kl} \left(\frac{2\varpi^{-i}x}{1-x}, \frac{-2\varpi^i\mu}{1-x} \right)$$

where

$$\text{ord} \left(\frac{2\varpi^{-i}x}{1-x} \right) + \text{ord} \left(\frac{-2\varpi^i\mu}{1-x} \right) = m-n-2m'.$$

When $m \geq n$, the sum of the ordinals of the arguments of the Kloosterman sum is $m-n \geq 0$. Hence the Kloosterman sum in (4.10) vanishes unless

$$\min\{-i+m, i-n\} \geq -1$$

i.e. $n-1 \leq i \leq m+1$ and

$$\mathcal{Kl} \left(\frac{2\varpi^{-i}x}{1-x}, \frac{-2\varpi^i\mu}{1-x} \right) = \begin{cases} 1-q^{-1}, & \text{when } n \leq i \leq m, \\ -q^{-1}, & \text{when } i = n-1, m+1. \end{cases}$$

Hence

$$\mathcal{N}_1(x, \mu) = \delta(\varpi)^n (n+1) q^n \{(m-n+1) - (m-n+3)q^{-1}\}.$$

When $m = n-1$, by Corollary 1, we have

$$\mathcal{Kl} \left(\frac{2\varpi^{-i}x}{1-x}, \frac{-2\varpi^i\mu}{1-x} \right) = \begin{cases} -q^{-1}, & \text{when } i = n-1, n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\mathcal{N}_1(x, \mu) = -2\delta(\varpi)^n (n+1) q^{n-1}.$$

When $0 \leq m \leq n-2$, the Kloosterman sum $\mathcal{Kl} \left(\frac{2\varpi^{-i}x}{1-x}, \frac{-2\varpi^i\mu}{1-x} \right)$ vanishes unless $m \equiv n \pmod{2}$ and $j = \frac{m+n}{2}$ since $m-n-2m' \leq -2$. When $m \equiv n \pmod{2}$, we have

$$\mathcal{N}_1(x, \mu) = \delta(\varpi)^n (n+1) q^{m'+n} \mathcal{Kl} \left(\frac{2\varpi^{-\frac{m+n}{2}}x}{1-x}, \frac{-2\varpi^{\frac{m+n}{2}}\mu}{1-x} \right).$$

Thus we have shown

$$\mathcal{I}(x, \mu) = \delta(\varpi)^{-m-2n} q^{-\frac{m}{2}-n} \mathcal{N}(x, \mu)$$

in every case, i.e. the equality (4.9) holds. \square

Thus by (4.4) and (4.9), we have established the desired matching of the unit elements of the Hecke algebras for the regular geometric terms.

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