ON CENTRAL CRITICAL VALUES OF THE DEGREE FOUR L-FUNCTIONS FOR GSp(4): A SIMPLE TRACE FORMULA

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To the memory of Hiroshi Saito

Abstract. We establish a simple relative trace formula for GSp(4) and inner forms with respect to Bessel subgroups to obtain a certain Bessel identity. From such an identity, one can hope to prove a formula relating central values of degree four spinor L-functions to squares of Bessel periods as conjectured by Böcherer. Under some local assumptions, we obtain nonvanishing results, i.e., a global Gross–Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$.

1. Introduction

Let E/F be a quadratic extension of number fields, π a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central character, and χ an idèle class character of E. Let $L(s, \pi \times \chi)$ denote the $\mathrm{GL}(2) \times \mathrm{GL}(2)$ Rankin–Selberg L-function associated to π times the theta lift θ_χ of χ to an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. In particular, if χ is the trivial character $\mathbb{1}_E$ of $\mathbb{A}_E^\times$, then

$$L(1/2, \pi \times \mathbb{1}_E) = L(1/2, \pi) L(1/2, \pi \otimes \kappa),$$

where κ is the quadratic idèle class character of F associated to E/F.

In [36], Waldspurger proved a celebrated formula for the twisted central value $L(1/2, \pi \times \chi)$ in terms of compact toric periods on a certain representation π_D which is a functorial lift of π. Namely, there is a unique quaternion algebra D/F determined by ϵ-factors such that $E \subset D$, π has a Jacquet–Langlands transfer π_D to $D^\times(\mathbb{A}_F)$, and the local Hom spaces $\mathrm{Hom}_{E_v^\times}(\pi_{D,v}, \chi_v) \neq 0$ for all v.

Fix π_D as above and consider the global periods

$$\mathcal{P}_D : \pi_D \to \mathbb{C}$$

given by

$$\mathcal{P}_D(\phi) = \int_{\mathbb{A}_E^\times / \mathbb{A}_F^\times} \phi(t) \chi^{-1}(t) \, dt.$$

Note $\mathcal{P}_D$ lies in the nonzero (in fact, one-dimensional) global Hom space $\mathrm{Hom}_{\mathbb{A}_E^\times}(\pi_D, \chi)$, though $\mathcal{P}_D$ may be zero or not. Then Waldspurger's formula is an expression of the form

$$\frac{|\mathcal{P}_D(\phi)|^2}{(\phi, \phi)} = \frac{1}{\zeta(2)} \prod_v \alpha_v(\phi, \pi, \chi) L(1/2, \pi \times \chi), \tag{1.1}$$

where $\phi \in \pi_D$, $(\cdot, \cdot)$ is an invariant scalar product on π_D, and the factors $\alpha_v(\phi, \pi, \chi)$, which are 1 for almost all v, are given by certain local integrals. In particular, one immediately gets that $P_D \not\equiv 0$ implies $L(1/2, \pi \times \chi) \neq 0$.

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Shortly thereafter, Jacquet also studied these L-values and periods using relative trace formulas, first for χ trivial in [19], then for arbitrary χ with a technically more involved trace formula in [20]. Specifically, Jacquet's trace formula—to be specific, the one in [19]—gives identities of the form

$$J_\pi(f) = J_{\pi_D}(f_D) \tag{1.2}$$

where $f \in C_c^\infty(\mathrm{GL}_2(\mathbb{A}_F))$ and $f_D \in C_c^\infty(D^\times(\mathbb{A}_F))$ are "matching functions," and J_π, J_{π_D} are certain spectral distributions related to $L(1/2,\pi)L(1/2,\pi\otimes\kappa)$ and products of periods of the form $\mathcal{P}_D(\phi_1)\mathcal{P}_D(\phi_2)$. These kinds of spectral distributions are now often called Bessel distributions, and equations of the form (1.2) Bessel identities. In particular, from this Bessel identity, Jacquet deduces that $L(1/2,\pi)L(1/2,\pi\otimes\kappa) \neq 0$ if and only if $\mathcal{P}_D \not\equiv 0$. Subsequently, Jacquet and Chen [21] use the Bessel identity from [20] to obtain a formula of the form

$$J_{\pi_D}(f_D) = c(f_D)L(1/2,\pi\times\chi). \tag{1.3}$$

For suitable test functions f_D, $J_{\pi_D}(f_D)$ is just $|P_D(\phi)|^2$ and one can realize (1.3) in the form of Waldspurger's formula (1.1). Indeed, in [27], the second author and Whitehouse used (1.3) to get explicit versions of Waldspurger's formula.

On the other hand, Waldspurger [35] had previously shown that L-values for elliptic modular forms are also related to Fourier coefficients of certain half-integral weight modular forms. Motivated by this, Böcherer [5] conjectured an L-value formula for Siegel modular forms of degree 2 in terms of certain Fourier coefficients, and proved his conjecture in special cases (cf. [9, Introduction]). We remark that the formula in [35] has also been reproven—in fact in a more general setting—by Baruch and Mao [4] using a relative trace formula.

By putting Böcherer's conjecture in a representation theoretic framework, the first author together with Shalika [9] realized this conjecture as a higher rank analogue of (1.1). Namely, they conjecture a relation between twisted central spinor L-values $L(1/2,\pi\times\chi) = L(1/2,\pi\otimes\theta_\chi)$ of cuspidal automorphic representations π of $\mathrm{GSp}_4(\mathbb{A}_F)$ and *Bessel periods* on inner forms (see Conjecture 1.1 below). They also conjecture two relative trace formulas—the first for trivial χ, the second for arbitrary χ—analogous to those in [19] and [20] which should resolve this generalization of Böcherer's conjecture. See Conjecture 1.2 below for the first trace formula.

Conjecture 1.1 essentially constitutes the $(\mathrm{SO}(5),\mathrm{SO}(2))$ case (for χ trivial) of the global Gross–Prasad conjectures (local and global conjectures for $(\mathrm{SO}(n{+}1),\mathrm{SO}(n))$ are made in [15], and local conjectures for $(\mathrm{SO}(2n+1),\mathrm{SO}(2m))$ are made in [16]). This is not strictly a generalization of Böcherer's conjecture, as Conjecture 1.1, or more generally the global Gross–Prasad conjectures, are only about nonvanishing, but the point is that Conjecture 1.2 should also yield a special value formula as in the GL(2) case. We remark that the global Gross–Prasad conjectures for $(\mathrm{SO}(n+1),\mathrm{SO}(n))$ have been recently made precise, in the sense of a conjectural special value formula, by Ichino and Ikeda [17].

As the first step in establishing these two relative trace formulas, [9] proves the fundamental lemma for the unit element of the Hecke algebra. In [7], the present authors proposed an alternative trace formula to the second one in [9], and proved the fundamental lemma for the unit element. Then in [8], the present authors with Shalika extended the fundamental lemma to the full Hecke algebra for the first trace formula proposed in [9] and the third one proposed in [7]. In the present paper, we apply these fundamental lemmas to establish a Bessel identity analogous to (1.2) for the first trace formula in [9] and deduce nonvanishing results, i.e., we show Conjectures 1.1 and 1.2 under some local assumptions. Partial results on global Gross–Prasad conjectures have previously been obtained in a general setting by Ginzburg, Jiang and Rallis [13], [14] using entirely different methods.

We now introduce the necessary notation to state our results precisely.

Notation. *Let F be a number field and let $\mathbb{A}$ be its ring of adeles. Let ψ be a non-trivial character of $\mathbb{A}/F$. Let E be a quadratic extension of F and let $\mathbb{A}_E$ be its ring of adeles. Let $\kappa = \kappa_{E/F}$ denote the quadratic character of $\mathbb{A}^\times/F^\times$ corresponding to the quadratic extension E/F in the sense of class field theory. Let σ denote the unique non-trivial element in $\mathrm{Gal}\,(E/F)$ and take $\eta \in E^\times$ such that $\eta^\sigma = -\eta$.*

If v is a nonarchimedean place of F, we denote by $\mathcal{O}_v$ the ring of integers of F_v, and by Ξ_v the characteristic function of $\mathrm{GSp}_4(\mathcal{O}_v)$.

For any algebraic group G, we will denote the center by Z.

1.1. **Setup.**

1.1.1. $\mathrm{GSp}\,(4)$ *and the Novodvorsky subgroups.* Let G be the group $\mathrm{GSp}\,(4)$, an algebraic group over F defined by

$$G = \left\{ g \in \mathrm{GL}\,(4) \mid {}^t g J g = \lambda\,(g)\, J,\ \ \lambda\,(g) \in \mathrm{GL}(1) \right\}, \quad \text{where } J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

Here ${}^t g$ denotes the transpose of g and $\lambda\,(g)$ is called the *similitude factor* of g.

Define the *upper and lower Novodvorsky (or split Bessel) subgroups*, resp. H and $\bar{H}$, of G by

$$H = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} : a, b \in \mathrm{GL}(1),\ X \in \mathrm{Sym}(2) \right\}$$

and

$$\bar{H} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} : a, b \in \mathrm{GL}(1),\ Y \in \mathrm{Sym}(2) \right\}.$$

Here $\mathrm{Sym}(2)$ denotes the group of symmetric 2×2 matrices.

1.1.2. *Quaternion similitude unitary groups and the Bessel subgroups.* For each $\epsilon \in F^\times$, let D_ϵ denote the quaternion algebra over F defined by

$$D_\epsilon = \left\{ \begin{pmatrix} a & b\epsilon \\ b^\sigma & a^\sigma \end{pmatrix} : a, b \in E \right\}.$$

We shall identify $a \in E$ with $\begin{pmatrix} a & 0 \\ 0 & a^\sigma \end{pmatrix} \in D_\epsilon$. We recall that $\{D_\epsilon\}_\epsilon$ gives a set of representatives for the isomorphism classes of central simple quaternion algebras over F containing E when ϵ runs over a set of representatives for $F^\times/\mathrm{N}_{E/F}\,(E^\times)$. Let $D_\epsilon \ni \alpha \mapsto \bar{\alpha} \in D_\epsilon$ denote the canonical involution of D_ϵ, i.e.,

$$\overline{\begin{pmatrix} a & b\epsilon \\ b^\sigma & a^\sigma \end{pmatrix}} = \begin{pmatrix} a^\sigma & -b\epsilon \\ -b^\sigma & a \end{pmatrix}.$$

We define the *quaternion similitude unitary group* G_ϵ of degree two over D_ϵ to be

$$G_\epsilon = \left\{ g \in \mathrm{GL}(2, D_\epsilon) : g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \mu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \, \mu(g) \in \mathrm{GL}(1) \right\}$$

where $g^* = \begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix}$ for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We recall that the G_ϵ's are inner forms of $G = \mathrm{GSp}(4)$. When $\epsilon = 1$, we have $D_1 \simeq \mathrm{Mat}_{2\times 2}(F)$ and $G = \alpha G_1 \alpha^{-1}$ in $\mathrm{GL}_4(E)$ where

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ \eta & -\eta & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \eta & -\eta \end{pmatrix}.$$

We define the *upper* (resp. *lower*) *(anisotropic) Bessel subgroup* R_ϵ (resp. $\bar R_\epsilon$) of G_ϵ by

$$R_\epsilon = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} : a \in E^\times, \, X \in D_\epsilon^- \right\},$$
$$\bar R_\epsilon = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} : a \in E^\times, \, Y \in D_\epsilon^- \right\},$$

where $D_\epsilon^- = \left\{ X \in D_\epsilon \mid X + \bar X = 0 \right\}$.

1.1.3. *Relative trace formula for G.* We define characters θ and ψ of $H(\mathbb{A})$ and $\bar H(\mathbb{A})$ by

$$\theta \left[\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \right] = \kappa(ab) \psi \left[\mathrm{tr} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X \right) \right]$$

and

$$\psi \left[\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \right] = \psi \left[\mathrm{tr} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y \right) \right].$$

For a cuspidal representation π of $G(\mathbb{A})/Z(\mathbb{A})$, we define the *upper and lower Novodvorsky periods* (with respect to θ^{-1} and ψ^{-1})

$$\mathcal{P} : \pi \to \mathbb{C}, \quad \mathcal{P}' : \pi \to \mathbb{C}$$

by

$$\mathcal{P}(\phi) = \mathcal{P}_{\theta^{-1}}(\phi) = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h) \theta^{-1}(h) \, dh \tag{1.4}$$

$$\mathcal{P}'(\phi) = \mathcal{P}'_{\psi^{-1}}(\phi) = \int_{Z(\mathbb{A})\bar H(F)\backslash \bar H(\mathbb{A})} \phi(\bar h) \psi^{-1}(\bar h) \, d\bar h. \tag{1.5}$$

The Novodvorsky periods necessarily vanish if π is not generic [6]. If π is generic, then these are the integrals that arise in Novodvorsky's $\mathrm{GSp}(4) \times \mathrm{GL}(1)$ integral representation [29] for the spinor L-functions $L(s, \pi)$ and $L(s, \pi, \otimes\kappa)$ when $s = \frac{1}{2}$. See Bump [6] for the unfolding and local unramified calculations, as well as Soudry

[32], [33] and Takloo-Bighash [34] for the local theory. In particular $\mathcal{P} \not\equiv 0$ is equivalent to $L(1/2, \pi \otimes \kappa) \neq 0$ and $\mathcal{P}' \not\equiv 0$ is equivalent to $L(1/2, \pi) \neq 0$.

Let $f \in C_c^\infty(G(\mathbb{A}))$. This gives rise to the associated kernel function

$$K(x, y) = K_f(x, y) = \sum_{\gamma \in Z(F)\backslash G(F)} \int_{Z(\mathbb{A})} f(x^{-1}\gamma y z)\, dz.$$

Then the relative trace formula in question will be derived from

$$J(f) = \int_{Z(\mathbb{A})\bar{H}(F)\backslash \bar{H}(\mathbb{A})} \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} K_f(\bar{h}, h)\psi(\bar{h})^{-1}\theta(h)\, d\bar{h}\, dh. \tag{1.6}$$

At least formally, the relative trace formula is an identity derived from the geometric and spectral expansions of $K(x, y)$, of the form

$$J(f) = \sum_{\gamma \in \bar{H}(F)\backslash G(F)/H(F)} J_\gamma(f) = \sum_{\pi \text{ cusp}} J_\pi(f) + J_{\mathrm{nc}}(f).$$

Here each $J_\gamma(f)$ is a certain relative orbital integral, $J_{\mathrm{nc}}(f)$ denotes the non-cuspidal contribution, and

$$J_\pi(f) = \sum_\phi \mathcal{P}'(\pi(f)\phi)\overline{\mathcal{P}(\phi)}$$

where π is a cuspidal automorphic representation of $G(\mathbb{A})/Z(\mathbb{A})$ and ϕ runs over an orthonormal basis for π (at least for suitable f and basis $\{\phi\}$; cf. (4.12) below). In particular $J_\pi \not\equiv 0$ if and only if $L(1/2, \pi)L(1/2, \pi \otimes \kappa) \neq 0$.

1.1.4. *Relative trace formula for G_ϵ.* Let τ and ξ be the characters of R_ϵ and $\bar{R}_\epsilon$ given by

$$\tau\left[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}\right] = \psi(\mathrm{tr}(-\eta X))$$

and

$$\xi\left[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}\right] = \psi(\mathrm{tr}(-\eta^{-1} Y)).$$

For a cuspidal representation π of $G_\epsilon(\mathbb{A})/Z(\mathbb{A})$, we define the *upper and lower Bessel periods* (with respect to τ^{-1} and ξ^{-1})

$$\mathcal{P}_\epsilon : \pi \to \mathbb{C}, \quad \mathcal{P}'_\epsilon : \pi \to \mathbb{C}$$

by

$$\mathcal{P}_\epsilon(\phi) = \mathcal{P}_{\epsilon, \tau^{-1}}(\phi) = \int_{Z(\mathbb{A})R_\epsilon(F)\backslash R_\epsilon(\mathbb{A})} \phi(r)\tau^{-1}(r)\, dr \tag{1.7}$$

$$\mathcal{P}'_\epsilon(\phi) = \mathcal{P}'_{\epsilon, \xi^{-1}}(\phi) = \int_{Z(\mathbb{A})\bar{R}_\epsilon(F)\backslash \bar{R}_\epsilon(\mathbb{A})} \phi(\bar{r})\xi^{-1}(\bar{r})\, d\bar{r}. \tag{1.8}$$

The representation π having a nonzero upper Bessel period (i.e., $\mathcal{P}_\epsilon \not\equiv 0$) is equivalent to having a nonzero lower Bessel period (i.e., $\mathcal{P}'_\epsilon \not\equiv 0$). This equivalence follows as

$$\mathcal{P}'_\epsilon(\phi) = \overline{\mathcal{P}_\epsilon(\pi(w_\eta)\phi)}, \quad \text{where} \quad w_\eta = \begin{pmatrix} 0 & -\eta^2 \\ 1 & 0 \end{pmatrix}.$$

Thus if $\mathcal{P}_\epsilon \not\equiv 0$, we simply say π *has a Bessel period (with respect to E).*

Let $f_\epsilon \in C_c^\infty(G_\epsilon(\mathbb{A}))$. This gives rise to the associated kernel function

$$K_\epsilon(x,y) = K_{f_\epsilon}(x,y) = \sum_{\gamma \in Z(F)\backslash G_\epsilon(F)} \int_{Z(\mathbb{A})} f_\epsilon(x^{-1}\gamma y z)\, dz.$$

Then the relative trace formula in question will be derived from

$$J_\epsilon(f_\epsilon) = \int_{Z(\mathbb{A})\bar{R}_\epsilon(F)\backslash \bar{R}_\epsilon(\mathbb{A})} \int_{Z(\mathbb{A})R_\epsilon(F)\backslash R_\epsilon(\mathbb{A})} K_{f_\epsilon}(\bar{r}, r)\xi(\bar{r})^{-1}\tau(r)\, d\bar{r}\, dr. \tag{1.9}$$

Ignoring convergence issues, (1.9) should have a geometric expansion of the form

$$J_\epsilon(f_\epsilon) = \sum_{\gamma_\epsilon \in \bar{R}_\epsilon(F)\backslash G_\epsilon(F)/R_\epsilon(F)} J_{\gamma_\epsilon}(f_\epsilon),$$

where the distributions $J_{\gamma_\epsilon}(f_\epsilon)$ are given by certain (relative) orbital integrals.

On the other hand, (1.9) should also have a spectral expansion of the form

$$J_\epsilon(f_\epsilon) = \sum_{\pi_\epsilon \text{ cusp}} J_{\pi_\epsilon}(f_\epsilon) + J_{\epsilon,\mathrm{nc}}(f_\epsilon)$$

where π_ϵ runs over the cuspidal automorphic representations of $G_\epsilon(\mathbb{A})/Z(\mathbb{A})$ and $J_{\epsilon,\mathrm{nc}}$ comprises the contribution from the non-cuspidal part of the spectrum. Namely, we have

$$J_{\pi_\epsilon}(f_\epsilon) = \sum_\phi \mathcal{P}'_\epsilon(\pi_\epsilon(f_\epsilon)\phi)\overline{\mathcal{P}_\epsilon(\phi)},$$

when ϕ runs over a suitable orthonormal basis for the space of π_ϵ. This implies that π_ϵ has a Bessel period if and only if $J_{\pi_\epsilon} \not\equiv 0$.

1.2. **Results.** In analogy with Jacquet's work [19], and in connection with Böcherer's conjecture, Shalika and the first author made the following conjectures.

Conjecture 1.1. ([9, Conjecture 1.10]) *Given a generic cuspidal representation π of $G(\mathbb{A})/Z(\mathbb{A})$ such that $L(1/2,\pi)L(1/2,\pi\otimes\kappa) \neq 0$, there exists a Jacquet–Langlands transfer π_ϵ of π to some $G_\epsilon(\mathbb{A})/Z(\mathbb{A})$ which has a Bessel period with respect to E.*

Conversely, given a cuspidal representation π_ϵ of $G_\epsilon(\mathbb{A})/Z(\mathbb{A})$ which has a Bessel period with respect to E, there exists a generic Jacquet–Langlands transfer π of π_ϵ to $G(\mathbb{A})/Z(\mathbb{A})$ such that $L(1/2,\pi)L(1/2,\pi\otimes\kappa) \neq 0$.

Here, by Jacquet–Langlands transfer, we mean that $\pi_v \simeq \pi_{\epsilon,v}$ for almost all v. While the existence of the global Jacquet–Langlands transfer for G/Z and G_ϵ/Z is not yet known, this should follow from the completion of Arthur's Book Project [3] (for split SO(5) and inner forms) or, at least in the cases of Conjecture 1.1, the relative trace formula below.

As mentioned above, this nonvanishing conjecture should be viewed as the global Gross–Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$ (and χ trivial). While this does not give a special value formula such as the ones conjectured by Böcherer, the point is that the following more general (if somewhat imprecise) conjecture should.

Conjecture 1.2. ([9, Conjecture 1.8], first relative trace formula identity) *For "matching" functions f and $(f_\epsilon)_\epsilon$, one has an identity of distributions*

$$J(f) = \sum_\epsilon J_\epsilon(f_\epsilon), \tag{1.10}$$

where these distributions are suitably regularized.

Here, for f to match with a family of functions $(f_\epsilon)_\epsilon$ ($\epsilon \in F^\times/N_{E/F}(E^\times)$) means the following. One defines a one-to-one correspondence between the set of "regular" double cosets $\bar{H}(F)\gamma H(F)$ for $G(F)$ and union over ϵ of the "regular" double cosets $\bar{R}_\epsilon(F)\gamma_\epsilon R_\epsilon(F)$ for G_ϵ. Then one says that the functions f and $(f_\epsilon)_\epsilon$ match if the orbital integrals $J_\gamma(f) = J_{\gamma_\epsilon}(f_\epsilon)$ are equal whenever γ corresponds to γ_ϵ. Roughly, the regular double cosets are the ones for which the orbital integrals as defined above are convergent. Then in general, one wants to regularize the *singular* (non-convergent) orbital integrals and show an equality of these regularized orbital integrals to deduce (1.10).

Leaving the singular orbital integrals aside, the main issue is to show the existence of sufficiently many matching functions. This can be easily reduced to showing the existence of *local matching functions*. In particular, one wants to choose $f_v = \Xi_v$ and $f_{\epsilon,v} = \Xi_v$ (when $G_\epsilon(F_v) \simeq G(F_v)$) at almost all v, so one needs to show the local Novodvorsky orbital integrals for Ξ_v equal the local Bessel orbital integrals for Ξ_v. This is known as the fundamental lemma for the unit element, and was established in [9].

Supposing now one has (1.10), one would like to deduce that

$$J_\pi(f) = J_{\pi_\epsilon}(f_\epsilon) \tag{1.11}$$

for suitable Jacquet-Langlands pairs π and π_ϵ. The fundamental lemma for the Hecke algebra [8] says that at almost all places we can vary our matching functions f and $(f_\epsilon)_\epsilon$ in the Hecke algebra. Thus the principle of infinite linear independence of characters (or, in our case, Bessel distributions) gives an equality of the form

$$\sum_{\pi\in\Pi} J_\pi(f) = \sum_\epsilon \sum_{\pi_\epsilon\in\Pi_\epsilon} J_{\pi_\epsilon}(f_\epsilon), \tag{1.12}$$

where Π and Π_ϵ denote certain near equivalence classes for π and π_ϵ. These near equivalence classes should be contained in the global L-packets of π and its transfers π_ϵ. This would follow, for instance, from the completion of Arthur's Book Project. Strong multiplicity one for generic representations of GSp(4) (proven by Jiang and Soudry [23] for F totally real) says the left hand side of (1.12) has at most one term. On the other hand, the weak form of the local Gross–Prasad conjectures say the right hand side of (1.12) has at most one term (the strong form of local Gross–Prasad says which ϵ and π_ϵ should appear). Hence one obtains (1.11), from which one should be able to obtain the desired L-value formula as in the GL(2) cases in [21], [27] and [4]. See also Lapid–Offen [25] and the recent work of W. Zhang [40] for instances of deducing L-value formulas from Bessel identities in higher-dimensional unitary cases.

We are now in a position to state our main results. To make our statements as simple as possible, we will assume strong multiplicity one for generic representations for arbitrary F, the near equivalence classes above are contained in global L-packets, and (the weak form of) the local Gross–Prasad conjectures for $(\mathrm{SO}(5), \mathrm{SO}(2))$, though our final statements in Section 5 do not require these. In any case, we expect these assumptions will be validated in the near future with the completion of Arthur's Book Project [3].

Suppose π is generic, locally tempered everywhere, and supercuspidal at some place split in E/F. Let ϵ, π_ϵ be such that π_ϵ is the unique Jacquet–Langlands transfer, assumed to exist and be automorphic, determined at all local places by the local Gross–Prasad conjectures so as to have nonvanishing local Bessel periods.

Theorem 1.1. *There exists a class of matching functions f and f_ϵ such that the Bessel identity* (1.11) *holds.*

See Theorem 5.1 for a more precise version of our main result. We note that our choice of matching functions guarantees the geometric and spectral expansions of our trace formulas are convergent without any regularization of integrals. To get from (1.11) to a special value formula, a detailed study of local Bessel distributions as in [21], [27], [4], [25] or [40] is needed. At present, we merely conclude

Corollary 1.2. *Suppose now that E/F is split at each archimedean place. Then*

$$L(1/2,\pi)L(1/2,\pi\otimes\kappa)\neq 0$$

if and only if π_ϵ has a Bessel period with respect to E.

See Corollaries 5.2 and 5.3 below. Thus we establish Conjectures 1.1 and 1.2 under certain assumptions. We remark that, by completely different methods, Ginzburg–Jiang–Rallis [14] make substantial progress towards the global Gross–Prasad conjecture for $(\mathrm{SO}(2n+1),\mathrm{SO}(2))$. However, they assume that their representations of $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2)$ transfer to cuspidal representations of $\mathrm{GL}(2n)$ and $\mathrm{GL}(2)$. Under these hypotheses, they obtain one direction of the global Gross–Prasad conjectures, and partial results for the converse direction. Our Corollary 1.2 establishes both directions of the global Gross–Prasad conjecture for the case $(\mathrm{SO}(5),\mathrm{SO}(2))$ (under our local hypotheses) in the case that the $\mathrm{SO}(2)$ representation is trivial, whence the $\mathrm{SO}(2)$ representation does not transfer to a cuspidal representation of $\mathrm{GL}(2)$. Thus, there is no overlap of this result with the results of [14].

We hope to remove our local assumptions and eventually obtain an L-value formula with future work on these trace formulas. We also remark that W. Zhang [39] also recently established a global Gross–Prasad conjecture for certain unitary groups under some local assumptions by using a simple relative trace formula.

In Section 2, we prove local matching for functions functions supported on the "regular sets" and the open Bruhat cell (the former is contained in the latter). We are also able to describe the behavior of the regular local orbital integrals near the singular set within the open Bruhat cells (i.e., the "germs" within the open Bruhat cells). This is done by reducing to a twisted version of the orbital integrals in [19], and performing a similar analysis as in [19].

In Section 3, we show for tempered representations, one can choose local functions supported in the regular set such that the local Bessel distributions are nonvanishing on these functions. This is similar to [18]. In Section 4, we derive the necessary simple relative trace formulas by choosing functions which are regularly supported at one place and supercusp forms at another place to ensure convergence of the geometric and spectral expansions of (1.6) and (1.9). Finally, in Section 5 we state and prove our global results.

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2. Some Matching Results

In this section, we will study orbital integrals and matching functions over a local field F of characteristic 0, though in Sections 2.1 and 2.2, we also allow F to be global. Throughout, E/F is a quadratic field extension. As above, D_ϵ will be a quaterion algebra (possibly split) over F.

In addition, we will only be considering the F-points of the relevant algebraic groups here, so for an algebraic group $\mathbb{G}$ over F, we simply write $\mathbb{G}$ for $\mathbb{G}(F)$.

2.1. **Double cosets for G.** Let F be local or global. Let

$$P = \left\{ \begin{pmatrix} A & \\ & \lambda {}^t A^{-1} \end{pmatrix} \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} : A \in \mathrm{GL}(2),\ \lambda \in \mathrm{GL}(1),\ X \in \mathrm{Sym}(2) \right\}$$

be the Siegel parabolic subgroup of G, and $\bar{P}$ be its transpose. Let U (resp. $\bar{U}$) be the unipotent radical of P (resp. $\bar{P}$). Consider the Bruhat decomposition

$$G = \bar{P}P \cup \bar{P}w_1 P \cup \bar{P}w_2 P$$

where

$$w_1 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Note

$$\bar{P}P = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G : A \in \mathrm{GL}(2) \right\}$$

is the large Bruhat cell, Pw_2P is the intermediate Bruhat cell and

$$\bar{P}w_2P = \left\{ \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \in G \right\}$$

is the small Bruhat cell.

If H_1 and H_2 are subgroups of G, by the action of $H_1 \times H_2$ on G, we mean the map given by left multiplication by H_1 and right multiplication by H_2, i.e., $(h_1, h_2) \in H_1 \times H_2$ corresponds to the map $g \mapsto h_1 g h_2$. While this is not technically a group action, this minor abuse of terminology makes notation slightly simpler.

We would like to determine the polynomial invariants for the double cosets $\bar{H}\backslash G/H$, i.e., the subring $F[G]^{\bar{H}\times H}$ of polynomials $p(x) \in F[G]$ such that $p(\bar{h}xh) = p(x)$ for $\bar{h} \in \bar{H}$, $h \in H$. Write $H = TU$ and $\bar{H} = T\bar{U}$ where T is the diagonal torus. Note that left multiplication by $\bar{U}$ and right multiplication by U preserve both the upper left 2×2 block $A(g)$ of an element $g \in G$, as well as the similitude factor $\lambda(g)$. In fact, $\lambda(g)$ and the entries of $A(g)$ generate all polynomial invariants for $\bar{U}\backslash G/U$. Thus we have a $\bar{U} \times U$-invariant morphism (of algebraic varieties) from G to $M(2) \times \mathrm{GL}(1)$ given by

$$g \mapsto (A(g), \lambda(g)),$$

where $M(2)$ denotes the full 2×2 matrix algebra. Further, the action of $T \times T$ on G induces the action

$$(A, \lambda) \mapsto (tAt', \det(tt')\lambda) \tag{2.1}$$

on $M(2) \times \mathrm{GL}(1)$ under the above morphism, where

$$\left(\begin{pmatrix} t & \\ & wtw \end{pmatrix}, \begin{pmatrix} t' & \\ & wt'w \end{pmatrix} \right) \in T \times T, \quad w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

It is then easy to see, that invariants of this action are given by

$$y(A,\lambda) = \lambda^{-1}ad \tag{2.2}$$

$$z(A,\lambda) = \lambda^{-1}bc \tag{2.3}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and these generate all polynomial invariants for the action of $T \times T$ on $M(2) \times \mathrm{GL}(1)$ given by (2.1). Consequently, the polynomial invariants for $\bar{H}\backslash G/H$ are generated by the two invariants

$$y(g) = y(A(g), \lambda(g)), \quad z(g) = z(A(g), \lambda(g)). \tag{2.4}$$

In particular, the map $g \mapsto (y(g), z(g))$ an $\bar{H} \times H$-invariant morphism $G \to \mathcal{S}$ where

$$\mathcal{S} = F \times F \tag{2.5}$$

is the parameter space for double cosets $\bar{H}\backslash G/H$ (more precisely, this parametrizes the double cosets $\bar{H}gH$ which are closed in G).

Let

$$H_g = \left\{(\bar{h}, h) \in \bar{H} \times H : \bar{h}gh = g\right\}$$

denote the stabilizer of g under the action of $\bar{H} \times H$. We say a double coset $\bar{H}gH$ is *relevant*, or simply that g is relevant, if

$$\psi(h_1)\theta(h_2) = 1 \quad \text{for all } (h_1, h_2) \in H_g.$$

We call $\bar{H}gH$ (or g) *regular* if

$$(y(g), z(g)) \in \mathcal{S}^{\mathrm{reg}} := F^\times \times F^\times - \Delta F^\times \tag{2.6}$$

Note both of these properties only depend upon the double coset of g. The relevant $(\bar{H}, H)$-double cosets are classified in [9, Prop. 6.4]. However, we will not need this complete classification, but only the following classification of relevant double cosets in $\bar{P}P$, which is elementary.

Let

$$n_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; n_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2).$$

For $x \in F - \{0, 1\}$, put

$$A(x) = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2).$$

If $A \in \mathrm{GL}(2)$ and $\lambda \in F^\times$, put

$$\delta(A, \lambda) = \begin{pmatrix} A & \\ & \lambda\, {}^tA^{-1} \end{pmatrix} \in G.$$

Note that $g \in \bar{P}P$ if and only if $(y(g), z(g)) \in \mathcal{S}^{\mathrm{big}} := F \times F - \Delta F$.

Proposition 2.1. *(1) If g is regular then $H_g = \left\{(z, z^{-1}) : z \in Z\right\}$ and g is relevant. Conversely, if $(y, z) \in \mathcal{S}^{\mathrm{reg}}$, then there is a unique double coset $\bar{H}gH$ with $(y(g), z(g)) = (y, z)$; a representative is given by*

$$\gamma(y, z) = \delta(A(y/z), 1/y).$$

(2) If $g \in \bar{P}P$ is relevant but not regular, then $H_g = \{(z, z^{-1}) : z \in Z\}$ and $(y(g), z(g))$ is of the form $(0, z)$ for $z \in F^\times$ or $(y, 0)$ for $y \in F^\times$. Conversely, given $(y, 0)$ (resp. $(0, z)$) with y (resp. z) in $F^\times$, there are two relevant double cosets HgH in $\bar{P}P$ with $(y(g), z(g)) = (y, 0)$ (resp. $(0, z)$), which are represented by

$$\gamma^{\pm}(y, 0) = \delta(n_{\pm}, y^{-1})$$
$$(\textit{resp.} \quad \gamma^{\pm}(0, z) = \delta(wn_{\pm}, z^{-1})\,).$$

2.2. **Double cosets for G_ϵ.** As in the previous section, F may be either local or global. Let N denote the norm map from either D_ϵ or E to F. Fix a set of representatives $\{\epsilon_1, \epsilon_2\}$ for $F^\times/N(E^\times)$ such that $\epsilon_1 \in N(E^\times)$ and $\epsilon_2 \notin N(E^\times)$. Thus G_{ϵ_1} is split and G_{ϵ_2} is not. Let $\epsilon \in \{\epsilon_1, \epsilon_2\}$.

Let P_ϵ denote the Siegel parabolic

$$P_\epsilon = \left\{ \begin{pmatrix} \alpha & \\ & \mu\bar{\alpha}^{-1} \end{pmatrix} \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} : \alpha \in D_\epsilon^\times,\, \mu \in F^\times,\, X \in D_\epsilon^- \right\}$$

and U_ϵ its unipotent radical. Denote their transposes by $\bar{P}_\epsilon$ and $\bar{U}_\epsilon$. The Bruhat decomposition for G_ϵ is of the form

$$G_\epsilon = \bar{P}_\epsilon P_\epsilon \cup \bar{P}_\epsilon w_3 P_\epsilon$$

if $\epsilon = \epsilon_2$ is and

$$G_\epsilon = \bar{P}_\epsilon P_\epsilon \cup \bar{P}_\epsilon w_3 P_\epsilon \cup \bar{P}_\epsilon w_4 P_\epsilon$$

if $\epsilon = \epsilon_1$. Here w_3 and w_4 are appropriate elements of the Weyl group.

The ring $F[G_\epsilon]^{\bar{R}_\epsilon \times R_\epsilon}$ of polynomial invariants for $\bar{R}_\epsilon \backslash G_\epsilon / R_\epsilon$ can be determined in a similar manner to the Novodvorsky case. Namely, the polynomial invariants for $\bar{U}_\epsilon \backslash G_\epsilon / U_\epsilon$ are generated by $(\alpha(g), \mu(g)) \in D_\epsilon \times \mathrm{GL}(1)$, where $\alpha(g)$ denotes the upper left entry of g (viewed as a matrix inside $\mathrm{GL}_2(D_\epsilon)$) and $\mu(g)$ denotes the similitude factor of g.

Write the Bessel subgroups $R_\epsilon = T_\epsilon U_\epsilon$ and $\bar{R}_\epsilon = T_\epsilon \bar{U}_\epsilon$, where $T_\epsilon \simeq E^\times$ is the torus of R_ϵ or $\bar{R}_\epsilon$. The action of $T_\epsilon \times T_\epsilon$ on G_ϵ induces the action

$$(s, t) \cdot (\alpha, \mu) = (s\alpha t, N(st)\mu), \quad s, t \in E^\times$$

on the invariant space $D_\epsilon \times \mathrm{GL}(1)$ for $\bar{U}_\epsilon \backslash G_\epsilon / U_\epsilon$. If we write $\alpha = \begin{pmatrix} a & b\epsilon \\ b^\sigma & a^\sigma \end{pmatrix} \in D_\epsilon$, $T_\epsilon \times T_\epsilon$ invariants under this action are

$$y_\epsilon(\alpha, \mu) = \mu^{-1} N(a), \quad z_\epsilon(\alpha, \mu) = \mu^{-1} \epsilon N(b). \tag{2.7}$$

Hence one can see all polynomial invariants for $\bar{R}_\epsilon \backslash G_\epsilon / R_\epsilon$ are generated by the invariants

$$y_\epsilon(g) = y_\epsilon(\alpha(g), \mu(g)), \quad z_\epsilon(\alpha, \mu) = z_\epsilon(\alpha(g), \mu(g)). \tag{2.8}$$

Thus $g \mapsto (y_\epsilon(g), z_\epsilon(g))$ is a $\bar{R}_\epsilon \times R_\epsilon$-invariant map from G to

$$\mathcal{S}_\epsilon = \left\{ (y, z) \in F \times F : yz = 0 \text{ or } \frac{z}{y} \in \epsilon N(E^\times) \right\}. \tag{2.9}$$

Denote by $R_{\epsilon,g}$ the stabilizer of g under the action of $\bar{R}_\epsilon \times R_\epsilon$. We say a double coset $\bar{R}_\epsilon g R_\epsilon$ (or simply g) is *relevant* if

$$\xi(\bar{r})\tau(r) = 1 \quad \text{for all } (\bar{r}, r) \in R_{\epsilon,g}. \tag{2.10}$$

We say $\bar{R}_\epsilon g R_\epsilon$ (or g) is *regular* if

$$(y_\epsilon(g), z_\epsilon(g)) \in \mathcal{S}_\epsilon^{\mathrm{reg}} := \{(y,z) \in \mathcal{S}_\epsilon : yz \neq 0 \text{ and } y \neq z\}. \tag{2.11}$$

Then g regular implies both that $g \in \bar{P}_\epsilon P$ and g is relevant. The following classification of relevant double cosets in $\bar{P}_\epsilon P_\epsilon$ is elementary (see [9, Prop. 5.2] for the full classification of relevant double cosets for G_{ϵ_1}).

For $x \in F$, let

$$\alpha_\epsilon(x) = \begin{pmatrix} 1 & \epsilon u \\ u^\sigma & 1 \end{pmatrix} \in D_\epsilon \tag{2.12}$$

where $u \in E$ such that $\epsilon N(u) = x$. While $\alpha_\epsilon(x)$ depends on a choice of u, the orbital integrals we define later will only depend upon x. We also put

$$\alpha_\epsilon(\infty) = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix} \in D_\epsilon.$$

For $\alpha \in D_\epsilon^\times$ and $\mu \in F^\times$, put

$$\delta_\epsilon(\alpha, \mu) = \begin{pmatrix} \alpha & \\ & \mu\bar{\alpha}^{-1} \end{pmatrix} \in G_\epsilon.$$

Note that $g \in \bar{P}_\epsilon P_\epsilon$ if and only if $\alpha(g) \in D_\epsilon^\times$. This is equivalent to $(y_\epsilon(g), z_\epsilon(g)) \in \mathcal{S}_\epsilon^{\mathrm{big}}$ where $\mathcal{S}_\epsilon^{\mathrm{big}} = \{(y,z) \in \mathcal{S}_\epsilon : y \neq z\}$.

Proposition 2.2. *(1) If g is regular, then $R_{\epsilon,g} = \{(z, z^{-1}) : z \in Z\}$ and g is relevant. Conversely, if $(y,z) \in \mathcal{S}_\epsilon^{\mathrm{reg}}$, then there is a unique double coset $\bar{R}_\epsilon g R_\epsilon$ with $(y_\epsilon(g), z_\epsilon(g)) = (y,z)$; a representative is given by*

$$\gamma_\epsilon(y,z) = \delta_\epsilon(\alpha_\epsilon(y/z), 1/y).$$

(2) If $g \in \bar{P}_\epsilon P_\epsilon$ is relevant but not regular, then $R_{\epsilon,g} = \{(t, t^{-1}) : t \in T_\epsilon\}$ and $(y_\epsilon(g), z_\epsilon(g))$ is of the form $(y, 0)$ for $y \in F^\times$ or $(0, z)$ for $z \in F^\times$. Conversely, given $(y,0)$ (resp. $(0,z)$) with y (resp. z) $\in F^\times$, there is exactly one relevant double coset $\bar{R}_\epsilon g R_\epsilon$ in $\bar{P}_\epsilon P_\epsilon$ with $(y_\epsilon(g), z_\epsilon(g)) = (y,0)$ (resp. $(0,z)$), which is represented by

$$\gamma_\epsilon(y,0) = \delta_\epsilon(\alpha_\epsilon(0), y^{-1}) = \delta_\epsilon(1, y^{-1})$$
$$(\textit{resp.} \quad \gamma_\epsilon(0,z) = \delta_\epsilon(\alpha_\epsilon(\infty), z^{-1})\,).$$

2.3. **Local orbital integrals for G.** Now assume F is local. Fix Haar measures dh and $d\bar{h}$ on the unimodular groups H and $\bar{H}$. Denote by G^{reg} the set of regular elements in G.

For $f \in C_c^\infty(G)$ and $g \in G$, we define the *local Novodvorsky (or split Bessel) orbital integral*

$$\mathcal{N}(g; f) = \int_{(\bar{H} \times H)/H_g} f(\bar{h} g h) \theta(h) \psi(\bar{h}) \, dh \, d\bar{h},$$

at least when this integral converges.

Lemma 2.3. *For $g \in G^{\mathrm{reg}}$, the map $\iota : (\bar{H} \times H)/H_g \to G$ given by $(\bar{h}, h) \mapsto \bar{h} g h$ is proper.*

Proof. Note that we can write ι as a composition of two maps

$$\iota_1 : (T \times T)/H_g \to \mathrm{GL}(2) \times \mathrm{GL}(1)$$
$$\left(\begin{pmatrix} t & \\ & wtw \end{pmatrix}, \begin{pmatrix} t' & \\ & wt'w \end{pmatrix}\right) \mapsto (tA(g)t', \det(tt')g)$$

and

$$\iota_2 : \mathrm{GL}(2) \times \mathrm{GL}(1) \times \bar{U} \times U \to G$$
$$(A, \lambda, \bar{u}, u) \mapsto \bar{u}\delta(A, \lambda)u.$$

Specifically

$$\iota(\bar{u}t, t'u) = \iota_2(\iota_1(t, t'), \bar{u}, u).$$

Hence it suffices to show ι_1 and ι_2 are proper. Both of these statements are elementary. □

Then the following is immediate.

Corollary 2.4. *For $g \in G^{\mathrm{reg}}$ and $f \in C_c^\infty(G)$, the local Novodvorsky orbital integral $\mathcal{N}(g; f)$ converges.*

For $(y, z) \in \mathcal{S}^{\mathrm{reg}}$ regular and $f \in C_c^\infty(G)$, put

$$\mathcal{N}(y, z; f) = \mathcal{N}(\gamma(y, z); f).$$

We want to analyze the behavior of the orbital integrals $\mathcal{N}(y, z; f)$ as functions of (y, z). First, we start with a simple special case.

We identify H/Z with the subgroup H_0 of H consisting of matrices of the form

$$\begin{pmatrix} a & & & \\ & 1 & & \\ & & 1 & \\ & & & a \end{pmatrix} \begin{pmatrix} 1 & Y \\ & 1 \end{pmatrix}.$$

Lemma 2.5. *The map from $G^{\mathrm{reg}} \to \bar{H} \times H_0 \times \mathcal{S}^{\mathrm{reg}}$ given by $\bar{h}\gamma(y, z)h \mapsto (\bar{h}, h, y, z)$ is continuous. If F is archimedean, it is also smooth.*

Proof. First note the map $G^{\mathrm{reg}} \to \mathrm{Sym}(2) \times \mathrm{Sym}(2) \times \mathrm{GL}(2)^{\mathrm{reg}} \times \mathrm{GL}(1)$

$$\begin{pmatrix} A & AY \\ XA & XAY + \lambda {}^tA^{-1} \end{pmatrix} \mapsto (X, Y, A, \lambda)$$

is continuous (and smooth if F archimedean). Here $\mathrm{GL}(2)^{\mathrm{reg}}$ denotes the set of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)$ such that $abcd \neq 0$. Then it suffices to note the map $\mathrm{GL}(2)^{\mathrm{reg}} \to \mathrm{GL}(1)^3$ given by

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c & \\ & 1 \end{pmatrix} \mapsto (a, b, c)$$

is continuous (smooth if F archimedean). □

Lemma 2.6. *The map*

$$C_c^\infty(G^{\mathrm{reg}}) \to C_c^\infty(\mathcal{S}^{\mathrm{reg}})$$
$$f \mapsto \phi_f(y, z) := \mathcal{N}(y, z; f)$$

is well-defined and surjective.

Proof. That $\phi_f(y,z) \in C^\infty(\mathcal{S}^{\rm reg})$ is clear. Hence to see the map is well defined, it suffices to observe that if Ω is a compact set in $G^{\rm reg}$, then $\{(y(g), z(g)) : g \in \Omega\}$ is a compact set in $\mathcal{S}^{\rm reg}$.

For the converse, let $\phi \in C_c^\infty(\mathcal{S}^{\rm reg})$. Let ϕ_1 and ϕ_2 be smooth compactly supported functions on $\bar{H}$ and H_0 such that

$$\int_{\bar{H}} \int_{H_0} \phi_1(\bar{h})\phi_2(h)\theta_1(\bar{h})\theta_2(h)\, dh\, d\bar{h} = 1.$$

Define f by

$$f(\bar{h}\gamma(y,z)h) = \phi_1(\bar{h})\Phi(y,z)\phi_2(h)$$

on $G^{\rm reg}$. Since the map $\bar{h}\gamma(y,z)h \mapsto (\bar{h}, h, y, z)$ is continuous (smooth if F is archimedean), we see that f is smooth. It is also clear that it must have compact support on $G^{\rm reg}$, and that $\phi_f = \phi$. □

Now let us consider functions supported on the big Bruhat cell. Write the Haar measures on H and $\bar{H}$ as $dh = dt\, du$ and $d\bar{h} = dt\, d\bar{u}$ where dt, du and $d\bar{u}$ are Haar measures on T, U and $\bar{U}$. For $f \in C_c^\infty(\bar{P}P)$, $A \in {\rm GL}(2)$ and $\lambda \in {\rm GL}(1)$, define

$$\Phi_f(A,\lambda) = \int_{\bar{U}} \int_U f(\bar{u}\delta(A,\lambda)u)\psi(\bar{u})\theta(u)\, d\bar{u}du. \tag{2.13}$$

Lemma 2.7. *The map*

$$C_c^\infty(\bar{P}P) \to C_c^\infty({\rm GL}(2) \times {\rm GL}(1))$$
$$f \mapsto \Phi_f$$

is well defined and surjective.

Proof. It is easy to see that $\Phi_f(A,\lambda)$ is a smooth function of ${\rm GL}(2) \times {\rm GL}(1)$ which vanishes if any entry of A is sufficiently large or λ lies outside of a compact set. So to show this map is well defined, it suffices to show $\Phi_f(A,\lambda)$ vanishes if $\det(A)$ is sufficiently close to 0. Given $g \in \bar{P}P$, write

$$g = \begin{pmatrix} 1 & \\ X & 1 \end{pmatrix} \begin{pmatrix} A & \\ & \lambda {}^tA^{-1} \end{pmatrix} \begin{pmatrix} 1 & Y \\ & 1 \end{pmatrix} = \begin{pmatrix} A & AY \\ XA & XAY + \lambda {}^tA^{-1} \end{pmatrix}.$$

Then the map from $\bar{P}P$ to ${\rm GL}(1)$ given by $\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \mapsto \det(g_4 - g_3 g_1^{-1} g_2)$ sends $g \mapsto \lambda^2 \det(A)^{-1}$. This map is continuous, so $\Phi(A,\lambda) = 0$ when $\det(A)$ lies outside of some fixed compact set.

To see surjectivity, given $\Phi \in C_c^\infty({\rm GL}(2) \times {\rm GL}(1))$, choose $\phi_1, \phi_2 \in C_c^\infty({\rm Sym}(2))$ such that

$$\int_{{\rm Sym}(2)} \int_{{\rm Sym}(2)} \phi_1(X)\phi_2(Y)\psi({\rm tr}(w(X+Y)))\, dX\, dY = 1$$

and set

$$f\left[\begin{pmatrix} 1 & \\ X & 1 \end{pmatrix} \delta(A,\lambda) \begin{pmatrix} 1 & Y \\ & 1 \end{pmatrix}\right] = \phi_1(X)\phi_2(Y)\Phi(A,\lambda).$$

□

For $\Phi \in C_c^\infty({\rm GL}(2) \times {\rm GL}(1))$, define the split toric orbital integral by

$$I(A,\lambda;\Phi) = \int_{(F^\times)^3} \Phi\left[\begin{pmatrix} a & \\ & b \end{pmatrix} A \begin{pmatrix} c & \\ & 1 \end{pmatrix}, abc\lambda\right] \eta(c)\, d^\times a\, d^\times b\, d^\times c, \tag{2.14}$$

at least if $(A, \lambda) \in \mathrm{GL}(2)^{\mathrm{reg}} \times \mathrm{GL}(1)$, i.e., $\delta(A, \lambda) \in G^{\mathrm{reg}}$. The point is that

$$\mathcal{N}(\delta(A, \lambda); f) = I(A, \lambda; \Phi_f) \tag{2.15}$$

for $f \in C_c^\infty(G)$ and $\delta(A, \lambda)$ regular. In particular $I(A, \lambda; \Phi_f)$ is convergent for $(A, \lambda) \in \mathrm{GL}(2)^{\mathrm{reg}} \times \mathrm{GL}(1)$. If $(y, z) \in \mathcal{S}^{\mathrm{reg}}$, we also put

$$I(y, z; \Phi) = I(A(z/y), 1/y; \Phi) \tag{2.16}$$

so that

$$\mathcal{N}(y, z; f) = I(y, z; \Phi_f). \tag{2.17}$$

The orbital integrals $I(A, \lambda; \Phi)$ on $\mathrm{GL}(2) \times \mathrm{GL}(1)$ are just twisted versions of the split toric integrals on $\mathrm{GL}(2)$ study by Jacquet in [19], and they can be analyzed in the same manner.

Proposition 2.8. *For $\Phi \in C_c^\infty(\mathrm{GL}(2) \times \mathrm{GL}(1))$ the function $I(y, z) = I(y, z; \Phi)$ is a smooth function of $\mathcal{S}^{\mathrm{reg}}$ satisfying the following properties:*

(i) $I(y, z)$ vanishes for y, z sufficiently large;

(ii) $I(y, z)$ vanishes for $y - z$ sufficiently small;

(iii) there exist functions $\zeta_1, \zeta_2 \in C_c^\infty(F^\times \times F - \Delta F^\times)$ such that for $z \in F^\times$ sufficiently small and all $y \in F^\times$,

$$I(y, z) = \zeta_1(y, z) + (1 + \eta(z/y))\zeta_2(y, z);$$

(iv) there exist functions $\xi_1, \xi_2 \in C_c^\infty(F \times F^\times - \Delta F^\times)$ such that for $y \in F^\times$ sufficiently small and all $z \in F^\times$,

$$I(y, z) = \xi_1(y, z) + (1 + \eta(y/z))\xi_2(y, z);$$

Proof. The smoothness and properties (i) and (ii) are immediate. Let us prove (iii) and (iv).

Write $\mathrm{GL}(2) \times \mathrm{GL}(1)$ as a union of two open sets

$$\Omega_1 = T^{(2)} N \bar{N} \times \mathrm{GL}(1)$$

$$\Omega_2 = T^{(2)} N w \bar{N} \times \mathrm{GL}(1)$$

where $T^{(2)}$ denotes the diagonal torus of $\mathrm{GL}(2)$, N is the standard upper unipotent subgroup of $\mathrm{GL}(2)$, and $\bar{N}$ its transpose. Now write $\Phi = \Phi_1 + \Phi_2$ where $\Phi_i \in C_c^\infty(\Omega_i)$. Let

$$\phi_i(A, \lambda) = \int_{(F^\times)^2} \Phi_i \left[\begin{pmatrix} a & \\ & b \end{pmatrix} A, ab\lambda \right] d^\times a \, d^\times b,$$

$$\Psi_1(u, v; \lambda) = \phi_1 \left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ v & 1 \end{pmatrix}, \lambda \right),$$

and

$$\Psi_2(u, v; \lambda) = \phi_2 \left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & \\ v & 1 \end{pmatrix}, \lambda \right).$$

Note $\Psi_i \in C_c^\infty(F \times F \times \mathrm{GL}(1))$. Since

$$\begin{aligned} A(x) \begin{pmatrix} c & \\ & 1 \end{pmatrix} &= \begin{pmatrix} c(1-x) & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & c^{-1}(1-x)^{-1}x \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-x & \\ & c \end{pmatrix} \begin{pmatrix} 1 & c(1-x)^{-1} \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & c^{-1} \\ & 1 \end{pmatrix}, \end{aligned}$$

and

$$\phi_1\left[\begin{pmatrix} a & \\ & b\end{pmatrix} A, \lambda\right] = \phi_1(A, (ab)^{-1}\lambda)$$

we have

$$\begin{aligned} I(y,z) &= \int_{F^\times} \phi_1\left[A(z/y)\begin{pmatrix} c & \\ & 1\end{pmatrix}; cy^{-1}\right]\eta(c)d^\times c + \int_{F^\times} \phi_2\left[A(z/y)\begin{pmatrix} c & \\ & 1\end{pmatrix}; cy^{-1}\right]\eta(c)d^\times c \\ &= \int_{F^\times} \Psi_1\left[\frac{z}{c(y-z)}, c; \frac{1}{y-z}\right]\eta(c)d^\times c + \int_{F^\times} \Psi_2\left[\frac{cy}{y-z}, c^{-1}; \frac{1}{y-z}\right]\eta(c)d^\times c. \end{aligned}$$

Then by [19, Lem. 3.2], there exist functions $\lambda_1, \lambda_2 \in C_c^\infty(F \times \mathrm{GL}(1))$ such that for $z \neq 0$ and $y \neq z$, one has

$$\int_{F^\times} \Psi_1\left[\frac{z}{c(y-z)}, c; \frac{1}{y-z}\right]\eta(c)d^\times c = \lambda_1\left(\frac{z}{y-z}, \frac{1}{y-z}\right) + \lambda_2\left(\frac{z}{y-z}, \frac{1}{y-z}\right)\eta\left(\frac{z}{y-z}\right)$$

By property (ii), there exists k such that $I(y,z;\Phi)$ vanishes if $|y-z| < k$, so we may assume $\lambda_i\left(z(y-z)^{-1}, (y-z)^{-1}\right) = 0$ if $|y-z| < k$. Put $\zeta_i(y,z) = \lambda_i\left(\frac{z}{y-z}, \frac{1}{y-z}\right)$. Then $\zeta_i \in C_c^\infty(F^\times \times F - \Delta F^\times)$. Furthermore, if $|y-z| \geq k$, then for z sufficently small, $\eta(z(y-z)^{-1}) = \eta(z/y)$. This proves (iii), and (iv) is similar. □

2.4. **Local orbital integrals for G_ϵ.** Let E/F be a quadratic extension of local fields. For $\epsilon \in \{\epsilon_1, \epsilon_2\}$, fix Haar measures $d\bar{r}$ and dr on $\bar{R}_\epsilon$ and R_ϵ. We also write $d\bar{r} = dt\, d\bar{u}$ and $dr = dt\, du$ where dt, $d\bar{u}$ and du are Haar measures on T_ϵ, $\bar{U}_\epsilon$ and U_ϵ. Denote by $G_\epsilon^{\mathrm{reg}}$ the set of regular elements of G_ϵ.

For $f \in C_c^\infty(G_\epsilon)$ and $g \in G_\epsilon$, define the *local (anisotropic) Bessel orbital integral*

$$\mathcal{B}_\epsilon(g; f) = \int_{(\bar{R}_\epsilon \times R_\epsilon)/R_{\epsilon,g}} f(\bar{r}gr)\xi(\bar{r})\tau(r)\, d\bar{r}\, dr, \tag{2.18}$$

at least when this integral converges.

Lemma 2.9. *For $f \in C_c^\infty(G_\epsilon)$ and $g \in G_\epsilon^{\mathrm{reg}}$, the local Bessel integral $B_\epsilon(g; f)$ converges.*

Proof. The proof is analogous to that for Corollary 2.4. □

For $(y,z) \in \mathcal{S}_\epsilon^{\mathrm{reg}}$ and $f \in C_c^\infty(G_\epsilon)$, let

$$\mathcal{B}_\epsilon(y,z;f) = \mathcal{B}_\epsilon(\gamma_\epsilon(y,z); f).$$

Lemma 2.10. *The map*

$$\begin{aligned} C_c^\infty(G_\epsilon^{\mathrm{reg}}) &\to C_c^\infty(\mathcal{S}_\epsilon^{\mathrm{reg}}) \\ f &\mapsto \phi_f(y,z) := \mathcal{B}_\epsilon(y,z;f) \end{aligned}$$

is well-defined and surjective.

Proof. The proof is similar to that for Lemma 2.6. □

For $f \in C_c^\infty(G_\epsilon)$, $\alpha \in D_\epsilon^\times$ and $\mu \in \mathrm{GL}(1)$, define

$$\Phi_f(\alpha,\mu) = \int_{\bar{U}_\epsilon}\int_{U_\epsilon} f(\bar{u}\delta_\epsilon(\alpha,\mu)u)\xi(\bar{u})\tau(u)\, d\bar{u}\, du. \tag{2.19}$$

Lemma 2.11. *The map*

$$C_c^\infty(\bar{P}_\epsilon P_\epsilon) \to C_c^\infty(D_\epsilon^\times \times \mathrm{GL}(1))$$
$$f \mapsto \Phi_f$$

is well defined and surjective.

Proof. The proof is similar to that for Lemma 2.7. □

For $\Phi \in C_c^\infty(D_\epsilon^\times \times \mathrm{GL}(1))$, define the anisotropic toric orbital integral by

$$I_\epsilon(\alpha, \mu; \Phi) = \int_{E^\times/F^\times} \int_{E^\times} \Phi\left[s\alpha t, N(st)\mu\right]\, d^\times s\, d^\times t \tag{2.20}$$

whenever (α, μ) is "relevant" in $D_\epsilon^\times \times \mathrm{GL}(1)$, i.e., $\delta_\epsilon(\alpha,\mu)$ is relevant in G_ϵ, i.e., α is not in $E^\times$ or $w_\epsilon E^\times$, where $w_\epsilon = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix} \in D_\epsilon^\times$. Then we have

$$\mathcal{B}_\epsilon(\delta(\alpha,\mu); f) = I_\epsilon(\alpha, \mu; \Phi_f). \tag{2.21}$$

For $(y,z) \in \mathcal{S}_\epsilon^{\mathrm{reg}}$, we also denote

$$I_\epsilon(y, z; \Phi) = I_\epsilon(\alpha_\epsilon(z/y), 1/y; \Phi) \tag{2.22}$$

so

$$\mathcal{B}_\epsilon(y, z; f) = I_\epsilon(y, z; \Phi_f). \tag{2.23}$$

We remark that while $\alpha_\epsilon(x)$ technically depends upon the choice of u in (2.12), i.e. (2.12) is ambiguous up to an element of $E^1 = \ker(N|_{E^\times})$, it is easy to see the integrals $B_\epsilon(\delta(\alpha(x), \mu); f)$ and $I_\epsilon(\alpha_\epsilon(x), \mu; \Phi)$ only depend upon x and not the choice of u such that $\epsilon N(u) = x$.

Proposition 2.12. *For $\Phi \in C_c^\infty(D_\epsilon^\times \times \mathrm{GL}(1))$, the function $I_\epsilon(y,z) = I_\epsilon(y,z;\Phi)$ is a smooth function on $\mathcal{S}_\epsilon^{\mathrm{reg}}$ satisfying the following properties:*

(i) $I_\epsilon(y,z)$ vanishes for y, z sufficiently large;

(ii) $I_\epsilon(y,z)$ vanishes for $y - z$ sufficiently small;

(iii) there exists $\zeta_\epsilon \in C_c^\infty(F^\times \times F)$ on such that $I_\epsilon(y,z) = \zeta_\epsilon(y,z)$ whenever z sufficiently close to 0 and $\frac{z}{y} \in \epsilon N(E^\times)$; and

(iv) there exists $\xi_\epsilon \in C_c^\infty(F \times F^\times)$ on such that $I_\epsilon(y,z) = \xi_\epsilon(y,z)$ whenever y sufficiently close to 0 and $\frac{y}{z} \in \epsilon N(E^\times)$.

Furthermore the functions ζ_ϵ and ξ_ϵ take the singular orbital integrals as values

$$\zeta_\epsilon(y, 0) = \mathrm{vol}(E^\times/F^\times) \int_{E^\times} \Phi(t, N(t)y^{-1})\, dt$$

$$\xi_\epsilon(0, z) = \mathrm{vol}(E^\times/F^\times) \int_{E^\times} \Phi(w_\epsilon t, \epsilon^{-1}N(t)z^{-1})\, dt.$$

Conversely, any function $\phi(y,z) \in C^\infty(\mathcal{S}^{\mathrm{reg}})$ satisfying (i)—(iv) is of the form $I_\epsilon(y,z;\Phi)$ for some $\Phi \in C_c^\infty(D_\epsilon^\times \times \mathrm{GL}(1))$.

Note properties (iii) and (iv) mean that $I_\epsilon(y,z)$ extends to a function in $C_c^\infty(\mathcal{S}_\epsilon^{\mathrm{big}})$.

Proof. Properties (i) and (ii) follows immediately from the compact support of Φ.

Now let us prove (iii). Note the change of variables $s \mapsto st^{-1}$ in (2.20) yields

$$I_\epsilon(y,z) = \int_{E^\times/F^\times} \int_{E^\times} \Phi\left[s\begin{pmatrix} 1 & \epsilon a^{-1}a^\sigma u \\ a(a^\sigma)^{-1}u^\sigma & 1 \end{pmatrix}, N(s)y^{-1}\right] d^\times a\, ds,$$

where $t = \begin{pmatrix} a & \\ & a^\sigma \end{pmatrix}$ and $\epsilon N(u) = z/y$.

First assume F is nonarchimedean. Since Φ has compact support, for any v sufficiently small,

$$\Phi \left[\alpha \begin{pmatrix} 1 & \epsilon v \\ v^\sigma & 1 \end{pmatrix}, N(\alpha) y^{-1} \right] = \Phi \left[\alpha, N(\alpha) y^{-1} \right]$$

for all $\alpha \in D_\epsilon^\times$. Thus, for $z/y \in \epsilon \mathcal{N}$ sufficiently small

$$I_\epsilon(y,z) = \mathrm{vol}(E^\times / F^\times) \int_{E^\times} \Phi(s, N(s) y^{-1}) ds,$$

so for $z \in F$ sufficiently small, we may define

$$\zeta_\epsilon(y,z) = \mathrm{vol}(E^\times / F^\times) \int_{E^\times} \Phi(s, N(s) y^{-1}) ds$$

and $\zeta_\epsilon(y,z) = 0$ otherwise. The compact support of ζ_ϵ follows from (i) and (ii).

Now suppose $F = \mathbb{R}$ and $E = \mathbb{C}$. Note $\mathcal{S}_\epsilon^{\mathrm{reg}} \subset \mathbb{R}^2$ is just the two quadrants $yz < 0$ if $\epsilon < 0$ or the two quadrants $yz > 0$ minus the line $y = z$ if $\epsilon > 0$. Let $\Omega = \{u \in \mathbb{C} : |u| < k\}$ be a disc in $\mathbb{C}$ such that $1 + w_\epsilon \Omega \subset D_\epsilon^\times$. Set

$$I'_\epsilon(u,y) = \int_{S^1} \int_{\mathbb{C}^\times} \Phi \left[s(1 + w_\epsilon u) t \right] \, ds \, dt$$

for $u \in \Omega, y \in \mathbb{R}^\times$. Then $I'_\epsilon(u,y)$ is a smooth function on $\Omega \times \mathbb{R}^\times$ satisfying

$$I'_\epsilon(u,y) = I_\epsilon(y,z), \quad \epsilon |u| = \frac{z}{y}$$

for $(y,z) \in S_\epsilon^{\mathrm{reg}}$ such that $|z/y| < k$, and the value of this function only depends upon $|u|$. Hence, when $\epsilon yz \geq 0$, we may define $\zeta_\epsilon(y,z) = I'_\epsilon(|z/y|, y)$ when y is not too small, a $\zeta_\epsilon(y,z) = 0$ if y is sufficiently small. Then Whitney's extension theorem implies ζ_ϵ extends to a smooth function of $\mathbb{R}^2$.

The proof of (iv) is similar.

For the converse, let $\phi \in C^\infty(\mathcal{S}_\epsilon^{\mathrm{reg}})$ satisfying (i)-(iv). Define

$$\Phi(\alpha, \mu) = \begin{cases} \phi(y,z) & \alpha \in \mathcal{O}_E^\times \alpha_\epsilon(z/y) \mathcal{O}_E^\times, \, \mu \in y^{-1} \mathcal{O}_F^\times \\ 0 & \text{else.} \end{cases}$$

Then it is clear

$$\phi(y,z) = I_\epsilon(y,z; c\Phi)$$

for $(y,z) \in \mathcal{S}_\epsilon^{\mathrm{reg}}$, where $c = \mathrm{vol}_Z(\mathcal{O}_E^\times \cap Z) / \mathrm{vol}_{E^\times}(\mathcal{O}_E^\times)^2$. Properties (i) and (ii) imply that Φ has compact support, while (iii) and (iv) guarantee smoothness. ☐

2.5. **Local matching results.** Roughly, functions f and f_ϵ on G and G_ϵ will be called matching functions if $\mathcal{N}(g; f) = \mathcal{B}_\epsilon(g_\epsilon; f_\epsilon)$ whenever g and g_ϵ represent matching double cosets, i.e., $y(g) = y_\epsilon(g_\epsilon)$ and $z(g) = z_\epsilon(g_\epsilon)$. However, the local orbital integrals depend upon the choice of double coset representatives, so we make the following definition for local matching.

Definition 2.13. *Let $f \in C_c^\infty(G)$ and $f_\epsilon \in C_c^\infty(G_\epsilon)$ for $\epsilon \in \{\epsilon_1, \epsilon_2\}$. We say the functions f and $(f_\epsilon)_\epsilon$* match *if*

$$\mathcal{N}(x, \lambda; f) = \mathcal{B}_\epsilon(u, \lambda; f_\epsilon) \quad \textit{when} \quad x = \epsilon N_{E/F}(u) \tag{2.24}$$

for all $\epsilon \in \{\epsilon_1, \epsilon_2\}$, $\lambda \in F^\times$, $x \in F - \{0,1\}$ and $u \in E^\times$ such that $\epsilon N_{E/F}(u) \neq 1$.

Proposition 2.14. *Suppose $f \in C_c^\infty(G^{\mathrm{reg}})$. Then there exist $f_\epsilon \in C_c^\infty(G_\epsilon^{\mathrm{reg}})$ for $\epsilon \in \{\epsilon_1, \epsilon_2\}$ such that $(f_\epsilon)_\epsilon$ matches f. Conversely, given $(f_{\epsilon_1}, f_{\epsilon_2}) \in C_c^\infty(G_{\epsilon_1}^{\mathrm{reg}}) \times C_c^\infty(G_{\epsilon_2}^{\mathrm{reg}})$, there exists $f \in C_c^\infty(G^{\mathrm{reg}})$ such that f matches $(f_\epsilon)_\epsilon$.*

Proof. Observing that $\mathcal{S}_{\epsilon_1}^{\mathrm{reg}} \cup \mathcal{S}_{\epsilon_2}^{\mathrm{reg}}$ is a decomposition of $\mathcal{S}^{\mathrm{reg}}$ into two disjoint open sets, this proposition is immediate from Lemmas 2.6 and 2.10. □

Proposition 2.15. *Suppose $f \in C_c^\infty(\bar{P}P)$. Then there exist $f_\epsilon \in C_c^\infty(\bar{P}_\epsilon P_\epsilon)$ for $\epsilon \in \{\epsilon_1, \epsilon_2\}$ such that $(f_\epsilon)_\epsilon$ matches f.*

Proof. This follows from Propositions 2.8 and 2.12. □

3. Local Bessel distributions

Let F be a local field of characteristic zero. In this section, we let G' denote G or G_ϵ, and π an irreducible admissible representation of G'. In the former case, set $R = H$, $\bar{R} = \bar{H}$, χ_1 the character ψ of $\bar{R}$, and χ_2 the character θ of R. If $G' = G_\epsilon$, put $R = R_\epsilon$, $\bar{R} = \bar{R}_\epsilon$, χ_1 the character ξ of $\bar{R}$ and χ_2 the character τ of R. Let ℓ_1 and ℓ_2 be linear forms on π satisfying

$$\ell_1(\pi(\bar{r})\phi) = \chi_1(\bar{r})\ell_1(\phi), \quad \bar{r} \in \bar{R}$$

and

$$\ell_2(\pi(r)\phi) = \chi_2(r)\ell_2(\phi), \quad r \in R.$$

We define the local Bessel distribution (w.r.t. ℓ_1, ℓ_2) to be

$$B_\pi(f) = \sum_\phi \ell_1(\pi(f)\phi)\bar{\ell_2}(\phi) \tag{3.1}$$

where $f \in C_c^\infty(G')$ and ϕ runs over an orthonormal basis for π. It is easy to see that $B_\pi(f) \not\equiv 0$ if and only if both ℓ_1 and ℓ_2 are nonzero.

When, ℓ_1 and ℓ_2 are nonzero, we want to show a regularity result about B_π, namely that it must be nonzero on some function with compact support in the subset G'^{reg} of regular elements of G'. We do this when π is tempered and F is nonarchimedean, by combining the argument in [18] with some estimates in [38].

Assume for the rest of this section that π is tempered, F is nonarchimedean and ℓ_1 and ℓ_2 are nonzero. This nonvanishing condition forces π to have trivial central character.

It is more convenient to work on G'/Z, so if $f \in C_c^\infty(G'/Z)$ we abuse notation to write $\pi(f)\phi = \int_{G'/Z} f(g)\pi(g)\phi$ and with this convention also use $B_\pi(f)$ to denote the expression in (3.1). Let U denote the upper Siegel unipotent of G' and, for $c \in \mathbb{Z}$, U_c the subgroup of U given by

$$U_c = \left\{ \begin{pmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{pmatrix} : x, y, z \in F,\ \mathrm{ord}(y) \geq c \right\}$$

if $G' = \mathrm{GSp}(4)$ and

$$U_c = \left\{ \begin{pmatrix} 1 & & x & \epsilon y \\ & 1 & y^\sigma & -x \\ & & 1 & \\ & & & 1 \end{pmatrix} : x, y \in E,\ x^\sigma = -x,\ \mathrm{ord}(x) \geq c \right\}$$

if $G' = G_\epsilon$. We let $\bar{U}$ (resp. $\bar{U}_c$) be the transpose of U (resp. U_c). We let $T = T_R/Z(G)$ where T denotes the maximal torus of R. Thus $R/Z \simeq TU$ and $\bar{R}/Z \simeq T\bar{U}$.

Fix an inner product $(\, , \,)$ on π.

Lemma 3.1. *There exists a function* Φ *on* G' *of the form*

$$\Phi(g) = (\pi(g)\phi_1, \phi_2)$$

for some $\phi_1, \phi_2 \in \pi$ *such that for sufficiently large* c *the identity*

$$B_\pi(f) = \int_{T\bar{U}_c} \int_{TU_c} \left(\int_{G'} f(g)\Phi(\bar{r}gr)dg \right) \chi_1^{-1}(\bar{r})\chi_2^{-1}(r)\, dg d\bar{r} dr \quad (\bar{r} \in T\bar{U}_c,\, r \in TU_c)$$

holds for any $f \in C_c^\infty(G'/Z)$.

Proof. By [38, Lem. 5.1 and Prop. 5.7], for some ϕ_1 and ϕ_2 in π, one has

$$\ell_1(\phi) = \int_{T\bar{U}_c} (\pi(\bar{r})\phi, \phi_1)\chi_1^{-1}(\bar{r})\, d\bar{r}, \quad \ell_2(\phi) = \int_{TU_c} (\pi(r)\phi, \phi_2)\chi_2^{-1}(r)\, dr$$

for c sufficiently large. Here the integrals are absolutely convergent and independent of c for c sufficiently large. Consequently,

$$\begin{aligned}
B_\pi(f) &= \sum_\phi \int_{T\bar{U}_c} (\pi(\bar{r})\pi(f)\phi, \phi_1)\chi_1^{-1}(\bar{r})\, d\bar{r} \int_{TU_c} (\phi_2, \pi(r)\phi)\chi_2(r)\, dr \\
&= \sum_\phi \int_{T\bar{U}_c} \int_{TU_c} (\pi(r^{-1})\phi_2, \phi)(\phi, \pi(f^*)\pi(\bar{r}^{-1})\phi_1)\chi_1^{-1}(\bar{r})\chi_2(r)\, d\bar{r}\, dr \\
&= \int_{T\bar{U}_c} \int_{TU_c} (\pi(r^{-1})\phi_2, \pi(f^*)\pi(\bar{r}^{-1})\phi_1)\chi_1^{-1}(\bar{r})\chi_2(r)\, d\bar{r}\, dr \\
&= \int_{T\bar{U}_c} \int_{TU_c} (\pi(\bar{r})\pi(f)\pi(r)\phi_2, \phi_1)\chi_1(\bar{r})^{-1}\chi_2(r)^{-1}\, d\bar{r}\, dr \\
&= \int_{T\bar{U}_c} \int_{TU_c} \left(\int_{G'/Z} f(g)(\pi(\bar{r}gr)\phi_2, \phi_1)dg \right) \chi_1(\bar{r})^{-1}\chi_2(r)^{-1}\, d\bar{r}\, dr.
\end{aligned}$$

Here we put $f^*(g) = \overline{f(g^{-1})}$. □

For $c \in \mathbb{Z}$, let

$$I_c(g; \Phi) = \int_{T\bar{U}_c} \int_{TU_c} \Phi(\bar{r}gr)\chi_1^{-1}(\bar{r})\chi_2^{-1}(r)\, d\bar{r}\, dr. \tag{3.2}$$

Proposition 3.2. *For* c *sufficiently large,*

$$B_\pi(f) = \int_{G'/Z} f(g)I_c(g; \Phi)\, dg \tag{3.3}$$

for all $f \in C_c^\infty(G'/Z)$, *and this integral is absolutely convergent.*

Proof. Note that if we knew the absolute convergence of the right hand side of (3.3), the result would follow from the previous lemma and applying Fubini's theorem. This is what we will prove.

We may assume f is of the form Kg_0K for a special maximal compact open subgroup K of G'/Z.

We let Ξ denote the Harish-Chandra Ξ function on G'/Z associated to K. Recall that this is defined as follows (cf. [37]). Let A_0 denote the maximal split torus of

G'/Z with centralizer M_0, and P_0 a minimal parabolic with Levi M_0. Let π_0 denote the normalized induction of the trivial representation from P_0 to G'/Z. Let e be the unique K-fixed vector in π_0 with $e(1) = 1$. We set $\Xi(g) = (\pi_0(g)e, e)$. In particular $\Xi(g) \geq 0$ for all $g \in G'/Z$.

First note that, since π is tempered, there exists a constant C such that

$$\Phi(g) \leq C\,\Xi(g)$$

for all $g \in G'/Z$. Then

$$\begin{aligned}\int_{G'/Z} |f(g)I_c(g;\Phi)| &\leq \int_K \int_K |I_c(k_1 g_0 k_2; \Phi)|\, dk_1\, dk_2 \\ &\leq C \int_K \int_K \int_{T\bar{U}_c} \int_{TU_c} \Xi(\bar{r} k_1 g_0 k_2 r)\, d\bar{r}\, dr\, dk_1\, dk_2 \\ &\leq C\,\Xi(g_0) \int_{T\bar{U}_c} \Xi(\bar{r})\, d\bar{r} \int_{TU_c} \Xi(r)\, dr,\end{aligned}$$

by the property that $\int_K \Xi(g_1 k g_2)\, dk = \Xi(g_1)\Xi(g_2)$ [37, Lem. II.1.3]. Hence it suffices to show

$$\int_{T\bar{U}_c} \Xi(\bar{r})\, d\bar{r} < \infty \tag{3.4}$$

and

$$\int_{TU_c} \Xi(r)\, dr < \infty. \tag{3.5}$$

The proofs are similar, and we will just explain the latter bound.

First suppose T is compact. Then it suffices to show

$$\int_{U_c} \Xi(u)\, du < \infty.$$

This is a special case of [38, 4.3(3)]. Now suppose T is split. In this case we apply [38, 4.3(4)], which says

$$\int_{TU_c} \Xi^T(t)\Xi(tu)\, dt\, du < \infty,$$

where Ξ^T is the Harish-Chandra function for T. However T being split implies $\Xi^T(t) = 1$ for all $t \in T$. □

We let G'^{reg} denote the set of regular elements of G', where the notion of regular elements for G and G_ϵ is as defined above.

Corollary 3.3. *There exists $f \in C_c^\infty(G'^{\mathrm{reg}})$ such that $B_\pi(f) \neq 0$.*

Proof. Since the complement $(G'^{\mathrm{reg}}/Z)^c$ is a closed subset of measure 0 in G'/Z, it is immediate from the above proposition that there exists $f \in C_c^\infty(G'^{\mathrm{reg}}/Z)$ such that $B_\pi(f) \neq 0$. Now note the map $C_c^\infty(G'^{\mathrm{reg}}) \to C_c^\infty(G'^{\mathrm{reg}}/Z)$ given by

$$f(g) \mapsto f^Z(g) = \int_Z f(zg)\, dz$$

is surjective and satisfies $B_\pi(f) = B_\pi(f^Z)$. □

4. Simple relative trace formulas

We return to the global setting and notation as in Section 1. In particular, E/F denotes a quadratic extension of number fields. Recall for a finite v, Ξ_v denotes the characteristic function of $G(\mathcal{O}_v)$.

4.1. **A simple trace formula for G_ϵ.** Let $f_\epsilon \in C_c^\infty(G_\epsilon(\mathbb{A}))$. We may assume f_ϵ has a factorization $f_\epsilon = \prod f_{\epsilon,v}$ where $f_{\epsilon,v} = \Xi_v$ for almost all v. Consider the distribution J_ϵ given by (1.9).

Since we do not deal with arithmetic results here, we need not be too careful about our choice of measures arising from (1.9). For simplicity, choose Haar measures $dz = \prod dz_v$, $dr = \prod dr_v$ and $d\bar{r} = \prod d\bar{r}_v$ on $Z(\mathbb{A})$, $R_\epsilon(\mathbb{A})$ and $\bar{R}_\epsilon(\mathbb{A})$ such that at all finite places, $Z(\mathcal{O}_v)$, $R_\epsilon(\mathcal{O}_v)$ and $\bar{R}_\epsilon(\mathcal{O}_v)$ all have volume 1.

One formally has the geometric decomposition

$$J_\epsilon(f_\epsilon) = \sum_{\gamma \in \bar{R}_\epsilon(F)\backslash G_\epsilon(F)/R_\epsilon(F)} J_{\epsilon,\gamma}(f_\epsilon), \tag{4.1}$$

where

$$J_{\epsilon,\gamma}(f_\epsilon) = \int_{(\bar{R}_\epsilon(\mathbb{A})\times R_\epsilon(\mathbb{A}))/Z(\mathbb{A})R_{\epsilon,\gamma}(F)} f_\epsilon(\bar{r}^{-1}\gamma r z)\, dz\, \xi(\bar{r})^{-1}\tau(r)\, d\bar{r}\, dr.$$

We can rewrite this as

$$J_{\epsilon,\gamma}(f_\epsilon) = \mathrm{vol}(Z(\mathbb{A})R_{\epsilon,\gamma}(F)\backslash R_{\epsilon,\gamma}(\mathbb{A}))\mathcal{B}_\epsilon(\gamma; f_\epsilon), \tag{4.2}$$

where $\mathcal{B}_\epsilon$ denotes the *global Bessel orbital integral* given by

$$\mathcal{B}_\epsilon(\gamma; f_\epsilon) = \int_{(\bar{R}_\epsilon(\mathbb{A})\times R_\epsilon(\mathbb{A}))/R_{\epsilon,\gamma}(\mathbb{A})} f_\epsilon(\bar{r}\gamma r)\xi(\bar{r})\tau(r)\, d\bar{r}\, dr. \tag{4.3}$$

Note that for relevant $\gamma \in \bar{P}_\epsilon(F)P_\epsilon(F)$ by Proposition 2.2, the volume appearing in (4.2) is finite, and equals 1 if γ is regular. Then we have a factorization into local Bessel integrals

$$\mathcal{B}_\epsilon(\gamma; f_\epsilon) = \prod_v \mathcal{B}_\epsilon(\gamma; f_{\epsilon,v}), \tag{4.4}$$

where the local Bessel integrals are defined in Section 2.4.

Suppose $\gamma = \gamma_\epsilon(u,\mu)$. Then, for almost all finite v, μ and $1 - N_{E/F}(u)$ are units in $\mathcal{O}_v$. For such v, $\mathcal{B}_\epsilon(\gamma; \Xi_{\epsilon,v}) = 1$ [9]. The hypothesis that $f_{\epsilon,v} = \Xi_{\epsilon,v}$ for almost all v together with Lemma 2.9 then implies the global Bessel orbital integral $\mathcal{B}_\epsilon(\gamma; f_\epsilon)$ converges for any regular γ. For such γ, recall $R_{\epsilon,\gamma} = Z$, so in fact

$$J_{\epsilon,\gamma}(f_\epsilon) = \mathcal{B}_\epsilon(\gamma; f_\epsilon).$$

Proposition 4.1. *Suppose $f_{\epsilon,v_0} \in C_c^\infty(G_\epsilon^{\mathrm{reg}}(F_{v_0}))$ for some place v_0. Then the geometric expansion* (4.1) *converges absolutely.*

Proof. First observe that for $\gamma \in G_\epsilon(F)$, the local Bessel orbital integral $\mathcal{B}_\epsilon(\gamma; f_{\epsilon,v_0}) = 0$ unless $\gamma \in G_\epsilon(F)^{\mathrm{reg}}$. Consequently, only regular γ can contribute to the sum in (4.1). For regular γ, we have already established the convergence of $J_{\epsilon,\gamma}(f_\epsilon) = \mathcal{B}_\epsilon(\gamma; f_\epsilon)$ in the discussion above. Hence, the proposition would follow from knowing only a finite number of γ contribute. We show this now.

Let $G_\epsilon(\mathbb{A})^{\mathrm{reg}} = \{g \in G_\epsilon(\mathbb{A}) : g_v \in G_\epsilon(F_v)^{\mathrm{reg}} \text{ for all } v\}$. Consider the continuous map

$$p_\epsilon : G_\epsilon(\mathbb{A})^{\mathrm{reg}} \to D_\epsilon^\times(\mathbb{A}) \times \mathbb{A}^\times \times \mathbb{A}^\times$$

given by

$$p_\epsilon \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (A, \det(A), \det(D - CA^{-1}B)).$$

Take

$$\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \mu\bar{\alpha}^{-1} \end{pmatrix} \in G_\epsilon(F)^{\mathrm{reg}},$$

where $\alpha = \alpha(u)$, and $g \in \bar{R}_\epsilon(\mathbb{A})\gamma R_\epsilon(A) \subset G_\epsilon(\mathbb{A})^{\mathrm{reg}}$. Write

$$g = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \gamma \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}.$$

Then

$$p_\epsilon(g) = (a\alpha b, N_{E/F}(ab)\det(\alpha), \mu^2 N_{E/F}(ab)\det(\alpha^{-1})).$$

Thus if g lies in a compact set of $G_\epsilon(\mathbb{A})^{\mathrm{reg}}$, we see the coordinates of $p_\epsilon(g)$ lie in compact sets of $\mathrm{GL}_2(\mathbb{A}_E)$, $\mathbb{A}^\times$ and $\mathbb{A}^\times$, respectively. Note

$$a\alpha b = a\alpha(u)b = \begin{pmatrix} ab & abu\epsilon \\ abu^\sigma & ab \end{pmatrix}$$

lying in a compact set implies both ab and u lie in compact sets of $\mathbb{A}_E^\times$. Since $u \in E^\times$, there are only finitely many possibilities for u. Looking at the product of the second and third coordinate of $p_\epsilon(g)$ then shows μ lies in a compact set of $\mathbb{A}^\times \cap F^\times$, whence a finite number of possibilities for μ. □

Now we consider the spectral decomposition of $J_\epsilon(f_\epsilon)$. It is well known that the kernel has a spectral expansion of the form

$$K_\epsilon(x, y) = \sum_{\pi_\epsilon} K_{\pi_\epsilon}(x, y) + K_{\epsilon,\mathrm{nc}}(x, y)$$

where π_ϵ runs through the cuspidal automorphic representations π_ϵ of $G_\epsilon(\mathbb{A})/Z(\mathbb{A})$ and $K_{\epsilon,\mathrm{nc}}(x, y)$ denotes the contribution from the noncuspidal spectrum. Here

$$K_{\pi_\epsilon}(x, y) = \sum_\phi (\pi_\epsilon(f_\epsilon^Z)\phi)(x)\overline{\phi(y)}, \tag{4.5}$$

where ϕ runs over an orthonormal basis for the space of π_ϵ, and f_ϵ^Z is the function in the space $C_c^\infty(G_\epsilon(\mathbb{A})/Z(\mathbb{A}))$ given by

$$f_\epsilon^Z(g) = \int_{Z(\mathbb{A})} f(zg)dz.$$

Since each $K_{\pi_\epsilon}(x, y)$ is of rapid decay in x and y, we have the absolute convergence of the *global Bessel distribution*

$$J_{\epsilon,\pi_\epsilon}(f_\epsilon) := \int_{Z(\mathbb{A})\bar{R}_\epsilon(F)\backslash\bar{R}_\epsilon(\mathbb{A})} \int_{Z(\mathbb{A})R_\epsilon(F)\backslash R_\epsilon(\mathbb{A})} K_{\epsilon,\pi_\epsilon}(\bar{r}, r)\xi(\bar{r})^{-1}\tau(r)\, d\bar{r}\, dr.$$

Let K be a maximal compact subgroup of $G_\epsilon(\mathbb{A})$ with $K = \prod K_v$ and $K_v = G_\epsilon(\mathcal{O}_v)$ for all finite v. Since the integrals here are over compact sets, (4.5) implies

$$J_{\epsilon,\pi_\epsilon}(f_\epsilon) = \sum_\phi \mathcal{P}'_\epsilon(\pi_\epsilon(f_\epsilon^Z)\phi)\mathcal{P}_\epsilon(\bar{\phi}), \tag{4.6}$$

where $\mathcal{P}_\epsilon$ and $\mathcal{P}'_\epsilon$ are defined as in (1.7) and (1.8), and ϕ is taken to run over an orthonormal basis.

Thus, at least formally, we should have a spectral decomposition of the form

$$J_\epsilon(f_\epsilon) = \sum_{\pi_\epsilon} J_{\epsilon,\pi_\epsilon}(f_\epsilon) + J_{\epsilon,\mathrm{nc}}(f_\epsilon).$$

For v nonarchimedean, we will call $f_\epsilon \in C_c^\infty(G_\epsilon(F_v))$ a *supercusp form* if for all proper parabolic subgroups of $G_\epsilon(F_v)$ with unipotent radical N, the integral

$$\int_N f^Z_{\epsilon,v}(g_1 n g_2)\, dn = 0$$

for all $g_1, g_2 \in G_\epsilon(F_v)$. In particular if $f^Z_{\epsilon,v}$ is the matrix coefficient of a supercuspidal representation π_v, then $f_{\epsilon,v}$ is a supercusp form.

Proposition 4.2. *Suppose $f_{\epsilon,v}$ is a supercusp form at some place v. Then the spectral expansion*

$$J_\epsilon(f_\epsilon) = \sum_{\pi_\epsilon} J_{\epsilon,\pi_\epsilon}(f_\epsilon) \tag{4.7}$$

converges absolutely.

Proof. First observe, that $K_{\epsilon,\mathrm{nc}}(x,y)$, whence $J_{\epsilon,\mathrm{nc}}(f_\epsilon)$, is zero ([31, Prop. 1.1]). Then

$$K_\epsilon(x,y) = K_{\epsilon,\mathrm{cusp}} = \sum_{\pi_\epsilon} K_{\pi_\epsilon}(x,y),$$

where π_ϵ runs over the cuspidal representations of $G_\epsilon(\mathbb{A})$ with trivial central character. This series converges absolutely and uniformly when x, y lie in compact sets of $Z(\mathbb{A})G_\epsilon(F)\backslash G_\epsilon(\mathbb{A})$, which implies our proposition. □

Just as the global orbital integrals on the geometric side factor into products of local orbital integrals, the global Bessel distributions on the spectral side factor into products of local Bessel distributions.

Namely, let V be the Hilbert space on which π_ϵ acts and V_∞ denote the set of smooth vectors. We can view the linear form $\mathcal{P}'_\epsilon$ (resp. $\bar{\mathcal{P}}_\epsilon$) as an element λ_ϵ (resp. λ'_ϵ) of the dual V^*_∞ (resp. conjugate dual V'_∞) of V_∞. We may also view $V \subseteq V^*_\infty$ and $V \subseteq V'_\infty$. Thus we can extend the inner product $(\cdot,\cdot)$ on $V \times V$ to $V^*_\infty \times V_\infty$ and $V_\infty \times V'_\infty$. We may define $\pi_\epsilon(f_\epsilon)\lambda_\epsilon \in V^*_\infty$ by

$$(\lambda_\epsilon, \pi_\epsilon(f_\epsilon)\phi) = (\pi_\epsilon(f_\epsilon)\lambda, \phi).$$

For f_ϵ smooth of compact support, in fact $\pi_\epsilon(f_\epsilon)\lambda_\epsilon \in V_\infty$. Then (4.6) above becomes

$$J_{\epsilon,\pi_\epsilon}(f_\epsilon) = \sum_\phi (\lambda_\epsilon, \pi_\epsilon(f_\epsilon)\phi)(\phi, \lambda'_\epsilon) = (\pi_\epsilon(f_\epsilon)\lambda_\epsilon, \lambda'_\epsilon). \tag{4.8}$$

By the local uniqueness of Bessel models (this is stated in [29] in the nonarchimedean case; see [1] for a proof of this, and [2] for the archimedean case), we know there exist factorizations $\lambda_\epsilon = \prod \lambda_{\epsilon,v}$ and $\lambda'_\epsilon = \prod \lambda'_{\epsilon,v}$ into local linear forms. Therefore

$$J_{\epsilon,\pi_\epsilon}(f_\epsilon) = \prod_v J_{\epsilon,\pi_{\epsilon,v}}(f_{\epsilon,v}), \tag{4.9}$$

has a factorization into local Bessel distributions

$$J_{\epsilon,\pi_{\epsilon,v}}(f_{\epsilon,v}) = (\pi_{\epsilon_v}(f^Z_{\epsilon,v})\lambda_{\epsilon,v}, \lambda_{\epsilon',v}), \tag{4.10}$$

which can also be expressed in the form (3.1). Here this local inner product is defined analogously to the global one, after factoring our global inner product on V into a product of local ones, normalized so that at almost all finite v where $G_{\epsilon,v} = G_v$ and $J_{\epsilon,\pi_{\epsilon,v}} \not\equiv 0$, we have $J_{\epsilon,\pi_{\epsilon,v}}(\Xi_v) = 1$.

From (4.8) and (4.10), it is clear that the global and local Bessel distributions are not identically zero if and only if the global and local Bessel periods are not identically zero.

4.2. **A simple trace formula for G.** Take a test function $f \in C_c^\infty(G(\mathbb{A}))$, which we assume to be factorizable $f = \prod f_v$ such that $f_v = \Xi_v$ for almost all v. Consider the relative trace formula given by

$$J(f) = \int_{Z(\mathbb{A})\bar{H}(F)\backslash\bar{H}(\mathbb{A})} \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} K(\bar{h}, h)\theta(h)\psi(\bar{h})^{-1}\, dh\, d\bar{h}$$

where

$$K(x, y) = K_f(x, y) = \int_{Z(F)\backslash Z(\mathbb{A})} \sum_{\gamma\in G(F)} f(x^{-1}\gamma yz)\, dz.$$

The measure $dz = \prod dz_v$ was already chosen above, and we choose Haar measures $dh = \prod dh_v$ and $d\bar{h} = \prod d\bar{h}_v$ such that $H(\mathcal{O}_v)$ and $\bar{H}(\mathcal{O}_v)$ have volume 1 for all finite v. We also assume the local archimedean measures dh_v and $d\bar{h}_v$ agree with the local archimedean measures dr_v and $d\bar{r}_v$ chosen above when $G_\epsilon(F_v) = G(F_v)$.

Similar to above, we formally have a geometric decomposition

$$J(f) = \sum_{\gamma\in\bar{H}(F)\backslash G(F)/H(F)} J_\gamma(f) \tag{4.11}$$

where

$$J_\gamma(f) = \mathrm{vol}(Z(\mathbb{A})H_\gamma(F)\backslash H_\gamma(\mathbb{A}))\mathcal{N}(\gamma; f),$$

and the *global Novodvorsky orbital integral* is

$$\mathcal{N}(\gamma; f) = \int_{(\bar{H}(\mathbb{A})\times H(\mathbb{A}))/H_\gamma(\mathbb{A})} f(\bar{h}\gamma h)\theta(h)\psi(\bar{h})\, dh\, d\bar{h}.$$

While the above volume is infinite in general, if $\gamma \in \bar{P}(F)P(F)$ is relevant then, by Proposition 2.1, $H_\gamma \simeq Z$ so the above volume is 1, i.e.,

$$J_\gamma(f) = \mathcal{N}(\gamma; f).$$

Proposition 4.3. *Suppose $f_v \in C_c^\infty(G(F_v)^{\mathrm{reg}})$ for some v. Then Equation* (4.11) *converges.*

Proof. The proof is similar to that for Proposition 4.1. □

Write the spectral decomposition of the kernel

$$K(x, y) = \sum_\pi K_\pi(x, y) + K_{\mathrm{nc}}(x, y),$$

where π runs over all cuspidal automorphic representations of $G(\mathbb{A})/Z(\mathbb{A})$. The *global Bessel distribution* for π is given by

$$J_\pi(f) = \int_{Z(\mathbb{A})\bar{H}(F)\backslash\bar{H}(\mathbb{A})} \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} K_\pi(\bar{h}, h)\theta(h)\psi(\bar{h})^{-1}\, dh\, d\bar{h}.$$

We can write

$$K_\pi(x,y) = \sum_\phi (\pi(f^Z)\phi)(x)\bar{\phi(y)},$$

where ϕ runs over an orthonormal basis for the space of π and $f^Z(g) = \int_Z f(zg)\,dg$. If f is in fact K-finite, one may choose a suitable basis $\{\phi\}$ such that this sum is in fact finite. At least in this case, one can expand

$$J_\pi(f) = \sum_\phi \mathcal{P}'(\pi(f^Z)\phi)\overline{\mathcal{P}(\phi)}, \tag{4.12}$$

and this sum converges absolutely. Let V_∞ be the space of smooth vectors for π, V_∞^* its dual and V'_∞ its conjugate dual. Regard the periods $\mathcal{P}'$ and $\mathcal{P}$ as elements $\lambda \in V_\infty^*$ and $\lambda' \in V'_\infty$. Then just as in (4.6), one can express $J_\pi(f)$ in terms of a pairing on $V_\infty \times V'_\infty$,

$$J_\pi(f) = (\pi(f^Z)\lambda, \lambda'). \tag{4.13}$$

In particular, $J_\pi \not\equiv 0$ if and only if $\mathcal{P}$ and $\mathcal{P}'$ are not identically zero.

As before, by 1-dimensionality of the spaces $\mathrm{Hom}_{\bar{H}(F_v)}(\pi_v, \psi_v)$ and $\mathrm{Hom}_{H(F_v)}(\pi_v, \theta_v)$, we know λ and λ' factor into local linear forms $\lambda = \prod \lambda_v$ and $\lambda' = \prod \lambda'_v$. Consequently, we have a factorization

$$J_\pi(f) = \prod J_{\pi_v}(f_v) \tag{4.14}$$

of $J_\pi(f)$ into local Bessel distributions

$$J_{\pi_v}(f_v) = (\pi_v(f_v^Z)\lambda_v, \lambda'_v),$$

where the local pairings are suitably normalized. We assume the local Bessel distributions are normalized so that $J_{\pi_v}(\Xi_v) = 1$ for almost all v such that $J_{\pi_v} \not\equiv 0$.

Proposition 4.4. *Suppose f_v is a supercusp form for some v. Then*

$$J(f) = \sum_\pi J_\pi(f),$$

where the sum on the right is absolutely convergent.

Proof. The proof is similar to that for Proposition 4.2. □

5. Global results

Let Σ_s (resp. Σ_i) denote the set of places of F which are split (resp. inert) in E. We assume ψ is unramified at each finite $v \in \Sigma_i$. Let ϵ denote an element of $F^\times/N_{E/F}(E^\times)$. When $\epsilon \in N_{E_v/F_v}(E_v^\times)$, we identify $G_\epsilon(F_v)$ with $G(F_v)$ as in Section 1.1.2. This identifies the local Bessel and Novodvorsky periods for $v \in \Sigma_s$.

For a finite odd v, denote by $\mathcal{H}_v$ the Hecke algebra for $G(F_v)$, i.e., set of bi-K_v-invariant functions in $C_c^\infty(G(F_v))$, where $K_v = G(\mathcal{O}_v)$.

Let $f = \prod f_v \in C_c^\infty(G)$ and $f_\epsilon = \prod f_{\epsilon,v} \in C_c^\infty(G_\epsilon)$ for each ϵ. We assume that the local functions f_v and $f_{\epsilon,v}$ are smooth of compact support, and equal the unit element Ξ_v of $\mathcal{H}_v$ for almost v. We also assume that $f_\epsilon = 0$ for all but finitely many ϵ.

We say regular double cosets $\bar{H}(F)\gamma H(F)$ and $\bar{R}_\epsilon\gamma_\epsilon R_\epsilon$ in $G(F)$ and $G_\epsilon(F)$ *match* if $(y(\gamma), z(\gamma)) = (y_\epsilon(\gamma_\epsilon), z_\epsilon(\gamma_\epsilon))$. We say f and $(f_\epsilon)_\epsilon$ are *global matching functions* if

$$\mathcal{N}(\gamma; f) = \mathcal{B}_\epsilon(\gamma_\epsilon; f_\epsilon)$$

whenever γ and γ_ϵ match. Since we may take regular γ and γ_ϵ of the form $\gamma(y,z)$ and $\gamma_\epsilon(y,z)$, f and $(f_\epsilon)_\epsilon$ are global matching functions if f_v and $(f_{\epsilon,v})_\epsilon$ are local matching functions for all v.

To state our global results, let us set the following notation. Let S be a finite set of places of F containing all infinite and even places, as well as two fixed finite places $v_1 \in \Sigma_s$ and $v_2 \in \Sigma_s \cup \Sigma_i$. For each $v \notin S$, let τ_v be an irreducible admissible unramified representation of $G(F_v)$. Let τ_{v_2} be a supercuspidal representation of $G(F_{v_1})$. Let Π (resp. Π_ϵ) be the set of cuspidal unitary automorphic representations π of $G(\mathbb{A})$, (resp. $G_\epsilon(\mathbb{A})$) such that $\pi_v \simeq \tau_v$ for all $v \notin S - \{v_1\}$ and π_v is tempered for all $v \in \Sigma_i$. Let Π^{gen} denote the subset of generic representations in Π.

Theorem 5.1. *Suppose $f \in C_c^\infty(G(\mathbb{A})))$ as above such that*

(i) f_{v_1} is a supercusp form;
(ii) $f_{v_2} \in C_c^\infty(G(F_{v_2})^{\mathrm{reg}})$;
(iii) at any $v \in \Sigma_i$, either $f_v \in \mathcal{H}_v$ where $v \notin S$ or $f_v \in C_c^\infty(\bar{P}_v P_v)$.

Then there exists a matching family $(f_\epsilon)_\epsilon$ such that

$$\sum_{\pi \in \Pi^{\mathrm{gen}}} J_\pi(f) = \sum_\epsilon \sum_{\pi_\epsilon \in \Pi_\epsilon} J_{\pi_\epsilon}(f_\epsilon). \tag{5.1}$$

Conversely, fix ϵ and let $f_\epsilon \in C_c^\infty(G_\epsilon(\mathbb{A}))$ as above such that

(i) f_{ϵ,v_1} is a supercusp form;
(ii) $f_{\epsilon,v_2} \in C_c^\infty(G_\epsilon(F_{v_2})^{\mathrm{reg}})$; and
(iii) at any $v \in \Sigma_i$, either $f_v \in \mathcal{H}_v$ where $v \notin S$ or $f_{\epsilon,v} \in C_c^\infty(G_\epsilon(F_v)^{\mathrm{reg}})$.

Then there exists a matching function $f \in C_c^\infty(G(\mathbb{A}))$ such that

$$\sum_{\pi \in \Pi^{\mathrm{gen}}} J_\pi(f) = \sum_{\pi_\epsilon \in \Pi_\epsilon} J_{\pi_\epsilon}(f_\epsilon). \tag{5.2}$$

Proof. Start with $f \in C_c^\infty(G(\mathbb{A}))$ as above. For any $v \in \Sigma_s$, then $\epsilon \in N(E_v^\times)$ and we put $f_{\epsilon,v} = f_v$ for all ϵ. These local functions match by our identification of $G_\epsilon(F_v)$ with $G(F_v)$.

For each $v \in \Sigma_i$, we will define a family $(f_{v,1}, f_{v,2})$ of local matching functions to f_v in the sense of Definition 2.13. Here $f_{v,i} \in C_c^\infty(G_{\epsilon_{v,i}}(F_v))$ where $\epsilon_{v,i} \in F_v^\times$ with $\epsilon_{v,1}$ being a norm from $E_v^\times$ and $\epsilon_{v,2}$ a non-norm. Then one simply sets $f_\epsilon = \prod f_{\epsilon,v}$ where $f_{\epsilon,v} = f_{v,i}$ according to whether ϵ is equivalent to $\epsilon_{v,1}$ or $\epsilon_{v,2}$ in $F_v^\times / N_{E_v/F_v}(E_v^\times)$. The local matching of functions then will imply the global matching of f with $(f_\epsilon)_\epsilon$.

For each $v \notin S$ with $v \in \Sigma_i$, put $f_{v,1} = f_v = \Xi_v$ and $f_{v,2} = 0$. These match by the fundamental lemma for the unit element [9]. If $v \notin S$ and $f_v \in \mathcal{H}_v$, we know $(f_{v,1}, 0)$ is a matching pair for some $f_{v,1} \in \mathcal{H}_v$ by the fundamental lemma for the Hecke algebra [8]. In particular, for almost all ϵ, we have $f_{\epsilon,v} = 0$ for some $v \in S$, i.e., for all but finitely many ϵ, $f_\epsilon = 0$.

Now suppose $v \in (S^c \cap \Sigma_i)$. If $f_v \in C_c^\infty(G^{\mathrm{reg}}(F_v))$, we let $(f_{v,1}, f_{v,2})$ be a matching pair of functions supported on $G_{\epsilon_{v,i}}(F_v)^{\mathrm{reg}}$ as in Proposition 2.14. If $f_v \in C_c^\infty(\bar{P}_v P_v))$, we let $(f_{v,1}, f_{v,2})$ be a matching pair of functions supported on $\bar{P}_{\epsilon_{v,i}} P_{\epsilon_{v,i}}$ as in Proposition 2.15. This completes the construction of a global matching family $(f_\epsilon)_\epsilon$.

Now by Propositions 4.1, 4.2, 4.3 and 4.4, we have

$$\sum_{\pi} J_\pi(f) = \sum_{\gamma} J_\gamma(f) = \sum_{\epsilon}\sum_{\gamma_\epsilon} J_{\gamma_\epsilon}(f_\epsilon) = \sum_{\epsilon}\sum_{\pi_\epsilon} J_{\pi_\epsilon}(f_\epsilon).$$

Here π and π_ϵ run over cuspidal automorphic representations of G and G_ϵ, and γ, γ_ϵ run over a set of representatives for the regular double cosets for G and G_ϵ.

By the fundamental lemma for the Hecke algebra [8] and infinite linear independence of Bessel distributions (cf. [24, p. 211] for Langlands' original argument, or [26, Lemma 4] for a more general version), one can reduce the equality between the first and last sums above to

$$\sum_{\pi\in\Pi} J_\pi(f) = \sum_{\epsilon}\sum_{\pi_\epsilon\in\Pi_\epsilon} J_{\pi_\epsilon}(f_\epsilon).$$

Furthermore, the Novodvorsky periods, whence $J_\pi(f)$, can only be nonzero for generic π. This yields the desired equality.

The proof of the converse direction is the same, with the minor exception that one does not use Proposition 2.15. □

We remark that Π^{gen} should contain at most one element. This is proved in [23] when F is totally real, and their argument should now apply to arbitrary F in light of [11]. This should also follow from Arthur's Book Project [3].

Corollary 5.2. *Suppose Π^{gen} contains exactly one representation π and E/F is split at all infinite places. If*

$$L(1/2,\pi)L(1/2,\pi\otimes\kappa) \neq 0,$$

then there exists $\pi_\epsilon \in \Pi_\epsilon$ such that π_ϵ has a Bessel period.

Proof. By Novodvorsky's integral representation, the nonvanishing of L-values implies the nonvanishing of Novodvorsky periods, so $J_\pi(f) \not\equiv 0$. Outside of a finite set $S_0 \supset S$, we have $J_{\pi_v}(\Xi_v) = 1$, and so at these places take $f_v = \Xi_v$. Then for each $v \in S_0 \cap \Sigma_s$, we take f_v to be any function such that $J_{\pi_v}(f_v) \neq 0$. Further, for any $v \in S_0 \cap \Sigma_i$, by Corollary 3.3, we may take $f_v \in C_c^\infty(G(F_v)^{\text{reg}})$ such that $J_{\pi_v}(f_v) \neq 0$. Then f satisfies the conditions of the theorem and $J_\pi(f) \neq 0$. By hypothesis, the left hand side of (5.1) just consists of $J_\pi(f)$, and hence (5.1) is nonzero, so some $J_{\pi_\epsilon}(f_\epsilon)$ is nonzero. □

Now fix ϵ. While Π_ϵ in general contains more than one representation, there should be at most one $\pi_\epsilon \in \Pi_\epsilon$ such that $J_{\pi_\epsilon} \not\equiv 0$. To see this, observe Π_ϵ should be contained in a single L-packet for G_ϵ. Namely, if $\pi_\epsilon, \pi'_\epsilon \in \Pi_\epsilon$, then they should transfer to the same representation of GL(4), whence should be in the same L-packet. Then local Gross–Prasad conjectures tell us that there is at most one $\pi_\epsilon \in \Pi_\epsilon$ possessing a Bessel period. These local Gross–Prasad conjectures are proven in our case when the residual characteristic is not 2 by Prasad–Takloo-Bighash [30] using the L-packets defined in [10] and [12], and for "generic L-packets" of more general orthogonal pairs by Waldspurger and Moeglin–Waldspurger [28] under some assumptions about L-packets and the stabilization of the trace formula.

We expect these results to be established in the near future with the completion of Arthur's Book Project [3].

Corollary 5.3. *Suppose there is exactly one $\pi_\epsilon \in \Pi_\epsilon$ which has a Bessel period and E/F is split at all infinite places. Then there exists $\pi \in \Pi^{\text{gen}}$ such that*

$$L(1/2, \pi)L(1/2, \pi \otimes \kappa) \neq 0.$$

Proof. The argument is the same as the previous corollary. □

We remark that one could remove the hypothesis of E/F split at infinity in both corollaries either if one knew a general smooth matching result at the archimedean places or if one had an archimedean analogue of Corollary 3.3. In the latter corollary, it is reasonable to expect the approach in [22] may allow one to relax this condition in the latter corollary to requiring that at each $v|\infty$, either E_v/F_v is split or $D_{\epsilon,v}$ is ramified.

References

[1] A. Aizenbud, D. Gourevitch, S. Rallis and G. Schiffman, Multiplicity one theorems. *Ann. of Math.* (2) 172 (2010), no. 2, 1407–1434.

[2] A. Aizenbud, D. Gourevitch, and E. Sayag, $(O(V \oplus F), O(V))$ is a Gelfand pair for any quadratic space V over a local field F. *Math. Z.* 261 (2009), no. 2, 239–244.

[3] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups.* Colloquium Publications 61, Amer. Math. Soc. (2013).

[4] E. Baruch and Z. Mao, Central value of automorphic L-functions. *Geom. Funct. Anal.* 17 (2007), no. 2, 333–384.

[5] S. Böcherer, Bemerkungen über die Dirichletreihen von Koecher und Maaß, *Math. Gottingensis Schrift.* SFB. Geom. Anal. Heft **68**, 1986.

[6] D. Bump, The Rankin-Selberg method: a survey, *Number theory, trace formulas and discrete groups (Oslo, 1987)*, 49–109, Academic Press, Boston, MA, 1989.

[7] M. Furusawa and K. Martin, On central critical values of the degree four L-functions for GSp(4): the fundamental lemma. II. *Amer. J. Math.* **133** (2011), no. 1, 197–233.

[8] M. Furusawa, K. Martin and J. A. Shalika, On central critical values of the degree four L-functions for GSp(4): the fundamental lemma. III. *Mem. Amer. Math. Soc.*, 225 (2013), no. 1057.

[9] M. Furusawa and J. A. Shalika, On central critical values of the degree four L-functions for GSp(4): the fundamental lemma, *Mem. Amer. Math. Soc.* **164** (2003), no. 782, x+139 pp.

[10] W.T. Gan and S. Takeda, The local Langlands conjecture for GSp(4). *Ann. of Math.* (2) **173** (2011), no. 3, 1841–1882.

[11] W. T. Gan, S. Takeda and Y. Qiu, The regularized Siegel–Weil formula (the second term identity) and the Rallis inner product formula. *Preprint.*

[12] W.T. Gan and W. Tatano, The local Langlands conjecture for GSp(4). II: the case of inner forms. *Preprint.*

[13] D. Ginzburg, D. Jiang and S. Rallis, On the nonvanishing of the central value of the Rankin-Selberg L-functions. J. Amer. Math. Soc. 17 (2004), no. 3, 679–722.

[14] D. Ginzburg, D. Jiang and S. Rallis, On the nonvanishing of the central value of the Rankin-Selberg L-functions. II. *Automorphic representations, L-functions and applications: progress and prospects*, 157–191, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[15] B. Gross and D. Prasad, On the decomposition of a representation of SO_n when restricted to SO_{n-1}. *Canad. J. Math.* 44 (1992), no. 5, 974–1002.

[16] B. Gross and D. Prasad, On irreducible representations of $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$. *Canad. J. Math.* 46 (1994), no. 5, 930950.

[17] A. Ichino and T. Ikeda, On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture, *Geom. Funct. Anal.* **19** (2010), no. 5, 1378–1425.

[18] A. Ichino and W. Zhang, Spherical characters for a strongly tempered pair. *Appendix to:* Fourier transform and the global Gan–Gross–Prasad conjecture for unitary groups, W. Zhang, *Preprint.*

[19] H. Jacquet, Sur un résultat de Waldspurger, *Ann. Sci. École Norm. Sup.* **19** (1986), 185–229.

[20] H. Jacquet, Sur un résultat de Waldspurger. II, *Compositio Math.* **63** (1987), 315–389.

[21] H. Jacquet and N. Chen. Positivity of quadratic base change L-functions. *Bull. Soc. Math. France*, 129(1):33–90, 2001.
[22] H. Jacquet and K. Martin, Shalika periods on $\mathrm{GL}_2(D)$ and GL(4). *Pacific J. Math.* **233** (2007), no. 2, 341–370.
[23] D. Jiang and D. Soudry, The multiplicity-one theorem for generic automorphic forms of GSp(4). *Pacific J. Math.*. **229** (2007), no. 2, 381–388.
[24] R. Langlands, Base change for GL(2). *Annals of Mathematical Studies*, **96**. Princeton University Press, Princeton, N.J.
[25] E. Lapid and O. Offen, Compact unitary periods. *Compos. Math.* 143 (2007), no. 2, 323–338.
[26] E. Lapid and J. Rogawski, Stabilization of periods of Eisenstein series and Bessel distributions on GL(3) relative to U(3). *Documenta Math.* **5** (2000), 317–350.
[27] K. Martin and D. Whitehouse, Central L-values and toric periods for GL(2). *Int. Math. Res. Not.* IMRN **2009**, no. 1, 141–191.
[28] C. Moeglin and J.-L. Waldspurger, La conjecture locale de Gross–Prasad pour les groupes spéciaux orthogonaux : le cas général. *Astérisque* **347** (2012), 167–216.
[29] M. E. Novodvorsky, Automorphic L-functions for symplectic group GSp(4), *Automorphic forms, representations and L-functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 87–95, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.
[30] D. Prasad and R. Takloo-Bighash, Bessel models for GSp (4). *J. Reine. Agnew. Math.* **655** (2011), 189–243.
[31] J. Rogawski, Representations of GL(n) and division algebras over a p-adic field. *Duke Math. J.* **50** (1983), no. 1, 161–196.
[32] D. Soudry, Rankin-Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$: local theory. *Mem. Amer. Math. Soc.* 105 (1993), no. 500, vi+100 pp.
[33] D. Soudry, On the Archimedean theory of Rankin-Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$. *Ann. Sci. Ecole Norm. Sup.* (4) 28 (1995), no. 2, 161–224.
[34] R. Takloo-Bighash. L-functions for the p-adic group GSp(4). *Amer. J. Math.* 122 (2000), no. 6, 1085–1120.
[35] J.-L. Waldspurger. Sur les coefficients de Fourier des formes modulaires de poids demi-entier. *J. Math. Pures Appl.* (9) 60 (1981), no. 4, 375–484.
[36] J.-L. Waldspurger, Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie, *Compositio Math.* **54** (1985), 173–242.
[37] J.-L. Waldspurger, La formule de Plancherel pour les groupes p-adiques. D'après Harish-Chandra. *J. Inst. Math. Jussieu*, (2003), no. 2, 235–333.
[38] J.-L. Waldspurger, Une formule intégrale reliée à la conjecture locale de Gross-Prasad, 2ème partie: extension au représentations tempérées. *Astérisque* **346** (2012), 171–312.
[39] W. Zhang, Fourier transform and the global Gross–Prasad conjecture for unitary groups. *Preprint.*
[40] W. Zhang, Automorphic period and the central value of Rankin-Selberg L-function, *J. Amer. Math. Soc.,* to appear.

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