

PERIODS AND NONVANISHING OF CENTRAL L -VALUES FOR $\mathrm{GL}(2n)$

BROOKE FEIGON, KIMBALL MARTIN, DAVID WHITEHOUSE

ABSTRACT. Let π be a cuspidal automorphic representation of $\mathrm{PGL}(2n)$ over a number field F , and η the quadratic idèle class character attached to a quadratic extension E/F . Guo and Jacquet conjectured a relation between the nonvanishing of $L(1/2, \pi)L(1/2, \pi \otimes \eta)$ for π of symplectic type and the nonvanishing of certain $\mathrm{GL}(n, E)$ periods. When $n = 1$, this specializes to a well-known result of Waldspurger. We prove this conjecture, and related global results, under some local hypotheses using a simple relative trace formula.

We then apply these global results to obtain local results on distinguished supercuspidal representations, which partially establish a conjecture of Prasad and Takloo-Bighash.

1. INTRODUCTION

One topic of recent interest is the study of how periods behave along functorial transfers, for instance among inner forms à la Gross–Prasad conjectures. The relative trace formula is an analytic tool developed for such problems. Here we apply a simple relative trace formula to show that certain periods behave as conjectured by Guo and Jacquet with respect to the Jacquet–Langlands lift under some local hypotheses. While these local hypotheses are somewhat restrictive, they are merely technical hypotheses used to simplify the trace formula and we can verify them sufficiently often to provide strong evidence for both the Guo–Jacquet conjecture and the feasibility of a trace formula approach.

Then we apply our global results to obtain local results on distinguished supercuspidal representations. These local results establish part of [PTB11, Conjecture 1] and [FM15, Conjecture 3], which concern local root number criteria for the existence of certain local linear forms and global periods.

1.1. Background. Let E/F be a quadratic extension of number fields, and η the associated quadratic idèle class character. Let \mathbf{A} be the adèles of F and \mathbf{A}_E the adèles of E . Let $X(E:F)$ denote the set of quaternion algebras D/F in which E embeds. Note the matrix algebra M_2 always lies in $X(E:F)$. For $D \in X(E:F)$, let $G_D = \mathrm{GL}_n(D)$. When D is fixed, we also write $G = G_D$. Put $G' = \mathrm{GL}_{2n}$. For each G_D , at almost all places v of F , we have $G_{D_v} \cong G'_v$. The Jacquet–Langlands correspondence in [BR10] associates to each discrete series representation π_D of $G_D(\mathbf{A})$ a discrete series representation $\pi' = \mathrm{JL}(\pi_D)$ of $G'(\mathbf{A})$ such that $\pi_{D_v} \cong \pi'_v$ for almost all v . Strong multiplicity one for these groups means this near local equivalence condition specifies a unique π' for each π_D , and vice versa when the inverse Jacquet–Langlands lift exists.

Date: February 17, 2017.

All of our representations are taken to be unitary with trivial central character, and we will also assume both π_D and π' are cuspidal.

We now describe the periods of interest. Let $H = H_D$ be the subgroup $\mathrm{GL}_n(E)$ of G_D and H' be the subgroup $\mathrm{GL}_n \times \mathrm{GL}_n$ of G' (all embeddings of these subgroups are conjugate, but we fix embeddings in Section 2). Denote by dh and dh' Haar measures on H and H' . We say π_D is H -distinguished if the linear form on π_D given by

$$\mathcal{P}_D(\varphi) = \int_{H(F)Z(\mathbf{A}) \backslash H(\mathbf{A})} \varphi(h) dh$$

is not identically zero. (Throughout Z denotes the center of the relevant group.) Now consider the linear forms on π' given by

$$\mathcal{P}'(\varphi) = \int_{H'(F)Z(\mathbf{A}) \backslash H'(\mathbf{A})} \varphi(h') dh', \quad \mathcal{P}'_\eta(\varphi) = \int_{H'(F)Z(\mathbf{A}) \backslash H'(\mathbf{A})} \varphi(h') \eta(\det(h')) dh'.$$

We say that π' is H' - (resp. (H', η) -) distinguished if the linear form \mathcal{P}' (resp. \mathcal{P}'_η) is not identically 0.

These periods are intimately connected with central L -values. Specifically, we have the following consequence of a result of Friedberg and Jacquet.

Theorem 1.1 ([FJ93]). *Suppose π' is cuspidal. Then π' is both H' - and (H', η) -distinguished if and only if (a) π' is of symplectic type, i.e., $L(s, \pi', \Lambda^2)$ has a pole at $s = 1$, and (b)*

$$L(1/2, \pi'_E) = L(1/2, \pi')L(1/2, \pi' \otimes \eta) \neq 0.$$

Here π'_E denotes the base change of π' to $G'(\mathbf{A}_E)$. We remark that being of symplectic type on GL_{2n} is equivalent to being in the image of the functorial lift from SO_{2n+1} (see Arthur's book [Art13]).

Given π' , let $X(E:F:\pi')$ denote the set of $D \in X(E:F)$ such that the inverse Jacquet–Langlands transfer $\pi_D = \mathrm{LJ}_D(\pi)$ of π' to G_D exists. In particular, $M_2 \in X(E:F:\pi')$. When $n = 1$, we have the following well-known theorem of Waldspurger. In this case, all π' are of symplectic type.

Theorem 1.2 ([Wal85]). *Suppose $n = 1$.*

- (1) *Fix $D \in X(E:F)$ and a cuspidal representation π of $G_D(\mathbf{A}) = D^\times(\mathbf{A})$. If π is H -distinguished, then $\pi' = \mathrm{JL}(\pi)$ is H' - and (H', η) -distinguished.*
- (2) *Let π' be a cuspidal representation of $G'(\mathbf{A}) = \mathrm{GL}_2(\mathbf{A})$. If π' is H' - and (H', η) -distinguished, then there exists a (unique) $D \in X(E:F:\pi')$ such that $\pi_D = \mathrm{LJ}_D(\pi')$ is H -distinguished.*

This result (and a twisted version) was originally proved by Waldspurger using the theta correspondence, and then by Jacquet [Jac86] using the relative trace formula. In fact, Waldspurger obtained an exact formula relating $L(1/2, \pi'_E)$ to $|\mathcal{P}_D(\varphi)|^2$ for any $\varphi \in \pi_D$. This formula, and its twisted version, has been the subject of much study and has many applications.

One might hope to generalize Waldspurger's result to higher rank groups. One such generalization is the orthogonal (Gan–)Gross–Prasad conjectures ([GP92], [GP94], [GGP12]) viewing Theorem 1.2 as a statement about $\mathrm{SO}_3 \times \mathrm{SO}_2$ L -values. Alternatively, remaining in the general linear framework, we have the following.

Conjecture 1.3 (Guo–Jacquet [Guo96]). (1) Fix $D \in X(E:F)$ and let π be a cuspidal representation of $G(\mathbf{A}) = G_D(\mathbf{A})$. If π is H -distinguished, then $\pi' = \mathbb{J}\mathbb{L}(\pi)$ is H' - and (H', η) -distinguished, i.e., π' is of symplectic type and $L(1/2, \pi'_E) \neq 0$. (2) Suppose n is odd, and π' is a cuspidal representation of $G'(\mathbf{A})$ such that π' is H' - and (H', η) -distinguished. Then there exists $D \in X(E:F:\pi')$ such that $\pi_D = \mathbb{L}\mathbb{J}_D(\pi')$ is H -distinguished.

Guo [Guo96] established the fundamental lemma for the unit element of the Hecke algebra for a relative trace formula to attack this conjecture.

We remark that the condition of n odd for (2) of the conjecture arises in a difference between the geometric decompositions of this trace formula in the n odd and n even case. Spectrally, the difference is related to the sign in the character identity $\chi_{\pi_v} = (-1)^n \chi_{\pi'_v}$ of the local Jacquet–Langlands correspondence.

The correspondence of geometric orbits in the trace formula suggests the D in (2) may be unique when n is odd. This suggests a local dichotomy principle as in the Gross–Prasad situation, i.e., when G_v is nonsplit, at most one of π_v and π'_v is locally $H(F_v)$ - or $H'(F_v)$ -distinguished (has a nonzero $H(F_v)$ - or $H'(F_v)$ -invariant linear form).

Local dichotomy when n is odd would also be consistent with spectral expectations. Let K/k be a quadratic extension of local fields and $D(k)$ the quaternion division algebra over k .

Conjecture 1.4. [PTB11, Conjecture 1] Let τ and τ' be irreducible admissible representations of $\mathrm{GL}_n(D(k))$ and $\mathrm{GL}_{2n}(k)$ which correspond via Jacquet–Langlands. If τ (resp. τ') is $\mathrm{GL}_n(K)$ -distinguished, then τ (resp. τ') is symplectic and $\epsilon(1/2, \tau_K) = (-1)^n$ (resp. $\epsilon(1/2, \tau'_K) = 1$). Moreover, these conditions are sufficient for distinction if τ (resp. τ') is discrete series.

In fact, [PTB11] gives a more general conjecture and proves the $n = 2$ case with the theta correspondence. Symplectic here means the local Langlands parameter has symplectic image in $\mathrm{GL}_{2n}(\mathbf{C})$.

This conjecture implies that (i) when n is odd, at most one of τ and τ' are locally $\mathrm{GL}_n(K)$ -distinguished; and (ii) when n is even and τ is discrete series, then τ is locally $\mathrm{GL}_n(K)$ -distinguished if and only if τ' also is. Hence, at least for discrete series representations, one should have local dichotomy precisely when n is odd.

On the other hand, one might ask for an analogue of (2) when n is even. Here it appears that extra conditions are needed to get D with π distinguished (cf. [PTB11], [FM15]). Moreover, when such a D exists, it need not be unique—this is suggested by (ii) and we will prove this below. We hope to discuss an analogue of (2) for n even in future work.

1.2. Main results. Building on Guo’s work, we establish a simple relative trace formula to prove our main global result.

Theorem 1.5. Suppose E/F is split at all archimedean places. Assume π is supercuspidal at some place which splits in E and H -elliptic at another place. Then Conjecture 1.3(1) holds, i.e., if $\pi = \pi_D$ is H -distinguished, then $L(1/2, \pi'_E) \neq 0$ and π' is symplectic.

This is Theorem 6.1 below. The condition of being H -elliptic at some place means that the associated local Bessel distribution is nonzero on the “ H -elliptic” set (see Section 4.2). This condition holds for many representations and will be discussed momentarily.

One might also hope to show the converse, Conjecture 1.3(2), when n is odd. This is more difficult due to the nature of the geometric correspondence in the relative trace formula. Still, in Proposition 6.2 we prove a converse result under some hypotheses, for n even or odd, though the hypotheses are weaker when n is odd.

Our final global result, Theorem 6.3, gives sufficient conditions for an H -period of π_{D_1} to transfer to an H -period of π_{D_2} , for two $D_1, D_2 \in X(E:F:\pi')$, when n is even. This tells us that when an analogue of Conjecture 1.3(2) for n even holds, the D should not be unique.

Since being globally distinguished implies being locally distinguished at each place, an appropriate global embedding result for locally distinguished representations allows us to conclude the following local results.

Let K/k be a quadratic extension of p -adic fields, $\eta_{K/k}$ the associated quadratic character, and $D(k)$ the quaternion division algebra over k .

Put $H(k) = \mathrm{GL}_n(K)$ and $H'(k) = \mathrm{GL}_n(k) \times \mathrm{GL}_n(k)$.

Theorem 1.6. *Let τ be a supercuspidal representation of $\mathrm{GL}_n(D(k))$ and τ' its Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(k)$.*

(a) *If τ is $H(k)$ -distinguished, then τ' is both $H'(k)$ - and $(H'(k), \eta_{K/k})$ -distinguished.*

(b) *Suppose n is even and τ' is also supercuspidal. If one of τ and τ' is both $H(k)$ -distinguished and $H(k)$ -elliptic, then the other also is.*

This is contained in Theorems 6.4 and 6.5 below, and establishes part of consequence (ii) of Conjecture 1.4 under an additional elliptic assumption.

Using a similar idea to [Pra07], we also obtain one direction of Conjecture 1.4 for supercuspidal representations of GL_{2n} :

Theorem 1.7. *Let τ' be a supercuspidal representation of $\mathrm{GL}_{2n}(k)$. If τ' is $\mathrm{GL}_n(K)$ -distinguished, then τ' is symplectic and $\epsilon(1/2, \tau'_K) = 1$.*

Let us now discuss the global result in more detail.

We expect that our approach to Theorem 1.5 should lead to a formula for the L -value $L(1/2, \pi'_E)$ in terms of the square periods $|\mathcal{P}_D(\varphi)|^2$, as in Waldspurger’s case. This was carried out with a relative trace formula for $n = 1$ by Jacquet–Chen [JC01] and the latter two authors [MW09]. In higher rank, Wei Zhang [Zha14a] also used a simple relative trace formula to obtain an L -value formula in the setting of the unitary Gan–Gross–Prasad conjectures under some local hypotheses.

When $n = 2$, this theorem can be thought of as a relation between the nonvanishing of certain periods and the nonvanishing of central spinor L -values for $\mathrm{GSp}(4)$. One direction of the $\mathrm{SO}(5) \times \mathrm{SO}(2)$ case of Gross–Prasad says the nonvanishing of these L -values should also be detected by Bessel periods. This is now known, e.g., [FM16]. See [FM15] for a discussion of the comparison between Bessel periods and $\mathrm{GL}(n, E)$ periods.

The idea of proof is similar to some other recent works such as [JM07], [FM14] and [Zha14b]. Before outlining the proof, let us highlight a couple of differences from these other works. First, unlike [JM07] and [FM14], this is valid in higher rank and we use Ramakrishnan’s mild Chebotarev result for $\mathrm{GL}(n)$ [Ram15] to avoid the need of the full fundamental lemma for the Hecke algebra. These are also features of [Zha14b], which was completed while we were finishing this project. Second, we need to show we can choose a test function which has H -elliptic support. There is no need for this in the cases treated

by [FM14] and [Zha14b], as the orbital integrals converge for a dense set of elements. In [JM07], this type of result was established at an archimedean place for $\mathrm{GL}_2(D)$ via explicit Lie algebra calculations. Here we impose this condition as the H -elliptic local hypothesis, but we expect this to hold for most H -distinguished representations. As evidence, we give a local proof of the existence of local H -elliptic supercuspidals when $E_v = \mathbf{Q}_2 \oplus \mathbf{Q}_2$ (Proposition 4.4), and use this to give a global proof of the existence of local H -elliptic representations for any local quadratic extension (Proposition 5.2). These types of results and proof appear somewhat novel. Some results in a similar vein have recently been obtained in other cases, such as [JM07, Proposition 9.6] and [Zha], but these have very different proofs. Related results in the group case have been well known for some time, e.g. [Rog83, Proposition 2.7].

1.3. Outline of method. Now we outline the proof of Theorem 1.5. Fix $D \in X(E:F)$. The trace formula on G_D is an expression of the form

$$(1.1) \quad I_D(f_D) = \sum_{\gamma_D} I_{D,\gamma_D}(f_D) = \sum_{\sigma_D} I_{D,\sigma_D}(f_D),$$

where f_D is a certain test function on $G_D(\mathbf{A})$, I_{D,γ_D} are certain orbital integrals indexed by double cosets $H(F) \backslash G_D(F) / H(F)$, and I_{D,σ_D} are certain spectral distributions, which for $\sigma_D = \pi_D$ involves \mathcal{P}_D and $\overline{\mathcal{P}_D}$. A key point is that $I_{D,\pi_D} \neq 0$ if and only if π_D is H -distinguished. Similarly, we have a trace formula on G' of the form

$$(1.2) \quad I'(f') = \sum_{\gamma'} I'_{\gamma'}(f') = \sum_{\sigma'} I'_{\sigma'}(f').$$

Here $I'_{\pi'} \neq 0$ if and only if π' is H' - and (H', η) -distinguished.

These trace formulas will not be convergent in general, but if we pick $f_D = \prod f_{D,v}$ and $f' = \prod f'_v$ so that at one place they are supported on “regular elliptic” double cosets and at another place they are matrix coefficients of a supercuspidals, then both sides converge absolutely.

One defines a correspondence among regular elliptic double cosets γ_D and γ' , and thus a notion of matching functions f_D and f' in the sense that $I_{D,\gamma_D}(f_D) = I'_{\gamma'}(f')$ for matching γ_D and γ' . Here each γ_D corresponds to a unique γ' , and no two double cosets γ_D correspond to the same γ' (for a fixed D when n is even, or among all D 's when n is odd). However not all γ' 's correspond to a γ_D when n is even. If $I'_{\gamma'}(f') = 0$ for such “bad” γ' , then for f' matching an f_D we can write

$$(1.3) \quad I_D(f_D) = I'(f')$$

for n even. When n is odd, there are no such bad (elliptic) γ' , but a single f' should match with a family $(f_{D'})$ as D' ranges over $X(E:F)$, and for such test functions we will have

$$(1.4) \quad \sum_{D' \in X(E:F)} I_{D'}(f_{D'}) = I'(f').$$

This should give the reader some sense of the differences between the n even and n odd case for Conjecture 1.3(2). Since we are just proving the other direction, we may take f' so that $I_{D'}(f_{D'}) = 0$ for all $D' \neq D$. Thus we may work with (1.3) in both the n odd and n even cases.

Now the standard thing to do is use the fundamental lemma for the Hecke algebra and linear independence of characters to deduce that $I_{D,\pi_D}(f) = I_{\pi'}(f')$. In our setup we do not yet know the fundamental lemma for the Hecke algebra, but it holds trivially at places where E/F splits. Instead, we use a result of Ramakrishnan [Ram15] which tells us that if two representations σ'_1 and σ'_2 of GL_{2n} are locally equivalent at almost all places where E/F splits, then σ'_2 is isomorphic to either σ'_1 or $\sigma'_1 \otimes \eta$. This yields

$$(1.5) \quad I_{D,\pi_D}(f_D) + I_{D,\pi_D \otimes \eta}(f_D) = I_{\pi'}(f') + I_{\pi' \otimes \eta}(f').$$

The main point is to show that we have sufficiently many pairs of matching functions $(f_D)_D$ and f' , which reduces to a question of proving the existence of local matching functions, as our geometric orbital integrals factor into local ones. Local matching comes for free when E_v/F_v is split, so we may just consider local matching at inert places.

When $f_{D,v}$ and f'_v are the unit elements of the Hecke algebra, this is the fundamental lemma proved by Guo [Guo96] (at almost all places). In Section 3, we prove the existence of matching functions supported on both certain dense and elliptic subsets of G_D and G' . Then in Section 4, we use an extension of another result of Guo [Guo98] on local integrability of local Bessel distributions of G' to say that at odd places it is enough to just consider functions f' supported on dense subsets of G' at odd nonarchimedean places. Here is where the assumption about being split at archimedean places arises. (We use [Zha15] to treat even places.)

The H -elliptic condition allows us to choose test functions that guarantee the convergence of the geometric sides, whereas the supercuspidal condition allows us to choose functions that give convergence of the spectral sides.

This matching is now enough to get Theorem 1.5, by showing that $I_{\pi_D \otimes \eta}(f_D)$ cannot cancel out $I_{\pi_D}(f_D)$ for all such f_D , i.e., the left hand side of (1.5) is nonzero for some f_D if π_D is H -distinguished, i.e., if $I_{\pi_D} \neq 0$.

1.4. Further remarks. After we completed an earlier draft, Chong Zhang [Zha15] used an idea of Wei Zhang [Zha14b] to get a smooth transfer result of the form: each $f_{D,v}$ has a matching f'_v for a nonarchimedean place v . While we use this at even places, one could also use this at all nonarchimedean places instead of our restricted smooth matching results in Section 3. However, we hope that our original approach may still be of interest as: (i) proving restricted smooth matching is much simpler than a full smooth matching result; (ii) it involves reducing relative orbital integrals to orbital integrals in [AC89] on lower rank groups, which may indicate an interesting connection between the trace formula in [AC89] and the relative trace formula here; and (iii) this approach may be useful in other situations where a complete smooth matching result is not known—e.g., the archimedean situation here or cases involving other groups.

Acknowledgements. We are grateful to Masaaki Furusawa, Hervé Jacquet, Fiona Mur-naghan, Dipendra Prasad, Dinakar Ramakrishnan and Alan Roche for helpful discussions. We thank Chong Zhang for pointing out a gap in an earlier version, and referees for useful comments. The first author was partially supported by National Science Foundation grant DMS-1201446, National Security Agency grant H98230-16-1-0017 and PSC-CUNY. The second author was partially supported by Simons Foundation Collaboration Grant 240605. The second and third authors were partially supported by National Science Foundation grant DMS-0758197.

2. NOTATION

Either F is a number field (Sections 5 and 6) or a local field (Sections 3 and 4), and E is a quadratic étale extension of F , i.e., either a quadratic field extension or, in Sections 4 and 5, possibly the split algebra $E = F \oplus F$. In the global (resp. local) case, η denotes the quadratic character of $F^\times \backslash \mathbf{A}^\times$ (resp. F^\times) corresponding to E/F by class field theory.

We denote the norm map from E to F by N . For an element $\alpha \in E$ we let $\bar{\alpha}$ denote the image of α under the non-trivial element of $\text{Gal}(E/F)$. We use the same notation for elements in $M_n(E)$. Denote by I_n the $n \times n$ identity matrix.

We set $G' = \text{GL}_{2n}$, viewed as an algebraic group over F , and we let $H' = \text{GL}_n \times \text{GL}_n$, which we view as a subgroup of G' via the embedding

$$(A_1, A_2) \mapsto \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

For $\varepsilon \in F^\times / (NE^\times)$, set

$$G_\varepsilon = \left\{ \begin{pmatrix} \alpha & \varepsilon\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \text{GL}_{2n}(E) : \alpha, \beta \in M_n(E) \right\}$$

and H_ε to be the image of $\text{GL}_n(E)$ in the block diagonal subgroup of G_ε . Note $G_\varepsilon \cong G_{D_\varepsilon} := \text{GL}_n(D_\varepsilon)$ where D_ε is the associated quaternion algebra

$$D_\varepsilon = \left\{ \begin{pmatrix} a & \varepsilon b \\ \bar{b} & \bar{a} \end{pmatrix} \in \text{GL}_2(E) : a, b \in E \right\}.$$

If ε is fixed, we often write G (resp. H) for G_ε (resp. H_ε).

3. ORBITAL INTEGRALS

In this section we prove the existence of matching functions for our relative trace formulas. A more complete matching result is now known by C. Zhang [Zha15], but was not available at the original writing of this paper. In addition, our approach may still be of interest as it is more elementary and may be useful for other situations, e.g., the archimedean case for the situation at hand.

The idea for our approach is to translate the matching in our case to matching between orbital integrals over conjugacy classes on GL_n and twisted orbital integrals for GL_n over a quadratic extension. Matching in this case is known by work of Arthur and Clozel [AC89], and we are able to deduce the existence of a large class of matching functions from their work. Throughout this section, F is a local nonarchimedean field of characteristic 0 and E is a quadratic field extension of F . While working locally, we often denote the F -rational points of an algebraic group \mathbb{G} over F simply by \mathbb{G} .

First we recall results of Guo [Guo96] on the matching of double cosets. Let

$$w = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

We consider the automorphism θ of G' of order 2 given by $\theta(g) = w^{-1}gw$. Then H' is the set of fixed points of θ . Let S' be the variety

$$S' = \{g\theta(g)^{-1} : g \in G'\},$$

and let $\rho : G' \rightarrow S' : g \mapsto g\theta(g)^{-1}$. The group G' acts on S' by twisted conjugation,

$$g \cdot s := gs\theta(g)^{-1},$$

so H' acts on S' by ordinary conjugation. With this action we have

$$\rho(xgh) = x \cdot \rho(g), \quad x, g \in G', h \in H'.$$

Hence ρ induces an isomorphism of G' -spaces between the symmetric space G'/H' (with G' acting by left translation) and S' . We define

$$\Gamma(S') = \{H'\text{-conjugacy classes } [s] \text{ in } S'\},$$

where $[s] = H' \cdot s$. Then the set $H' \backslash G' / H'$ of H' double cosets in G' is identified with $\Gamma(S')$. We set

$$\Gamma^{\text{ss}}(S') = \{\text{semisimple } H'\text{-conjugacy classes in } S'\}.$$

We define an element $s \in S'$ to be $(\theta\text{-})$ regular if s is semisimple and $[s]$ in $\Gamma(S')$ has maximal dimension among the elements in $\Gamma(S')$. We denote this set by S'^{reg} . We define an element $s \in S'$ to be $(\theta\text{-})$ elliptic (regular) if s is regular and the centralizer of s in H' is an elliptic torus. We denote this set by S'^{ell} . Then we define

$$\Gamma^{\text{reg}}(S') = \{[s] \in \Gamma(S') : s \in S'^{\text{reg}}\}, \quad \Gamma^{\text{ell}}(S') = \{[s] \in \Gamma(S') : s \in S'^{\text{ell}}\},$$

$$G'^{\text{reg}} = \{g \in G' : \rho(g) \in S'^{\text{reg}}\}, \quad \text{and} \quad G'^{\text{ell}} = \{g \in G' : \rho(g) \in S'^{\text{ell}}\}.$$

Correspondingly, we call an H' double coset of G' regular or elliptic if the associated H' -class in S' is. We let $\Gamma_{H'}^{\text{reg}}(G')$ (resp. $\Gamma_{H'}^{\text{ell}}(G')$) denote the set of regular (resp. elliptic) H' double cosets in G' . We take $\Gamma^{\text{reg}}(\text{GL}_n(F))$ (resp. $\Gamma^{\text{ell}}(\text{GL}_n(F))$) to be the regular (resp. elliptic) semisimple conjugacy classes in $\text{GL}_n(F)$. (Note when we say regular (resp. elliptic), we mean θ -regular (resp. θ -elliptic) double cosets if we are talking about G' and regular (resp. elliptic) in the usual sense if we are talking about $\text{GL}_n(F)$.)

For $A \in M_n$, let

$$g'(A) = \begin{pmatrix} I_n & A \\ I_n & I_n \end{pmatrix} \in G'.$$

By [Guo96, Lemma 1.3],

$$(3.1) \quad \Gamma_{H'}^{\text{reg}}(G') = \{[g'(A)] : A \in \Gamma^{\text{reg}}(\text{GL}_n(F)), I_n - A \in \text{GL}_n(F)\}$$

and

$$\Gamma_{H'}^{\text{ell}}(G') = \left\{ [g'(A)] : A \in \Gamma^{\text{ell}}(\text{GL}_n(F)), I_n - A \in \text{GL}_n(F) \right\},$$

where $[g'] = H'g'H'$ for $g' \in G'$.

Now we look at the double cosets on G_ε for a fixed $\varepsilon \in F^\times$. Let $\tau \in F^\times$ such that $E = F(\sqrt{\tau})$, let

$$w_\varepsilon = \begin{pmatrix} \sqrt{\tau}I_n & 0 \\ 0 & -\sqrt{\tau}I_n \end{pmatrix} \in G_\varepsilon,$$

and let θ_ε denote the automorphism of G_ε defined by $\theta_\varepsilon(g) = w_\varepsilon g w_\varepsilon^{-1}$. As before H_ε is the set of fixed points of θ_ε and we define

$$S_\varepsilon = \{g\theta_\varepsilon(g)^{-1} : g \in G_\varepsilon\}.$$

Then G_ε acts on S_ε by twisted conjugation,

$$g \cdot_\varepsilon s := gs\theta_\varepsilon(g)^{-1}.$$

Define $\rho_\varepsilon : G_\varepsilon \rightarrow S_\varepsilon$ by $\rho(g) = g\theta_\varepsilon(g)^{-1}$, which identifies $G_\varepsilon/H_\varepsilon$ with S_ε as G_ε -spaces. In particular $H_\varepsilon \backslash G_\varepsilon / H_\varepsilon$ is identified with the set of H_ε -conjugacy classes in S_ε .

Define $\Gamma(S_\varepsilon)$, $\Gamma^{\text{ss}}(S_\varepsilon)$, $\Gamma^{\text{reg}}(S_\varepsilon)$, $\Gamma^{\text{ell}}(S_\varepsilon)$, $S_\varepsilon^{\text{reg}}$ and $S_\varepsilon^{\text{ell}}$ similar to above. Let

$$G_\varepsilon^{\text{reg}} = \{g \in G_\varepsilon : \rho_\varepsilon(g) \in S_\varepsilon^{\text{reg}}\} \quad \text{and} \quad G_\varepsilon^{\text{ell}} = \{g \in G_\varepsilon : \rho_\varepsilon(g) \in S_\varepsilon^{\text{ell}}\}.$$

Also define $\Gamma_{H_\varepsilon}^{\text{reg}}(G_\varepsilon)$ and $\Gamma_{H_\varepsilon}^{\text{ell}}(G_\varepsilon)$ similarly to the case of G' .

We say $g_1, g_2 \in \text{GL}_n(E)$ are twisted conjugate if there exists $g \in \text{GL}_n(E)$ such that $g_2 = gg_1\bar{g}^{-1}$. Let $\Gamma^{\text{tw}}(\text{GL}_n(E))$ denote the set of twisted conjugacy classes in $\text{GL}_n(E)$. By [AC89, Lemma 1.1] there is an injective norm map

$$\mathcal{N} : \Gamma^{\text{tw}}(\text{GL}_n(E)) \rightarrow \Gamma(\text{GL}_n(F))$$

defined as follows. Let $A \in \text{GL}_n(E)$. Then $A\bar{A} \in \text{GL}_n(E)$ is conjugate in $\text{GL}_n(E)$ to an element $B \in \text{GL}_n(F)$, which is unique up to conjugation in $\text{GL}_n(F)$. One defines $\mathcal{N}A$ as the conjugacy class of B in $\text{GL}_n(F)$.

We say $g \in \text{GL}_n(E)$ is regular (resp. elliptic) twisted if $\mathcal{N}g$ is regular (resp. regular elliptic) in $\text{GL}_n(F)$. Let $\Gamma^{\text{reg,tw}}(\text{GL}_n(E))$ (resp. $\Gamma^{\text{ell,tw}}(\text{GL}_n(E))$) denote the set of regular (resp. elliptic) twisted conjugacy classes in $\text{GL}_n(E)$. For $A \in M_n(E)$ let

$$g_\varepsilon(A) = \begin{pmatrix} I_n & \varepsilon A \\ \bar{A} & I_n \end{pmatrix}.$$

Then, by [Guo96, Lemma 1.7],

$$(3.2) \quad \Gamma_{H_\varepsilon}^{\text{reg}}(G_\varepsilon) = \{[g_\varepsilon(A)] : A \in \Gamma^{\text{reg,tw}}(\text{GL}_n(E)), I_n - \varepsilon A\bar{A} \in \text{GL}_n(E)\}$$

and

$$\Gamma_{H_\varepsilon}^{\text{ell}}(G_\varepsilon) = \{[g_\varepsilon(A)] : A \in \Gamma^{\text{ell,tw}}(\text{GL}_n(E)), I_n - \varepsilon A\bar{A} \in \text{GL}_n(E)\}$$

where $[g_\varepsilon] = H_\varepsilon g_\varepsilon H_\varepsilon$ for $g_\varepsilon \in G_\varepsilon$.

We have defined varieties $S_\varepsilon \subset G_\varepsilon \subset G'(E)$ and $S' \subset G' \subset G'(E)$. By [Guo96, Proposition 1.3], given a semisimple element $s \in S_\varepsilon$ there exists $h \in H_\varepsilon$ such that $h^{-1}sh \in S'$. This yields an embedding,

$$\iota_\varepsilon : \Gamma^{\text{ss}}(S_\varepsilon) \hookrightarrow \Gamma^{\text{ss}}(S').$$

According to [Guo96, page 117] this extends to an embedding $\Gamma(S_\varepsilon) \hookrightarrow \Gamma(S')$. The map ι_ε gives an injection of $\Gamma^{\text{reg}}(S_\varepsilon)$ into $\Gamma^{\text{reg}}(S')$, and of $\Gamma^{\text{ell}}(S_\varepsilon)$ into $\Gamma^{\text{ell}}(S')$.

The injection ι_ε induces an embedding,

$$\iota_\varepsilon : \Gamma_{H_\varepsilon}^{\text{reg}}(G_\varepsilon) \hookrightarrow \Gamma_{H'}^{\text{reg}}(G')$$

by

$$(3.3) \quad \iota_\varepsilon([g_\varepsilon(A)]) = [g'(\varepsilon \mathcal{N}A)].$$

and thus by restriction $\iota_\varepsilon : \Gamma_{H_\varepsilon}^{\text{ell}}(G_\varepsilon) \hookrightarrow \Gamma_{H'}^{\text{ell}}(G')$.

When n is odd, by [Guo96, Lemma 1.8],

$$(3.4) \quad \Gamma_{H'}^{\text{ell}}(G') = \bigsqcup_{\varepsilon \in F^\times / NE^\times} \iota_\varepsilon \left(\Gamma_{H_\varepsilon}^{\text{ell}}(G_\varepsilon) \right)$$

and

$$(3.5) \quad \bigsqcup_{\varepsilon \in F^\times / NE^\times} \iota_\varepsilon \left(\Gamma_{H_\varepsilon}^{\text{reg}}(G_\varepsilon) \right) \subset \Gamma_{H'}^{\text{reg}}(G').$$

When n is even

$$(3.6) \quad \iota_{\varepsilon_1} \left(\Gamma_{H_{\varepsilon_1}}^{\text{ell}}(G_{\varepsilon_1}) \right) = \iota_{\varepsilon_2} \left(\Gamma_{H_{\varepsilon_2}}^{\text{ell}}(G_{\varepsilon_2}) \right)$$

for any $\varepsilon_1, \varepsilon_2 \in F^\times$.

3.1. Local orbital integrals for G' . For $g \in G'$, let

$$H'_g = \{(h_1, h_2) \in H' \times H' : h_1^{-1}gh_2 = g\}$$

denote the stabilizer of g under the action of $H' \times H'$. We call a double coset $H'gH'$ (or the element g) *relevant* if the map

$$H'_g \rightarrow \mathbf{C} : (h_1, h_2) \mapsto \eta(\det h_2)$$

is trivial.

Fix a Haar measure on $\text{GL}_n(F)$ and use this to give $H' = \text{GL}_n(F) \times \text{GL}_n(F)$ the product measure. For each double coset $H'gH'$, we fix a (left) Haar measure on H'_g . Then for $f \in C_c^\infty(G')$ and relevant $g \in G'$ we define the orbital integral

$$(3.7) \quad I'_g(f) = \int_{H'_g \backslash H' \times H'} f(h_1^{-1}gh_2) \eta(\det h_2) dh_1 dh_2,$$

provided it converges.

For $F' \in C_c^\infty(\text{GL}_n(F))$ and $X \in \text{GL}_n(F)$ we define the orbital integrals over conjugacy classes of $\text{GL}_n(F)$ by

$$O_X(F') = \int_{T'_X \backslash \text{GL}_n(F)} F'(g^{-1}Xg) dg,$$

where T'_X denotes the centralizer of X in $\text{GL}_n(F)$. We will specify the Haar measure on T'_X for certain X in the proof of Lemma 3.2.

Now we will relate the orbital integrals on G' to the orbital integrals on $\text{GL}_n(F)$.

Define the open subset of M_n ,

$$\mathcal{U}' = \left\{ X \in M_n : \begin{pmatrix} I_n & X \\ I_n & I_n \end{pmatrix} \in G' \right\}.$$

Consider the mapping from $\mathcal{U}' \rightarrow G'$ by

$$X \mapsto g'(X) = \begin{pmatrix} I_n & X \\ I_n & I_n \end{pmatrix}.$$

Given $f \in C_c^\infty(G')$ we define a smooth function F'_f on \mathcal{U}' by

$$F'_f(X) = \int_{(\text{GL}_n(F))^3} f \left(\begin{pmatrix} A_1^{-1} & A_1^{-1}XA_2B \\ A_2^{-1} & B \end{pmatrix} \right) \eta(\det(A_2B)) dA_1 dA_2 dB$$

when this integral converges.

For $F'_f(X)$ to be nice, we want to look at functions f supported on the subset

$$(3.8) \quad G'^{\text{main}} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G' : A, B, C, D \in \text{GL}_n(F) \right\}.$$

Put $\mathcal{U}'^{\text{main}} = \mathcal{U}' \cap \text{GL}_n(F)$ and $\mathcal{U}'^{\text{ell}} = \mathcal{U}' \cap \text{GL}_n(F)^{\text{ell}}$, where $\text{GL}_n(F)^{\text{ell}}$ denotes the set of regular elliptic elements in $\text{GL}_n(F)$. Note the mapping $X \mapsto g'(X)$ maps $\mathcal{U}'^{\text{main}}$ to G'^{main} and maps $\mathcal{U}'^{\text{ell}}$ to G'^{ell} .

Now we prove F'_f is defined and smooth on $\mathcal{U}'^{\text{main}}$.

Lemma 3.1. *Let $f \in C_c^\infty(G')$ and $X \in \mathcal{U}'^{\text{main}}$. Then the following statements hold:*

- (1) $F'_f(X)$ is a convergent integral.
- (2) $F'_f(X) = 0$ if $|\det(I_n - X)|$ is sufficiently small (in terms of an explicit constant that depends on f).

Proof. We will prove that $F'_f(X)$ is an integral over compact sets. Since f is compactly supported on G' there exists a compact set Ω_f in $M_n(F)$ such that if

$$f\left(\begin{pmatrix} A_1 & A_1 X A_2^{-1} B \\ A_2 & B \end{pmatrix}\right) \neq 0$$

then $A_1, A_2, B, A_1 X A_2^{-1} B \in \Omega_f$. It remains to prove that the determinant of each variable of integration is bounded away from zero in the support of f . First we note that since Ω_f is a compact subset of M_n there exists a $c_f > 0$ such that if $g \in \Omega_f$ then $|\det g| < c_f$. Now we note that since f is compactly supported in G' there exists a $c'_f > 0$ such that if $f(g) \neq 0$ then $|\det g| > c'_f$. Finally we note that

$$\begin{pmatrix} A_1 & A_1 X A_2^{-1} B \\ A_2 & B \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} I_n & X \\ I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & A_2^{-1} B \end{pmatrix}.$$

Combining these facts we see that if

$$f\left(\begin{pmatrix} A_1 & A_1 X A_2^{-1} B \\ A_2 & B \end{pmatrix}\right) \neq 0$$

then

$$|\det(A_1)|, |\det(A_2)|, |\det(B)| < c_f, \quad |\det(A_1) \det(X) \det(B)| < c_f |\det(A_2)|$$

and

$$c'_f < |\det(I_n - X)| |\det(A_1)| |\det(B)|.$$

Thus, in the support of f , $|\det(A_1)|$ and $|\det(B)|$ are bounded below by $\frac{c'_f}{c_f} |\det(I_n - X)|^{-1}$ and $|\det(A_2)|$ is bounded below by $\frac{(c'_f)^2 |\det(X)|}{c_f^3 |\det(I_n - X)|^2}$.

To prove the second statement we note that the integrand defining $F'_f(X)$ is identically zero unless $|\det(I_n - X)| > \frac{c'_f}{c_f^2}$. □

Lemma 3.2. *Let $f \in C_c^\infty(G'^{\text{main}})$ (resp. $f \in C_c^\infty(G'^{\text{ell}})$). Then*

- (1) $F'_f \in C_c^\infty(\mathcal{U}'^{\text{main}})$ (resp. $F'_f \in C_c^\infty(\mathcal{U}'^{\text{ell}})$); and
- (2) for $X \in \mathcal{U}'^{\text{main}}$, $I'_{g'(X)}(f) = O_X(F'_f)$.

Proof. For the first part we note that, by the definition of G'^{main} (see (3.8)), for $f \in C_c^\infty(G'^{\text{main}})$ there exists a compact subset K_f of $\text{GL}_n(F)$ such that if

$$f\left(\begin{pmatrix} A_1 & A_1 X A_2^{-1} B \\ A_2 & B \end{pmatrix}\right) \neq 0$$

then $A_1, A_1 X A_2^{-1} B, A_2, B \in K_f$. Hence if $F'_f(X) \neq 0$ then $X \in K_f^{-1} K_f K_f^{-1} K_f$. The result for G'^{main} now follows by applying this fact and the second result from the previous lemma. The result for G'^{ell} follows from the result for G'^{main} .

We now proceed to prove the equality of orbital integrals under the mapping by a straightforward calculation. First we note that for $X \in \mathcal{U}'^{\text{main}}$,

$$(3.9) \quad H_{g'(X)} = \left\{ \left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \right) : t \in T'_X \right\}.$$

Consequently, $g'(X)$ is relevant and $H_{g'(X)}$ is unimodular. Normalize the measure on T'_X so that it is compatible with this isomorphism between T'_X and $H_{g'(X)}$. Thus

$$\begin{aligned} I'_{g'(X)} &= \int_{H'_{g'(X)} \backslash H' \times H'} f \left(h_1^{-1} \begin{pmatrix} I_n & X \\ I_n & I_n \end{pmatrix} h_2 \right) \eta(\det h_2) dh_1 dh_2 \\ &= \int_{H'_{g'(X)} \backslash H' \times H'} f \left(\begin{pmatrix} A_1^{-1} B_1 & A_1^{-1} X B_2 \\ A_2^{-1} B_1 & A_2^{-1} B_2 \end{pmatrix} \right) \eta(\det B_1 B_2) dA_1 dA_2 dB_1 dB_2. \end{aligned}$$

By the change of variables $A_1 \mapsto B_1 A_1$ and $B_2 \mapsto B_1 A_2 B_2$ the previous line equals

$$\begin{aligned} \int_{T'_X \backslash \text{GL}_n(F)} \int_{(\text{GL}_n(F))^3} f \left(\begin{pmatrix} A_1^{-1} & A_1^{-1} B_1^{-1} X B_1 A_2 B_2 \\ A_2^{-1} & B_2 \end{pmatrix} \right) \eta(\det(A_2 B_2)) dA_1 dA_2 dB_1 dB_2 \\ = \int_{T'_X \backslash \text{GL}_n(F)} F'_f(B_1^{-1} X B_1) dB_1 = O_X(F'_f). \end{aligned}$$

□

Lemma 3.3. *The map from $C_c^\infty(G'^{\text{main}}) \rightarrow C_c^\infty(\mathcal{U}'^{\text{main}})$ defined by $f \mapsto F'_f$ is surjective. The restriction of this map also gives a surjection $C_c^\infty(G'^{\text{ell}}) \rightarrow C_c^\infty(\mathcal{U}'^{\text{ell}})$.*

Proof. Given $\varphi \in C_c^\infty(\mathcal{U}'^{\text{main}})$ let $\varphi_0, \varphi_1 \in C_c^\infty(\text{GL}_n(F))$ be such that

$$\int_{\text{GL}_n(F)} \varphi_0(g) dg = 1 \quad \text{and} \quad \int_{\text{GL}_n(F)} \varphi_1(g) \eta(\det g) dg = 1.$$

Then define

$$f \left(\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} \begin{pmatrix} I_n & X \\ I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & \\ & B \end{pmatrix} \right) = \varphi_0(A_1) \varphi_0(A_2) \varphi_1(B) \varphi(X).$$

We extend f to all of G'^{main} by defining f to be zero on $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ if $A^{-1} B D^{-1} C \notin \mathcal{U}'^{\text{main}}$.

It is clear that $f \in C_c^\infty(G'^{\text{main}})$ and $F'_f = \varphi$.

The elliptic case is similar. □

3.2. Local orbital integrals for G . Fix $\varepsilon \in F^\times$ and throughout this subsection let $G = G_\varepsilon$.

For $g \in G$, let

$$H_g = \{(h_1, h_2) \in H \times H : h_1^{-1} g h_2 = g\}$$

denote the stabilizer of g under the action of $H \times H$. Fix a Haar measure dh on the group $H \cong \text{GL}_n(E)$, and one on each stabilizer H_g . For $f \in C_c^\infty(G)$ we define the orbital

integral

$$(3.10) \quad I_g(f) = \int_{H_g \backslash H \times H} f(h_1^{-1} g h_2) dh_1 dh_2,$$

when convergent.

For $F \in C_c^\infty(\mathrm{GL}_n(E))$ and $X \in \mathrm{GL}_n(E)$, the twisted orbital integral on $\mathrm{GL}_n(E)$ is

$$TO_X(F) = \int_{T_X \backslash \mathrm{GL}_n(E)} F(g^{-1} X \bar{g}) dg,$$

where T_X denotes the twisted centralizer of X in $\mathrm{GL}_n(E)$, that is,

$$T_X = \{g \in \mathrm{GL}_n(E) : g^{-1} X \bar{g} = X\}.$$

We specify a measure on T_X for certain X in the proof of Lemma 3.5.

Let

$$G^{\mathrm{main}} = \left\{ \begin{pmatrix} \alpha & \varepsilon\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G : \alpha, \beta \in \mathrm{GL}_n(E) \right\}.$$

Consider the open subset of $M_n(E)$,

$$\mathcal{U} = \mathcal{U}_\varepsilon = \left\{ X \in M_n(E) : \begin{pmatrix} I_n & \varepsilon X \\ \bar{X} & I_n \end{pmatrix} \in G \right\}.$$

We also set $\mathcal{U}^{\mathrm{main}} = \mathcal{U}_\varepsilon^{\mathrm{main}} = \mathcal{U} \cap \mathrm{GL}_n(E)$ and $\mathcal{U}^{\mathrm{ell}} = \mathcal{U}_\varepsilon^{\mathrm{ell}} = \mathcal{U} \cap \mathrm{GL}_n(E)^{\mathrm{ell}, \mathrm{tw}}$, where $\mathrm{GL}_n(E)^{\mathrm{ell}, \mathrm{tw}}$ denotes the set of twisted elliptic elements of $\mathrm{GL}_n(E)$. Define a mapping from $\mathcal{U} \rightarrow G$ by

$$X \mapsto g(X) = g_\varepsilon(X) = \begin{pmatrix} I_n & \varepsilon X \\ \bar{X} & I_n \end{pmatrix}.$$

Note that this mapping restricted to $\mathcal{U}^{\mathrm{main}}$ maps to G^{main} . Now we can define a mapping of test functions.

Given $f \in C_c^\infty(G)$, we define a smooth F_f on \mathcal{U} by

$$F_f(X) = \int_{\mathrm{GL}_n(E)} f \left(\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} g(X) \right) d\alpha,$$

when this integral converges.

Lemma 3.4. *Let $f \in C_c^\infty(G)$ and $X \in \mathcal{U}^{\mathrm{main}}$. Then*

- (1) $F_f(X)$ is a convergent integral; and
- (2) $F_f(X) = 0$ if $|\det(I_n - \varepsilon X \bar{X})|$ is sufficiently small (in terms of an explicit constant that depends on f).

The proof is very similar to, but simpler than, the proof of Lemma 3.1, so we omit it.

Lemma 3.5. *Let $f \in C_c^\infty(G^{\mathrm{main}})$ (resp. $C_c^\infty(G^{\mathrm{ell}})$), then $F_f \in C_c^\infty(\mathcal{U}^{\mathrm{main}})$ (resp. $C_c^\infty(\mathcal{U}^{\mathrm{ell}})$). Furthermore, for $X \in \mathcal{U}^{\mathrm{main}}$, $I_{g(X)}(f) = TO_X(F_f)$.*

Proof. The first statement is similar to the case of G' .

For the equality of orbital integrals, first we note that for $X \in \mathcal{U}^{\mathrm{main}}$

$$(3.11) \quad H_{g(X)} = \left\{ \left(\begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \right) : t \in T_X \right\}.$$

Similar to before, use this isomorphism with T_X to transport the measure from $H_{g(X)}$ to T_X for $X \in \mathcal{U}^{\text{main}}$. Now we proceed by a straightforward calculation,

$$\begin{aligned} I_{g(X)}(f) &= \int_{H_{g(X)} \backslash H \times H} f(h_1^{-1}g(X)h_2) dh_1 dh_2 \\ &= \int_{T_X \backslash \text{GL}_n(E) \times \text{GL}_n(E)} f\left(\begin{pmatrix} \alpha_1^{-1}\alpha_2 & \varepsilon\alpha_1^{-1}X\bar{\alpha}_2 \\ \bar{\alpha}_1^{-1}\bar{X}\alpha_2 & \bar{\alpha}_1^{-1}\bar{\alpha}_2 \end{pmatrix}\right) d\alpha_1 d\alpha_2. \end{aligned}$$

With a change of variables sending $\alpha_1 \mapsto \alpha_2\alpha_1$ the previous line equals

$$\begin{aligned} \int_{T_X \backslash \text{GL}_n(E)} \int_{\text{GL}_n(E)} f\left(\begin{pmatrix} \alpha_1^{-1} & \varepsilon\alpha_1^{-1}\alpha_2^{-1}X\bar{\alpha}_2 \\ \bar{\alpha}_1^{-1}\bar{\alpha}_2^{-1}\bar{X}\alpha_2 & \bar{\alpha}_1^{-1} \end{pmatrix}\right) d\alpha_1 d\alpha_2 \\ = \int_{T_X \backslash \text{GL}_n(E)} F_f(g^{-1}X\bar{g}) dg = TO_X(F_f). \end{aligned}$$

□

Lemma 3.6. *The map from $C_c^\infty(G^{\text{main}}) \rightarrow C_c^\infty(\mathcal{U}^{\text{main}})$ defined by $f \mapsto F_f$ is surjective, and similarly for $C_c^\infty(G^{\text{ell}}) \rightarrow C_c^\infty(\mathcal{U}^{\text{ell}})$.*

Proof. Given $\varphi \in C_c^\infty(\mathcal{U}^{\text{main}})$ let $\varphi_0 \in C_c^\infty(\text{GL}_n(E))$ be such that

$$\int_{\text{GL}_n(E)} \varphi_0(g) dg = 1.$$

Then define

$$f\left(\begin{pmatrix} \alpha & \\ & \bar{\alpha} \end{pmatrix} \begin{pmatrix} I_n & \varepsilon X \\ \bar{X} & I_n \end{pmatrix}\right) = \varphi_0(\alpha)\varphi(X).$$

It is clear that $f \in C_c^\infty(G^{\text{main}})$ and $F_f = \varphi$.

The elliptic case is similar. □

3.3. Local matching. Fix a set of representatives $\{\varepsilon_1, \varepsilon_2\}$ for F^\times/NE^\times such that $\varepsilon_1 \in NE^\times$, $\varepsilon_2 \notin NE^\times$. Now we will define the notion of matching functions, for which the reader should recall the correspondence of regular double cosets given by (3.3).

We also need to use compatible measures. Namely, our orbital integrals depend upon a choice of measures on H and H' as well as on stabilizers H_{g_ε} and $H'_{g'}$. The choice of measures on H and H' is not important for the general notion of matching functions, as one can just scale functions appropriately. However, for global applications it will be convenient to assume the following: if $[g'] = \iota_\varepsilon([g_\varepsilon])$ with g' regular, then $H_{g_\varepsilon} \cong H'_{g'}$, and we use measures compatible with this isomorphism.

Definition 3.7. *Let n be even and fix $\varepsilon \in F^\times$. Let $f' \in C_c^\infty(G')$ and $f_\varepsilon \in C_c^\infty(G_\varepsilon)$. We say that f' and f_ε are matching functions if*

$$I'_{g'}(f') = \begin{cases} I_{g_\varepsilon}(f_\varepsilon) & \text{if } [g'] = \iota_\varepsilon([g_\varepsilon]) \text{ for } [g_\varepsilon] \in \Gamma_{H_\varepsilon}^{\text{reg}}(G_\varepsilon), \\ 0 & \text{if } [g'] \in \Gamma_{H'}^{\text{reg}}(G') \setminus \iota_\varepsilon(\Gamma_{H_\varepsilon}^{\text{reg}}(G_\varepsilon)). \end{cases}$$

When n is odd, recall the disjointedness of regular double cosets of G' corresponding to G_{ε_1} versus G_{ε_2} from (3.5).

Definition 3.8. Let n be odd. Let $f' \in C_c^\infty(G')$ and $f_\varepsilon \in C_c^\infty(G_\varepsilon)$ for $\varepsilon \in \{\varepsilon_1, \varepsilon_2\}$. We say that f' and $(f_\varepsilon)_\varepsilon$ are matching functions if

$$(3.12a) \quad I'_{g'}(f') = \begin{cases} I_{g_\varepsilon}(f_\varepsilon) & \text{if } [g'] = \iota_\varepsilon([g_\varepsilon]) \text{ for } [g_\varepsilon] \in \Gamma_{H_\varepsilon}^{\text{reg}}(G_\varepsilon), \varepsilon \in \{\varepsilon_1, \varepsilon_2\}, \\ 0 & \text{if } [g'] \in \Gamma_{H'}^{\text{reg}}(G') \setminus \bigsqcup_\varepsilon \iota_\varepsilon(\Gamma_{H_\varepsilon}^{\text{reg}}(G_\varepsilon)). \end{cases}$$

When n is odd and f' matches $(f_{\varepsilon_1}, 0)$ or $(0, f_{\varepsilon_2})$, we may simply say f' matches f_{ε_1} or f_{ε_2} . We only need to consider f' matching a pair $(f_{\varepsilon_1}, f_{\varepsilon_2})$ for Conjecture 1.3(2).

We first extend the matching of orbital integrals over (twisted) conjugacy classes for $\text{GL}_n(F)$ from [AC89]. Denote by $\text{GL}_n(F)^{\text{reg}}$ (resp. $\text{GL}_n(E)^{\text{reg,tw}}$) the set of regular elements of $\text{GL}_n(F)$ (resp. twisted regular elements of $\text{GL}_n(E)$). For $\gamma \in \text{GL}_n(F)$, denote by $[\gamma]$ its conjugacy class.

Proposition 3.9. Fix $\varepsilon \in \{\varepsilon_1, \varepsilon_2\}$.

- (1) Fix $\varphi \in C_c^\infty(\text{GL}_n(E))$ (resp. $C_c^\infty(\text{GL}_n(E)^{\text{ell,tw}})$). Then there exists $\varphi' \in C_c^\infty(\text{GL}_n(F))$ (resp. $C_c^\infty(\text{GL}_n(F)^{\text{ell}})$) such that for $\gamma \in \text{GL}_n(F)^{\text{reg}}$,

$$O_\gamma(\varphi') = \begin{cases} TO_\delta(\varphi) & \text{if } \gamma = \varepsilon\delta\bar{\delta} \text{ for } \delta \in \text{GL}_n(E), \\ 0 & \text{if } [\gamma] \notin \varepsilon\mathcal{N}\text{GL}_n(E). \end{cases}$$

- (2) Suppose n is odd and $\varphi' \in C_c^\infty(\text{GL}_n(F))$ (resp. $C_c^\infty(\text{GL}_n(F)^{\text{ell}})$) such that $O_\gamma(\varphi') = 0$ if $[\gamma] \notin \varepsilon_1\mathcal{N}\text{GL}_n(E) \cup \varepsilon_2\mathcal{N}\text{GL}_n(E)$. Then there exist $\varphi_{\varepsilon_1}, \varphi_{\varepsilon_2} \in C_c^\infty(\text{GL}_n(E))$ (resp. $C_c^\infty(\text{GL}_n(E)^{\text{ell,tw}})$) such that

$$(3.13) \quad O_\gamma(\varphi') = \begin{cases} TO_\delta(\varphi_{\varepsilon_1}) & \text{if } \gamma = \varepsilon_1\delta\bar{\delta} \text{ for } \delta \in \text{GL}_n(E)^{\text{reg,tw}}, \\ TO_\delta(\varphi_{\varepsilon_2}) & \text{if } \gamma = \varepsilon_2\delta\bar{\delta} \text{ for } \delta \in \text{GL}_n(E)^{\text{reg,tw}}. \end{cases}$$

- (3) Suppose n is even and fix $\varphi' \in C_c^\infty(\text{GL}_n(F))$ (resp. $C_c^\infty(\text{GL}_n(F)^{\text{ell}})$) such that $O_\gamma(\varphi') = 0$ for $[\gamma] \notin \varepsilon\mathcal{N}\text{GL}_n(E)$. Then there exists $\varphi \in C_c^\infty(\text{GL}_n(E))$ (resp. $C_c^\infty(\text{GL}_n(E)^{\text{ell,tw}})$) such that

$$O_\gamma(\varphi') = TO_\delta(\varphi), \quad \text{if } \gamma = \varepsilon\delta\bar{\delta} \text{ for } \delta \in \text{GL}_n(E)^{\text{reg,tw}}.$$

Proof. First we prove part (1) for $\varphi \in C_c^\infty(\text{GL}_n(E))$. We may assume $\varepsilon_1 = 1$. If $\varepsilon = \varepsilon_1$ then this is contained in Proposition 3.1 in Chapter 1 of [AC89]. Denote this matching function by φ'_1 . For $\varepsilon = \varepsilon_2$ let $\varphi'_2(g) = \varphi'_1(\varepsilon_2^{-1}g)$. Then $O_g(\varphi'_2) = O_{\varepsilon_2^{-1}g}(\varphi'_1)$. Hence $O_{\varepsilon_2\delta\bar{\delta}}(\varphi'_2) = TO_\delta(\varphi)$ and $O_\gamma(\varphi'_2) = 0$ for $[\gamma] \notin \varepsilon_2\mathcal{N}\text{GL}_n(E)$.

Now suppose $\varphi \in C_c^\infty(\text{GL}_n(E)^{\text{ell,tw}})$. Consider the map $s : \text{GL}_n(F) \rightarrow F^n$ given by the coefficients of the characteristic polynomial, i.e., if $A \in \text{GL}_n(F)$ has characteristic polynomial $x^n + \sum c_i x^i$, put $s(A) = (c_0, \dots, c_{n-1})$. Let \mathbf{s}^{ell} be the image of $\text{GL}_n(F)^{\text{ell}}$ under s . Similarly, define $s_\varepsilon : \text{GL}_n(E) \rightarrow F^n$ by $s_\varepsilon(A) = s(\varepsilon\mathcal{N}A)$ and $\mathbf{s}_\varepsilon^{\text{ell}} = s_\varepsilon(\text{GL}_n(E)^{\text{ell,tw}})$. Then s and s_ε are continuous and $\mathbf{s}_\varepsilon^{\text{ell}} \subset \mathbf{s}^{\text{ell}}$. We may view the orbital integrals $\varphi \mapsto TO_*(\varphi)$ and $\varphi' \mapsto O_*(\varphi')$ as maps $C_c^\infty(\text{GL}_n(E)^{\text{ell,tw}}) \rightarrow C_c^\infty(\mathbf{s}_\varepsilon^{\text{ell}})$ and $C_c^\infty(\text{GL}_n(F)^{\text{ell}}) \rightarrow C_c^\infty(\mathbf{s}^{\text{ell}})$. These maps are surjective, which gives the desired matching in (1).

Next suppose n is odd, and consider part (2) first for arbitrary $\varphi' \in C_c^\infty(\text{GL}_n(F))$. Write $\varphi' = \varphi'_1 + \varphi'_2$ where φ'_i has support in the set of elements whose determinant lies in $\varepsilon_i\mathcal{N}E^\times$. Then the orbital integrals of φ'_1 vanish off the norms, so by [AC89, Proposition 3.1], there exists φ_1 such that $O_{\delta\bar{\delta}}(\varphi'_1) = TO_\delta(\varphi_1)$. Let $\tilde{\varphi}'_2(g) = \varphi'_2(\varepsilon_2g)$, whose orbital

integrals also vanish off the norms. Then there exists φ_2 such that $O_{\varepsilon_2\delta\bar{\delta}}(\varphi'_2) = O_{\delta\bar{\delta}}(\tilde{\varphi}'_2) = TO_{\delta}(\varphi_2)$, which is our desired matching.

For the case of elliptic support in (2), we use the fact that the sets $\varepsilon_1\mathcal{N}\mathrm{GL}_n(E)^{\mathrm{ell},\mathrm{tw}}$ and $\varepsilon_2\mathcal{N}\mathrm{GL}_n(E)^{\mathrm{ell},\mathrm{tw}}$ are disjoint and open (when regarded as subsets) in $\mathrm{GL}_n(F)^{\mathrm{ell}}$ and their union is equal to $\mathrm{GL}_n(F)^{\mathrm{ell}}$ (cf. [Guo96, proof of Lemma 1.8]). Then argue as in the elliptic case of (1).

Part (3) is similar to (2) for general $\varphi' \in C_c^\infty(\mathrm{GL}_n(F))$. The elliptic case of (3) is similar to that of (1), observing that the vanishing condition implies the orbital integral map $\varphi' \mapsto O_*(\varphi')$ gives a function in $C_c^\infty(\mathfrak{s}^{\mathrm{ell}})$ with support in $\mathfrak{s}_\varepsilon^{\mathrm{ell}}$. \square

Now we deduce our local matching results, both for functions with support in “main” sets and in the elliptic sets.

Proposition 3.10. *Fix $\varepsilon \in F^\times$ and $f \in C_c^\infty(G_\varepsilon^{\mathrm{main}})$ (resp. $C_c^\infty(G_\varepsilon^{\mathrm{ell}})$). Then there exists $f' \in C_c^\infty(G'^{\mathrm{main}})$ (resp. $C_c^\infty(G'^{\mathrm{ell}})$) such that*

$$I'_{g'(\gamma)}(f') = \begin{cases} I_{g_\varepsilon(\delta)}(f) & \text{if } \gamma = \varepsilon\delta\bar{\delta} \text{ for } \delta \in \mathcal{U}_\varepsilon^{\mathrm{main}}, \\ 0 & \text{if } [\gamma] \notin \varepsilon\mathcal{N}\mathcal{U}_\varepsilon^{\mathrm{main}}. \end{cases}$$

In particular, f' matches with f .

Proof. The arguments for $G_\varepsilon^{\mathrm{main}}$ and $G_\varepsilon^{\mathrm{ell}}$ are identical, except for the use of different parts of Proposition 3.9. We just write the argument down for $G_\varepsilon^{\mathrm{main}}$.

Let $f \in C_c^\infty(G_\varepsilon^{\mathrm{main}})$. Then by Lemma 3.5, $F_f \in C_c^\infty(\mathcal{U}^{\mathrm{main}})$ and for all $X \in \mathcal{U}^{\mathrm{main}}$, $I_{g(X)}(f) = TO_X(F_f)$. By Proposition 3.9 there exists a $\varphi' \in C_c^\infty(\mathrm{GL}_n(F))$ such that for $X \in \mathrm{GL}_n(E)^{\mathrm{reg},\mathrm{tw}}$, $TO_X(F_f) = O_{\varepsilon X\bar{X}}(\varphi')$ and $O_Y(\varphi') = 0$ for $Y \notin \varepsilon\mathcal{N}\mathrm{GL}_n(E)$. By Corollary 3.13 of Chapter 1 in [AC89], these orbital integrals are equal up to a sign for any $X \in \mathrm{GL}_n(E)$. Since $F_f \in C_c^\infty(\mathcal{U}^{\mathrm{main}})$, there exists a c such that $F_f(X)$ and hence also $TO_X(F_f)$ vanish for X such that $|\det(I_n - \varepsilon X\bar{X})| > c$. Thus $O_{X'}(\varphi') = 0$ unless $|\det(I_n - X')| > c$. Let $\mathbf{1}_c$ be the characteristic function of

$$\{X' \in \mathcal{U}'^{\mathrm{main}} : |\det(I_n - X')| > c\}$$

and set $\tilde{\varphi}' = \varphi' \cdot \mathbf{1}_c$. Then $\tilde{\varphi}' \in C_c^\infty(\mathcal{U}'^{\mathrm{main}})$ and $O_{X'}(\tilde{\varphi}') = O_{X'}(\varphi')$ for all $X' \in \mathcal{U}'^{\mathrm{main}}$. By Lemma 3.3 there exists an $f' \in C_c^\infty(G'^{\mathrm{main}})$ such that $\tilde{\varphi}' = F_{f'}$ and $I'_{g'(X')}(f') = O_{X'}(\tilde{\varphi}')$. \square

We also want converse matching results.

Proposition 3.11. *Let n be odd and let $f' \in C_c^\infty(G'^{\mathrm{main}})$ (resp. $C_c^\infty(G'^{\mathrm{ell}})$) satisfying the vanishing condition (3.12b). Then there exist $f_\varepsilon \in C_c^\infty(G_\varepsilon^{\mathrm{main}})$ (resp. $C_c^\infty(G_\varepsilon^{\mathrm{ell}})$) for $\varepsilon = \varepsilon_1, \varepsilon_2$ such that $(f_\varepsilon)_\varepsilon$ and f' are matching.*

Note (3.12b) is vacuously satisfied when f' has elliptic support by (3.4).

Proof. Let $f' \in C_c^\infty(G'^{\mathrm{main}})$. By Lemma 3.2, we can consider $\varphi' = F_{f'} \in C_c^\infty(\mathrm{GL}_n(F))$. Then apply Proposition 3.9 to get the existence of $\varphi_{\varepsilon_1}, \varphi_{\varepsilon_2} \in C_c^\infty(\mathrm{GL}_n(E))$ that satisfy (3.13). We now apply Lemma 3.6 and complete the proof as before to find $f_{\varepsilon_1}, f_{\varepsilon_2}$. The elliptic case is similar. \square

Proposition 3.12. *Let n be even, $\varepsilon \in F^\times$ and $f' \in C_c^\infty(G'^{\text{main}})$ (resp. $C_c^\infty(G'^{\text{ell}})$) such that $I'_{g'(X)}(f') = 0$ for $X \notin \varepsilon \mathcal{N} \text{GL}_n(E)$. Then there exists $f \in C_c^\infty(G_\varepsilon^{\text{main}})$ (resp. $C_c^\infty(G_\varepsilon^{\text{ell}})$) such that f and f' are matching.*

Proof. This proof is similar to the proof of the previous proposition. \square

Corollary 3.13. *Let n be even, fix $i \in \{1, 2\}$, and let $f_{\varepsilon_i} \in C_c^\infty(G_{\varepsilon_i}^{\text{ell}})$. Then for $j \in \{1, 2\}$, there exists $f_{\varepsilon_j} \in C_c^\infty(G_{\varepsilon_j}^{\text{ell}})$ such that f_{ε_i} and f_{ε_j} are matching.*

Here f_{ε_1} matching f_{ε_2} means they both match a single $f' \in C_c^\infty(G')$. We do not know an analogue of this corollary for the main or regular sets, which is what forces us to make elliptic assumptions in Theorems 6.3 and 6.5.

Proof. This result follows from Propositions 3.10 and 3.12, and (3.6). \square

Remark 3.1. *The above matching results for the elliptic set can be carried out similarly at archimedean places.*

By work of Guo [Guo96] the fundamental lemma is known for the unit element in the Hecke algebra. As we will need this result for the global comparison, we state it here.

Proposition 3.14. [Guo96] *Let E/F be an unramified quadratic extension of local nonarchimedean fields with odd residual characteristic. Let Ξ' be the characteristic function of $G'(\mathcal{O})$ and Ξ_{ε_1} the characteristic function of $G_{\varepsilon_1}(\mathcal{O})$, where \mathcal{O} is the integer ring of F . Assume the measures on H and H' are such that $\text{vol}(H(\mathcal{O})) = \text{vol}(H'(\mathcal{O}))$. Then Ξ' and Ξ_{ε_1} are matching functions.*

4. LOCAL BESSEL DISTRIBUTIONS

In this section, F is a local field of characteristic zero (possibly archimedean), E is a quadratic étale extension of F (possibly $F \oplus F$), and $D = D_\varepsilon$ is a quaternion algebra of F which splits over E . We allow for the possibility that D is split, i.e., $G := G_\varepsilon = G'$. Further, if F is archimedean, we assume E/F is split.

Let π be an irreducible admissible unitary representation of G with trivial central character. Let λ be an H -invariant linear form on π . Since (G, H) is a Gelfand pair ([JR96] for E/F split nonarchimedean, [AG09] for E/F split archimedean, and [Guo97] for E/F inert nonarchimedean), λ is unique up to scaling. If π has a nonzero H -invariant linear form, we say π is H -distinguished.

We define the *local Bessel distribution* on G for π with respect to λ to be

$$B_\pi(f) = \sum_{\varphi} \lambda(\pi(f)\varphi) \overline{\lambda(\varphi)}$$

for $f \in C_c^\infty(G)$, where φ runs over an orthonormal basis for π . Note $B_\pi \equiv 0$ if and only if λ is zero. Local Bessel distributions are also sometimes referred to as *spherical characters* in the literature.

Now let π' be an irreducible admissible unitary representation of G' with trivial central character. If E/F is split, we may identify G with G' , H with H' , and define the local Bessel distribution on G' for π' as above. Assume E/F is inert. Let λ_1 be an H' -invariant linear form on π and let λ_2 be an $(H', \eta \circ \det)$ -equivariant linear form on π' . Since (G', H') is a Gelfand pair, λ_1 and λ_2 are unique up to scaling (note λ_2 is the same

as an H' -invariant linear form on $\pi' \otimes \eta$). We define the *local Bessel distribution* on G' for π' with respect to (λ_1, λ_2) to be

$$B_{\pi'}(f') = \sum_{\varphi} \lambda_1(\pi'(f')\varphi) \overline{\lambda_2(\varphi)},$$

where φ runs over an orthonormal basis for π' and $f' \in C_c^\infty(G')$. As before, $B_{\pi'} \equiv 0$ if and only if λ_1 or λ_2 is zero.

4.1. Generalities. We will now establish some results we will need about $B_\pi(f)$ and $B_{\pi'}(f')$.

Note that, by definition, the distribution $B_\pi(f)$ is bi- H -invariant. Similarly, if E/F is inert, the distribution $B_{\pi'}(f')$ is left H' -invariant and is right $(H', \eta \circ \det)$ -equivariant.

From now on we assume F is nonarchimedean and that $B_\pi \neq 0$. We say a distribution B on G is locally integrable if there is a locally integrable function b on G such that $B(f) = \int_G f(g)b(g) dg$ for all $f \in C_c^\infty(G)$.

Proposition 4.1. *Suppose the residual characteristic of F is odd. Then $B_\pi(f)$ is locally integrable. In particular, for any dense open subset $X \subset G$, there exists $f \in C_c^\infty(X)$ such that $B_\pi(f) \neq 0$.*

This is a minor extension of [Guo98].

Proof. This result was proved in [Guo98] under the additional hypotheses that G is split. The proof of [Guo98] in the case D is ramified goes through similarly. We outline this now. We remark Rader–Rallis [RR96] showed (in great generality) that $B_\pi(f)$ is locally integrable on G^{reg} .

We drop the ε subscript from the notation in the previous section, e.g., $S = S_\varepsilon$. Fix $s \in S$ semisimple and let $x \in \rho^{-1}(s)$. Let G_s be the connected component of the stabilizer of s in G , and $H_s = G_s \cap H$. Let U_x be the set of $g \in G_s$ such that the map $H \times G_s \times H \rightarrow G$ given by $(h_1, g, h_2) \mapsto h_1 x g h_2$ is submersive at $(1, g, 1)$. This is an open bi- H_s -invariant neighborhood of 1 in G_s . Further the image Ω_x of U_x is an open bi- H -invariant neighborhood of x in G . By standard Harish-Chandra theory, the restriction of B_π to $C_c^\infty(\Omega_x)$ may be viewed as an H_s -invariant distribution Θ_x on U_x .

For rather general symmetric spaces, Rader–Rallis [RR96] proved a germ expansion theorem for spherical characters when $x = 1$, which was extended to arbitrary x by Guo [Guo98, Theorem 2.1]. This germ expansion expresses Θ_x , in a neighborhood of 1 in U_x as a linear combination of Fourier transforms $\hat{\Lambda}$ of H_s -invariant distributions Λ on \mathfrak{s}_s supported in $\mathfrak{N}_{\mathfrak{s}_s}$. Here \mathfrak{s}_s is the Lie algebra of $S_s = G_s/H_s$, and $\mathfrak{N}_{\mathfrak{s}_s}$ is the subset of nilpotent elements. This reduces the problem to showing the $\hat{\Lambda}$'s are locally integrable.

Note the Lie algebra of G can be written as

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha & \varepsilon\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in M(n, E) \right\}.$$

Consider the subspaces

$$\mathfrak{h} = \left\{ \begin{pmatrix} \alpha & \\ & \bar{\alpha} \end{pmatrix} \right\}, \quad \text{and} \quad \mathfrak{s} = \left\{ \begin{pmatrix} & \varepsilon\beta \\ \bar{\beta} & \end{pmatrix} \right\}.$$

Note \mathfrak{h} is the Lie algebra of H , and \mathfrak{s} plays the role of the Lie algebra for S .

In the case that D is split, Guo [Guo98, Proposition 4] shows that the representation $(H_{\mathfrak{s}}, \mathfrak{s}_{\mathfrak{s}})$ is isomorphic to a product of representations of the form (G_0, \mathfrak{g}_0) and $(H(n_i), \mathfrak{s}(n_i))$. Here, G_0 is a certain reductive group, and $H(n_i)$ and $\mathfrak{s}(n_i)$ denote the corresponding H and \mathfrak{s} for $G(n_i) = \mathrm{GL}(n_i, D)$. It is not hard to show the same statement is true when D is nonsplit (cf. [Zha15, Proposition 4.7]).

Harish-Chandra showed each nilpotent orbit in \mathfrak{g}_0 has a G -invariant measure with locally integrable Fourier transform. To complete the proof, one needs to show the analogous statement for pairs (H, \mathfrak{s}) , i.e., each nilpotent orbit in \mathfrak{s} has an H -invariant measure with locally integrable Fourier transform. Guo achieves this by proving certain integral formulas for distributions Λ on \mathfrak{s} given by nilpotent orbital integrals and their Fourier transforms $\hat{\Lambda}$, and showing that $\hat{\Lambda}$ is locally integrable using a Weyl integration formula.

Since the representations (H, \mathfrak{s}) are isomorphic in the cases where D is split and where D is ramified (the action is given by twisted conjugation of $\mathrm{GL}(n, E)$ on $M(n, E)$), and the description of the nilpotent orbits of \mathfrak{s} is the same in both cases (cf. [Guo97]), Guo's proof extends to the case where D is ramified without difficulty. \square

Lemma 4.2. *Put $G^{\pm} = \{g \in G : \eta(\det g) = \pm 1\}$. Suppose $\pi \not\cong \pi \otimes \eta$ (hence E/F is not split). Then B_{π} is neither supported on G^+ nor G^- .*

Proof. Note B_{π} and $B_{\pi \otimes \eta}$ are linearly independent (cf. [FLO12, Lemma 2.2]). For $f \in C_c^{\infty}(G)$, put $f^{\eta}(g) = \eta(\det g)f(g)$. Note that $\pi(f^{\eta}) = (\pi \otimes \eta)(f)$. Thus $B_{\pi}(f^{\eta}) = \kappa B_{\pi \otimes \eta}(f)$ for some $\kappa \in \mathbf{C}$ where $\kappa = 1$ if the same H -invariant linear form λ is chosen for both B_{π} and $B_{\pi \otimes \eta}$. However if $f \in C_c^{\infty}(G^+)$, then $f^{\eta} = f$. Hence if B_{π} is supported on G^+ , we would have $B_{\pi}(f) = B_{\pi}(f^{\eta}) = \kappa B_{\pi \otimes \eta}(f)$ for all $f \in C_c^{\infty}(G)$, a contradiction. The case of G^- is similar. \square

Lemma 4.3. *Suppose $\pi \not\cong \pi \otimes \eta$ and F has odd residual characteristic. Then for any open dense $X \subset G$ and any $c \in \mathbf{C}^{\times}$, there exists $f \in C_c^{\infty}(X)$ such that $B_{\pi}(f) \neq c B_{\pi \otimes \eta}(f)$.*

Proof. Put $X^{\pm} = X \cap G^{\pm}$. By Proposition 4.1 and Lemma 4.2, we know there exist $f_{\pm} \in C_c^{\infty}(X^{\pm})$ such that $B_{\pi}(f_{\pm}) \neq 0$. Since $B_{\pi}(f_{\pm}) = \pm B_{\pi \otimes \eta}(f_{\pm})$, we can choose constants c_{\pm} such that $f = c_+ f_+ + c_- f_-$ satisfies the desired property. \square

4.2. Elliptic support of Bessel distributions. For use in our simple trace formula, we in fact want to know something stronger about our Bessel distributions B_{π} —that they often do not vanish on the H -elliptic set for H -distinguished π . Namely, we will say π (not necessarily a priori H -distinguished) is *H -elliptic* if there exists an H -invariant functional λ on π such that the associated Bessel distribution $B_{\pi}(f) \neq 0$ for some $f \in C_c^{\infty}(G^{\mathrm{ell}})$. In this section, we give a local proof of the following.

Proposition 4.4. *There exist H -elliptic (simple) supercuspidal representations when $E = \mathbf{Q}_2 \oplus \mathbf{Q}_2$.*

We show this by reducing the problem to showing the nonvanishing of an elliptic orbital integral for a supercuspidal form. In the next section, we will use this result together with global methods to show the existence of H -elliptic representations for a general E .

In Section 3, we defined local orbital integrals $I_g(f)$ for $f \in C_c^{\infty}(G)$, which converge for $g \in G^{\mathrm{ell}}$. Here it is more convenient to work with orbital integrals for functions

$\Phi \in C_c^\infty(G/Z)$, for which we consider the orbital integral

$$I^Z(g; \Phi) = \int_{H/Z} \int_{H/Z} \Phi(h_1 g h_2) dh_1 dh_2.$$

(Here we do not bother to quotient out by H_g , which has finite volume for elliptic g .) Note any such Φ is of the form $\Phi(g) = \int_Z f(gz) dz$ in which case we have

$$I^Z(g; \Phi) = cI_g(f)$$

for some $c \neq 0$. Hence $I^Z(g; \Phi)$ converges for $g \in G^{\text{ell}}$ and is nonzero if and only if $I_g(f)$ is nonzero. On G^{ell} , $I^Z(g; \Phi)$ is a smooth function.

Lemma 4.5. *Suppose π is a supercuspidal representation of G over a p -adic field F . Then π is H -elliptic if and only if the orbital integral $I^Z(g; \Phi) \neq 0$ for some $g \in G^{\text{ell}}$ and some matrix coefficient Φ of π .*

Proof. Let Φ be any matrix coefficient of π . Then [Mur08, Theorem 6.1] tells us

$$D_\Phi(f) = \int_{H/Z} \int_{H/Z} \int_G f(g) \Phi(h_1 g h_2) dg dh_1 dh_2$$

defines a bi- H -invariant distribution on G . For f supported on G^{ell} , we have absolute convergence of the orbital integrals and can write

$$(4.1) \quad D_\Phi(f) = \int_G f(g) I^Z(g; \Phi) dg.$$

Suppose π is H -elliptic. Then by [Zha16, Corollary 1.11] $B_\pi = D_\Phi$ for some matrix coefficient Φ of π (this also follows from Theorem 6.1 and Lemma 8.3 of [Mur08] when $G = \text{GL}(2n)$). Since $B_\pi \neq 0$, $D_\Phi \neq 0$. Then (4.1) implies $I^Z(g; \Phi) \neq 0$ for some $g \in G^{\text{ell}}$.

Now suppose $I^Z(g; \Phi) \neq 0$ for some $g \in G^{\text{ell}}$. Since $I^Z(g; \Phi)$ is locally constant on G^{ell} , we may choose f with small support around g to get $D_\Phi(f) \neq 0$. In particular, $D_\Phi \neq 0$, so we must have that π is H -distinguished (cf. [Mur08, Theorem 6.1]) and, by the uniqueness of H -invariant linear forms, $D_\Phi = cB_\pi$ for some nonzero c . Hence $B_\pi(f) \neq 0$. \square

Now we will show the existence of H -elliptic supercuspidal representations. Let us assume F is nonarchimedean, $E = F \oplus F$, so $G = \text{GL}(2n)$ and $H = \text{GL}(n) \times \text{GL}(n)$.

We first recall some facts about simple supercuspidal representations of G . See [KL15] for more details.

Let \mathcal{O} be the integers of F with maximal ideal $\mathfrak{p} = \varpi\mathcal{O}$, and residue field $\mathbb{F}_q = \mathcal{O}/\mathfrak{p}$ of order q . Let $K = K_{2n}$ be the subgroup of $G(\mathcal{O})$ consisting of matrices which are upper unipotent mod \mathfrak{p} , and let $J = ZK$. Fix a nontrivial character ψ of $\mathbb{F}_q = \mathcal{O}/\mathfrak{p}$ and $t_1, \dots, t_{2n} \in \mathbb{F}_q^\times$. Now define a character χ of J by $\chi|_Z = 1$ and

$$\chi(k) = \psi(t_1 x_1 + t_2 x_2 + \dots + t_{2n} x_{2n})$$

for $k \in K$ of the form

$$k \equiv \begin{pmatrix} 1 & x_1 & * & \cdots & * \\ 0 & 1 & x_2 & * & * \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & x_{2n-1} \\ \varpi x_{2n} & 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}},$$

i.e., if $k = (k_{ij})_{ij}$, then $x_i = k_{i+1,i}$ for $1 \leq i \leq 2n-1$ and $\varpi x_{2n} = k_{1,2n}$.

Let $\pi_\chi = c\text{-Ind}_J^G \chi$. This is a direct sum of $2n$ irreducible supercuspidal representations $\pi_{\chi,1}, \dots, \pi_{\chi,2n}$, which are the simple supercuspidals associated to χ . Next we want to define a matrix coefficient Φ of π_χ . This will be a sum of matrix coefficients for the simple supercuspidals $\pi_{\chi,i}$.

For our purposes, we call $A \in M_m(F)$ a permutation matrix if it has exactly one nonzero entry in each row and column. If e_1, \dots, e_m denotes the standard basis of F^m , then A permutes the lines Fe_i , and we think of A as representing the element of $\sigma_A \in S_m$, the symmetric group on $\{1, 2, \dots, m\}$, given by $\sigma_A(i) = j$ if $A \cdot Fe_i = Fe_j$.

Let $w = w_{2n} \in \text{GL}_{2n}(F)$ be the 0-1 permutation matrix associated to the product of 2-cycles $(2i, 2n-2i+1)$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Note we can inductively define w_{2n} by

$$w_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad w_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \text{and} \quad w_{2n} = \begin{pmatrix} 1 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ & & \boxed{w_{2n-4}} & & \\ 0 & 1 & & 0 & 0 \\ 0 & 0 & & 0 & 1 \end{pmatrix}$$

for $n > 2$. Consider the conjugate subgroups $K' = K'_{2n} = w_{2n}K_{2n}w_{2n}$ and $J' = ZK'$. For $g \in J'$, let $\chi'(g) = \chi(wgw)$. Let $\Phi(g) = 1_{J'}(g)\chi'(g)$, which is a matrix coefficient for π_χ . The reason to work with this Φ rather than $\Phi_J(g) = 1_J(G)\chi(g)$ is that $\chi|_{H \cap J}$ is nontrivial, which forces the integrals $I^Z(g; \Phi_J)$ to vanish (when convergent).

To see $\chi'|_{H \cap J'} = 1$, we can inductively write the subgroups K'_{2n} of $\text{GL}_{2n}(\mathcal{O})$ as

$$K'_2 = \left\{ k \equiv \begin{pmatrix} 1 & x_1 \\ \varpi x_2 & 1 \end{pmatrix} \pmod{\mathfrak{p}} \right\}, \quad K'_4 = \left\{ k \equiv \begin{pmatrix} 1 & * & x_1 & * \\ 0 & 1 & 0 & x_3 \\ 0 & x_2 & 1 & * \\ \varpi x_4 & 0 & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}} \right\},$$

and, for $n > 2$,

$$K'_{2n} = \left\{ k \equiv \begin{pmatrix} 1 & * & & & x_1 & * \\ 0 & 1 & & & 0 & x_{2n-1} \\ \hline 0 & * & & & 0 & * \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & * & & & 0 & * \\ 0 & x_{2n-2} & & & 0 & * \\ \hline 0 & * & & & 1 & * \\ \varpi x_{2n} & 0 & & & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}} : k_{2n-4} \in K_{2n-4} \right\},$$

where we write k_{2n-4} in the form

$$k_{2n-4} \equiv \begin{pmatrix} 1 & x_3 & * & \cdots & * \\ 0 & 1 & x_4 & \cdots & * \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & x_{2n-3} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}}.$$

As before x_i 's and $*$'s denote arbitrary elements of \mathcal{O} , and with k in the above form, we have

$$\chi'(k) = \psi(t_1x_1 + t_2x_2 + \cdots + t_{2n}x_{2n}).$$

It is now clear that all the x_i 's appear in the upper right (for i odd) or lower left (for i even) $n \times n$ block of k , so $\chi'|_{H \cap J'} = 1$.

Let $k_0 \in K'_{2n}$ be the element where all x_i 's and diagonal entries are 1, and all other entries are 0. We will now show that $k_0 \in G^{\text{ell}}$. Write $k_0 = \begin{pmatrix} I_n & X \\ Y & I_n \end{pmatrix}$ where all four blocks are $n \times n$. It is easy to see that k_0 is (θ) -elliptic if and only if $XY \in \text{GL}_n(F)^{\text{ell}}$. The inductive description of K'_{2n} implies that X and Y are permutation matrices, hence so is XY .

We claim that XY represents an n -cycle in S_n . Since exactly one entry is ϖ and the others are 1, this will imply XY is similar to the matrix

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \varpi & & & 0 \end{pmatrix},$$

whence has characteristic polynomial $\lambda^n - (-1)^n \varpi$, and must therefore be elliptic.

First we show the claim for n odd. In this case, $X = Y$ so $XY = X^2$, so it suffices to check that X is an n -cycle. Note that for $1 \leq j \leq n$,

$$Xj = \begin{cases} n & j = 1 \\ n - j & j \text{ even} \\ n - j + 2 & j \text{ odd} > 1. \end{cases}$$

Consequently, we see X represents the permutation

$$(1, n, 2, (n-2), 4, (n-4), \dots, (n-1))$$

which has order n .

Now suppose $n = 2m$. Then

$$Xj = \begin{cases} n - j & j \text{ odd} \\ n - j + 2 & j \text{ even} \end{cases} \quad \text{and} \quad Yj = \begin{cases} n & j = 1 \\ 1 & j = n \\ n - j & j < n \text{ even} \\ n - j + 2 & j > 1 \text{ odd.} \end{cases}$$

Consequently, XY represents the n -cycle

$$(1, 2, 4, 6, \dots, n, (n-1), (n-3), (n-5), \dots, 3).$$

Thus our claim is justified, and k_0 is indeed elliptic.

Lemma 4.6. *Let $h_1, h_2 \in H$ such that $h_1 k_0 h_2 \in K'$. Then $\chi'(h_1 k_0 h_2) = \psi(u_1 t_1 + \cdots + u_{2n} t_{2n})$ for some units $u_i \in \mathcal{O}^\times$.*

Proof. Write $h_1 = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$ and $h_2 = \begin{pmatrix} B_1^{-1} & \\ & B_2^{-1} \end{pmatrix}$. Then

$$h_1 k_0 h_2 = \begin{pmatrix} A_1 B_1^{-1} & A_1 X B_2^{-1} \\ A_2 Y B_1^{-1} & A_2 B_2^{-1} \end{pmatrix}.$$

By the structure of K' , we need $A_1 B_1^{-1}, A_2 B_2^{-1} \in \mathrm{GL}_n(\mathcal{O})$. For $A \in M_n(F)$, write $v(A)$ for $v(\det A)$, where v is the valuation. Then $v(A_1) = v(B_1)$ and $v(A_2) = v(B_2)$. On the other hand for $h_1 k_0 h_2 \in K'$, looking at the upper right and lower left blocks of $h_1 k_0 h_2$, we need $v(A_1 X B_2^{-1}) \geq 0$ and $v(A_2 Y B_1^{-1}) \geq 1$. Since $v(X) = 0$ and $v(Y) = 1$, this means $v(A_1) \geq v(B_2) = v(A_2)$ and $v(A_2) \geq v(B_1) = v(A_1)$, i.e., $v(A_1) = v(A_2) = v(B_1) = v(B_2)$. Thus the upper right block of $h_1 k_0 h_2$ must have valuation 0, and the lower left block must have valuation 1.

Write $h_1 k_0 h_2 = \begin{pmatrix} U & X' \\ Y' & V \end{pmatrix}$. Now we can use the inductive expression for K'_{2n} to see that $v(X') = 0$ and $v(Y') = 1$ implies that all of the x_i -coordinates for $h_1 k_0 h_2$ must be units in \mathcal{O} . This is evident for x_1, \dots, x_{2n-1} ; for x_{2n} , take the determinant of Y' by expanding in cofactors along the first column (or last row). \square

Corollary 4.7. *Suppose $q = 2$. Then $\chi'(g) = 1$ for any $g \in J' \cap H k_0 H$. Consequently, $I^Z(k_0; \Phi) \neq 0$.*

Proof. When $q = 2$, any element of \mathcal{O}^\times is 1 mod \mathfrak{p} , which means, with notation as in the lemma above, if $g = z h_1 k_0 h_2$ with $z \in F^\times$, then $\sum u_i t_i \equiv 0 \pmod{\mathfrak{p}}$. This proves the first statement, which means $I^Z(k_0; \Phi)$ is just the (nonzero finite) volume of $\{(h_1, h_2) \in H/Z \times H/Z : h_1 k_0 h_2 \in J'\}$. \square

Since Φ is a sum of matrix coefficients for $\pi_{\chi,1}, \dots, \pi_{\chi,2n}$, this means that when $q = 2$ (so we have no choice of t_i 's and there is a unique χ), one of the simple supercuspidals $\pi_{\chi,1}, \dots, \pi_{\chi,2n}$ is H -elliptic, completing the proof of Proposition 4.4.

5. A SIMPLE RELATIVE TRACE FORMULA

Return to the global situation, i.e., F is a number field and D a quaternion algebra over F which splits over a quadratic étale extension E/F . We allow for the possibility that $E = F \oplus F$ when $D = M_2(F)$, i.e. G and H may be G' and H' , in order to treat both trace formulas of interest simultaneously.

Let θ be the inner automorphism of G fixing H as defined locally. The notions of (θ) -regular and (θ) -elliptic elements of $G(F)$ are defined similarly as in the local case. If E is a field, put $\chi(h) = \eta(\det h)$. Otherwise put $\chi(h) = 1$.

We choose Haar measures $dz = \prod dz_v$ and $dh = \prod dh_v$ on $Z(\mathbf{A})$ and $H(\mathbf{A})$ such that at all finite places $Z(\mathcal{O}_v)$ and $H(\mathcal{O}_v)$ have volume 1.

For $f \in C_c^\infty(G(\mathbf{A}))$, we define the kernel

$$K(x, y) = \sum_{\gamma \in Z(F) \backslash G(F)} \int_{Z(\mathbf{A})} f(zx^{-1}\gamma y) dz,$$

and consider the (partial) distribution given by

$$(5.1) \quad I(f) = \int_{H(F)Z(\mathbf{A}) \backslash H(\mathbf{A})} \int_{H(F)Z(\mathbf{A}) \backslash H(\mathbf{A})} K(h_1, h_2) \chi(h_2) dh_1 dh_2,$$

when convergent. For $\gamma \in G(F)^{\text{ell}}$, recall from Section 3.1 that γ is relevant. Choose a Haar measure on $H_\gamma(\mathbf{A})$ and put

$$I_\gamma(f) = \text{vol}(Z(\mathbf{A})H_\gamma(F) \backslash H_\gamma(\mathbf{A})) \int_{H_\gamma(\mathbf{A}) \backslash (H(\mathbf{A}) \times H(\mathbf{A}))} f(h_1^{-1} \gamma h_2) \chi(h_2) dh_1 dh_2,$$

where $H_\gamma = \{(h_1, h_2) \in H \times H : h_1 \gamma h_2 = \gamma\} \cong T_X$ if $\gamma = g(X)$. We note that $I_\gamma(f)$ factors as a finite global constant and a product of local orbital integrals,

$$(5.2) \quad I_\gamma(f) = \text{vol}(Z(\mathbf{A})H_\gamma(F) \backslash H_\gamma(\mathbf{A})) \prod_v I_\gamma(f_v).$$

For π a cuspidal automorphic representation of $G(\mathbf{A})$ with $\omega_\pi = 1$, let

$$K_\pi(x, y) = \sum_\varphi \pi(f) \varphi(x) \overline{\varphi(y)}$$

where φ runs over an orthonormal basis for the space of π , and put

$$I_\pi(f) = \int_{H(F)Z(\mathbf{A}) \backslash H(\mathbf{A})} \int_{H(F)Z(\mathbf{A}) \backslash H(\mathbf{A})} K_\pi(h_1, h_2) \chi(h_2) dh_1 dh_2.$$

The terms $I_\gamma(f)$ and $I_\pi(f)$ are convergent, as will be explained in the proof below.

For $v < \infty$, put $\Xi_v = 1_{\text{GL}_{2n}(\mathcal{O}_v)}$.

A function $f_v \in C_c^\infty(G(F_v))$ is said to be a supercusp form if $\int_{N_v} f_v(gnh) = 0$, for all $g, h \in G(F_v)$ and all unipotent radicals N_v of proper parabolic subgroups of $G(F_v)$. If $\Phi_v \in C_c^\infty(G(F_v)/Z(F_v))$ is a matrix coefficient for a supercuspidal representation π_v , then there exists a supercusp form f_v such that $\Phi_v(g) = \int_{Z(F_v)} f_v(gz) dz$. In this case, we say f_v is essentially a matrix coefficient for π_v .

Proposition 5.1. *Let $f = \prod f_v \in C_c^\infty(G(\mathbf{A}))$ such that (i) $f_v = \Xi_v$ at almost all v ; (ii) at some finite place v_1 of F , f_{v_1} is a supercusp form; and (iii) at some place v_2 of F , $f_{v_2} \in C_c^\infty(G(F_{v_2})^{\text{ell}})$. Then*

$$(5.3) \quad \sum_{\gamma \in G(F)^{\text{ell}}} I_\gamma(f) = \sum_{\pi \text{ cusp}} I_\pi(f),$$

where π runs over cuspidal representations of $G(\mathbf{A})$ with trivial central character. Here both sides are absolutely convergent.

Proof. First observe for $\gamma \in G(F)^{\text{ell}}$, we formally have a factorization into local orbital integrals $I_\gamma(f) = c_\gamma \prod_v I_\gamma(f_v)$ as in (5.2). See equations (3.7) and (3.10) for the definition of $I_\gamma(f_v)$. (For v split, $I_\gamma(f_v)$ is defined by (3.7) with η trivial. The archimedean orbital integrals are defined in the same way as the nonarchimedean ones.) For a fixed γ , at almost all v , $I_\gamma(f_v)$ simply reduces to either the twisted orbital integral on $\text{GL}_n(E_v)$ or the usual orbital integral on $\text{GL}_n(F_v)$ for the unit element of the Hecke algebra. Since each local orbital integral converges, the global integrals $I_\gamma(f)$ are absolutely convergent by the convergence of the elliptic terms appearing in the trace formula in [AC89].

Hence, at least formally, by (iii), $I(f)$ equals the left hand side of (5.3). We can justify this by showing that at most finitely many $I_\gamma(f)$ are nonzero. It suffices to show that at most finitely many elliptic H -conjugacy classes of

$$S(\mathbf{A}) = \{g\theta(g)^{-1} : g \in G(\mathbf{A})\}$$

lie in a given compact subset Ω of $S(\mathbf{A})$. This follows from the fact that only finitely many elliptic conjugacy classes of $\mathrm{GL}_n(E)$ intersect a given compact subgroup of $\mathrm{GL}_n(\mathbf{A}_E)$.

Lastly, it is well known that condition (ii) implies $K(x, y) = \sum K_\pi(x, y)$ where π runs over cuspidal representations, that and each $K_\pi(x, y)$ is rapidly decreasing. This makes $I(f) = \sum_\pi I_\pi(f)$, where the sum is absolutely convergent. \square

We use this result to get the existence of many $H(F_v)$ -elliptic representations.

Proposition 5.2. *Suppose k is a local field of characteristic 0, let K/k be a quadratic étale extension, and let $D(k)$ be the split or non-split quaternion algebra over k , which we take to be split if K/k is. There exist irreducible admissible unitary representations τ of $\mathrm{GL}_n(D(k))$ which are $\mathrm{GL}_n(K)$ -elliptic.*

Proof. We may globalize k , K and $D(k)$ to F , E and D such that (a) these local algebras are the localizations of the corresponding global algebras at some place v_1 , (b) there is another place v_2 of F over which D splits and such that there exists an $H(F_{v_2})$ -elliptic supercuspidal representation τ_2 of $G(F_{v_2})$ (Proposition 4.4), and (c) there is some infinite place $v_3 \neq v_1$ such that D_{v_3} is split.

Choose a test function $f = \prod f_v$ as follows. Let f_{v_1} be the characteristic function of an open compact subset of $G(F_{v_1})^{\mathrm{ell}}$. By Lemma 4.5 we can take f_{v_2} to be essentially a matrix coefficient of τ_2 such that $I_\gamma(f_{v_2})$ is nonzero for any $\gamma \in \Omega_{v_2}$, where Ω_{v_2} is some open subset of $G(F_{v_2})^{\mathrm{ell}}$. At all other finite v , choose f_v to be a characteristic function of some compact subset of $G(F_v)$ such that $f_v = \Xi_v$ outside of some finite set of places S . The archimedean choices will be made below.

Let $C \subset G(\mathbf{A}^\infty)$ be the support of $f^\infty = \prod_{v < \infty} f_v$. Note $Z(\mathbf{A}^\infty)C \cap \mathrm{SL}_n(D(\mathbf{A}^\infty))$ is open in $\mathrm{SL}_n(D(\mathbf{A}^\infty))$. Strong approximation for $\mathrm{SL}_n(D)$ for indefinite D tells us $\mathrm{SL}_n(D(F))$ is dense in $\mathrm{SL}_n(D(\mathbf{A}^\infty))$, so there exists $\gamma \in G(F) \cap Z(\mathbf{A}^\infty)C \subset G(F)^{\mathrm{ell}}$.

Thus, for any such γ , we must have $I_\gamma(f_v) \neq 0$ at any v where f_v is a characteristic function. The only other finite place to consider is v_2 , but we can guarantee $I_\gamma(f_{v_2}) \neq 0$ by taking $\gamma \in \Omega_{v_2}$. Fix one such γ . For $v | \infty$, choose f_v such that $I_\gamma(f_v) \neq 0$ and $I_{\gamma_1}(f) = 0$ for any $\gamma_1 \in G(F) - H(F)\gamma H(F)$. This is possible by taking the support of archimedean f_{v_3} small enough, since only a finite number of global geometric terms $I_{\gamma_1}(f)$ can be nonzero as explained in the proof of the previous proposition.

At almost all places, we have $I_\gamma(f_v) \geq \mathrm{vol}((H_\gamma \backslash H \times H)(\mathcal{O}_v))$, and the product over v is nonzero. Hence, for f and γ as above, $I(f) = I_\gamma(f) \neq 0$. By (5.3), $I_\pi(f) \neq 0$ for some cuspidal π . By the uniqueness of local $H(F_v)$ -invariant functionals, $I_\pi(f)$ factors into a product of local Bessel distributions $B_{\pi_v}(f_v)$, and thus $B_{\pi_{v_1}}(f_{v_1}) \neq 0$. \square

6. MAIN RESULTS

Let G , H , G' and H' be as in the introduction and choose measures as in the previous section. Assume E is a field which is split at each archimedean place. On G , we keep the same notation for the (partial) distributions I_* defined in the previous section; on G' , we

denote them with primes, i.e., by I'_* . Write Σ_s for the set of places of F split in E and Σ_s^c for the set of places of F inert or ramified in E .

Recall all representations are assumed to be unitary with trivial central character.

6.1. Global results.

Theorem 6.1. *Fix $D \in X(E:F)$ and say $G = G_D$. Suppose π is an H -distinguished cuspidal automorphic representation of $G(\mathbf{A})$, which is supercuspidal at some finite place v_1 where E/F is split and $H(F_{v_2})$ -elliptic at another place v_2 . Let $\pi' = \mathbf{JL}(\pi)$ be the Jacquet–Langlands transfer to $G'(\mathbf{A})$. Then π' is H' -distinguished and (H', η) -distinguished.*

Proof. Since π is H -distinguished, $I_\pi \neq 0$. We will choose a nice test function $f = \prod f_v \in C_c^\infty(G(\mathbf{A}))$ such that $I_\pi(f) \neq 0$. By the uniqueness of H -invariant linear forms on π , we can factor $I_\pi(f) = \prod B_{\pi_v}(f_v)$ where the B_{π_v} 's are local Bessel distributions attached to certain linear functionals $\lambda_v = \lambda_{1,v} = \lambda_{2,v}$ as in Section 4.

At v_1 , we may take f_{v_1} to be essentially a matrix coefficient (in the sense of Section 5) of π_{v_1} such that $B_{\pi_{v_1}}(f_{v_1}) \neq 0$. At v_2 , we may take $f_{v_2} \in C_c^\infty(G(F_{v_2})^{\text{ell}})$ such that $B_{\pi_{v_2}}(f_{v_2}) \neq 0$.

There exists a finite set of places S , including all archimedean places, such that for $v \notin S$, $B_{\pi_v}(\Xi_v) \neq 0$. Enlarge S if necessary so that it contains all even places and all places where E or D ramifies. For $v \notin S$, take $f_v = \Xi_v$. Away from S , choose λ_v so that $B_{\pi_v}(\Xi_v) = 1$ to ensure convergence of the factorization of $I_\pi(f)$. Now consider $v \in S - \{v_1, v_2\}$. If $v \in \Sigma_s$, take any f_v such that $B_{\pi_v}(f_v) \neq 0$. If $v \notin \Sigma_s$, we may choose $f_v \in C_c^\infty(G(F_v)^{\text{main}})$ such that $B_{\pi_v}(f_v) \neq 0$ by Proposition 4.1. This defines f such that $I_\pi(f) \neq 0$.

Now we will get a matching f' . For regular γ and γ' whose double cosets correspond at v , choose measures on $H_\gamma(F_v)$ and $H_{\gamma'}(F_v)$ which are compatible. Whenever E/F is split, we can identify $G(F_v)$ and $G'(F_v)$ so that $H(F_v) = H'(F_v)$. At such places, take $f'_v = f_v$. When $v \notin S$ is inert, the function $f'_v = \Xi_v$ matches $f_v = \Xi_v$ by Guo's fundamental lemma (Proposition 3.14, again identifying $G(F_v)$ with $G'(F_v)$). When $v \in S$ is inert (and thus nonarchimedean by assumption), we know there exists a matching f'_v for f_v by Proposition 3.10 when v is odd; for any nonarchimedean v this follows from [Zha15]. We may also assume $f'_{v_2} \in C_c^\infty(G'(F_{v_2})^{\text{ell}})$. Let $f' = \prod f'_v \in C_c^\infty(G'(\mathbf{A}))$. By the equality of the global volumes of stabilizers for matching elliptic elements, we have that f' matches f globally, in the sense that $I_\gamma(f) = I'_{\gamma'}(f')$ for each regular matching γ and γ' .

Therefore, Proposition 5.1 implies

$$(6.1) \quad \sum_{\sigma \text{ cusp}} I_\sigma(f) = \sum_{\sigma' \text{ cusp}} I'_{\sigma'}(f').$$

Let S^c denote the complement of S . For $v \in \Sigma_s \cap S^c$, we may vary f_v in the Hecke algebra and retain (6.1). Therefore, the principle of linear independence of characters (see [LR00, Lemma 4]) implies

$$(6.2) \quad \sum_{\sigma \in \Pi} I_\sigma(f) = \sum_{\sigma' \in \Pi'} I'_{\sigma'}(f')$$

where Π (resp. Π') denotes the set of cuspidal representations of $G(\mathbf{A})$ (resp. $G'(\mathbf{A})$) which, at each $v \in \Sigma_s \cap S^c$, are isomorphic to π_v . A result of Ramakrishnan for $\text{GL}(n)$ [Ram15] tells us that if $\tau_1, \tau_2 \in \Pi'$, then $\tau_2 \cong \tau_1$ or $\tau_2 \cong \tau_1 \otimes \eta$. Hence by strong

multiplicity one for G' , we have $\Pi' = \{\pi', \pi' \otimes \eta\}$. Using strong multiplicity one for G and the Jacquet–Langlands transfer to G' , we can also apply [Ram15] to get that $\Pi = \{\pi, \pi \otimes \eta\}$. Thus (6.2) becomes

$$(6.3) \quad I_\pi(f) + I_{\pi \otimes \eta}(f) = I'_{\pi'}(f') + I'_{\pi' \otimes \eta}(f').$$

Since $I'_{\pi'} \neq 0$ if and only if $I'_{\pi' \otimes \eta} \neq 0$ if and only if π' is H' - and (H', η) -distinguished, we want to show $I_\pi(f) + I_{\pi \otimes \eta}(f) \neq 0$. If $\pi \cong \pi \otimes \eta$, then $I_{\pi \otimes \eta}(f) = I_\pi(f) \neq 0$. If not, it is a priori possible that $I_{\pi \otimes \eta}(f) = -I_\pi(f)$. However, by Lemma 4.3, we may choose $f_{v_3} \in C_c^\infty(G(F_{v_3})^{\text{main}})$ at some odd non-split place v_3 to ensure that $I_{\pi \otimes \eta}(f) \neq -I_\pi(f)$. \square

Remark 6.1. *While in light of [Zha15] we do not need to appeal to Proposition 3.10 for matching at odd places, we use it in the argument because (i) [Zha15] was not available at the time of the first version of our paper, and (ii) we hope this approach may provide a simpler way to get global results in situations where smooth matching is not known. Without using [Zha15], the above argument still goes through with the additional hypotheses that at each even place v either v is split or π_v is $H(F_v)$ -elliptic.*

Proposition 6.2. *Let π' be a cuspidal automorphic representation of $G'(\mathbf{A})$ with trivial central character such that π' is both H' - and (H', η) -distinguished, π' is supercuspidal at a split place v_1 , and π' is $H(F_{v_2})$ -elliptic at another split place v_2 . Assume that, for each v inert in E , at least one of the following holds:*

- (a) v is an odd place at which E/F is unramified and $B_{\pi'_v}(\Xi_v) \neq 0$;
- (b) $B_{\pi'_v}$ is not identically zero on $C_c^\infty(X_v)$ where

$$X_v = \left\{ g \in G'(F_v)^{\text{main}} : [g] \in \bigcup_{\varepsilon \in F_v^\times / NE_v^\times} \iota_\varepsilon(\Gamma^{\text{ss}}(G_\varepsilon(F_v))) \right\}.$$

Then, there exists $D \in X(E:F:\pi')$ such that the Jacquet–Langlands transfer π_D to $G_D(\mathbf{A})$ is H -distinguished.

Note (a) holds for almost all v and (b) is automatically satisfied when n is odd and π'_v is H'_v -elliptic. In particular, this establishes Conjecture 1.3(2) under some local hypotheses on π' (admittedly, stronger hypotheses than one would like).

Proof. The proof is similar to the previous case. We just explain where details differ.

Factor $I_{\pi'}(f')$ into local Bessel distributions $B_{\pi'_v}(f'_v)$ as defined in Section 4. We choose f'_v such that $B_{\pi'_v}(f'_v) \neq 0$ at each place v with the conditions that (i) $f'_v = \Xi_v$ outside of some finite set of places S containing v_1 and v_2 (assume S is small enough so every $v \in S \cap \Sigma_S^c$ satisfies (b)); (ii) f'_{v_1} is a supercusp form; (iii) $f'_{v_2} \in C_c^\infty(G'(F_{v_2})^{\text{ell}})$; and (iv) at any $v \in S \cap \Sigma_S^c$ satisfying (b), $f'_v \in C_c^\infty(X_v)$.

By Propositions 3.14 and 3.11 or 3.12 (see also the partial converse in [Zha15]), for each v there is a pair of matching functions $(f_{v,\varepsilon_1}, f_{v,\varepsilon_2})$ that satisfy the following conditions: (i) $(f_{v_1,\varepsilon_1}, f_{v_1,\varepsilon_2}) = (f'_{v_1}, 0)$; and (ii) f_{v_2,ε_1} has elliptic support.

For $D_\varepsilon \in X(E:F)$ we let $f_{D_\varepsilon} = \prod_v f_{v,\varepsilon}$. Denote the distributions I and I_{π_D} defined in the previous section on G_D by I_D and I_{D,π_D} respectively. Thus we have

$$(6.4) \quad \sum_D I_D(f_D) = I'(f'),$$

where in fact the sum on the left is finite. Again, by [Ram15] and strong multiplicity one, one gets

$$(6.5) \quad \sum_{D \in X(E:F:\pi')} I_{D,\pi_D}(f_D) + I_{D,\pi_D \otimes \eta}(f_D) = I'_{\pi'}(f') + I'_{\pi' \otimes \eta}(f').$$

By Lemma 4.3, the right hand side may be chosen nonzero, so that for at least one such D , we have $I_{D,\pi_D} \neq 0$ or $I_{D,\pi_D \otimes \eta} \neq 0$. But these conditions are both equivalent to π_D being H -distinguished. \square

The following result tells us that when an analogue of Conjecture 1.3(2) for n even holds, the D should not be unique.

Theorem 6.3. *Suppose n is even and $D_1, D_2 \in X(E:F)$. Let π_{D_1} and π_{D_2} be cuspidal automorphic representations of $G_{D_1}(\mathbf{A})$ and $G_{D_2}(\mathbf{A})$ such that $\mathrm{JL}(\pi_{D_1}) \cong \mathrm{JL}(\pi_{D_2})$. Suppose π_{D_1} is (i) supercuspidal at a place v_1 such that $D_1(F_{v_1}) \cong D_2(F_{v_1})$ and (ii) $H(F_v)$ -elliptic for v such that $D_1(F_v) \not\cong D_2(F_v)$. Then if π_{D_1} is H -distinguished, so is π_{D_2} .*

Proof. Assume $D_1 \not\cong D_2$. Again the proof is similar to the previous cases and we just explain where details differ. Here we directly compare the trace formulas on G_{D_1} and G_{D_2} . We construct matching $f_{1,v} \in C_c^\infty(G_{D_1}(F_v))$ and $f_{2,v} \in C_c^\infty(G_{D_2}(F_v))$ such that $I_{g_{\varepsilon_1}(X)}(f_{1,v}) = I_{g_{\varepsilon_2}(X)}(f_{2,v})$ for all $X \in \Gamma^{\mathrm{ell}, \mathrm{tw}}(\mathrm{GL}_n(E))$, and the global orbital integrals vanish on non-elliptic terms. Let $S = \{v : D_1(F_v) \not\cong D_2(F_v)\}$. We choose $f_1 \in C_c^\infty(G_{D_1}(\mathbf{A}))$ such that (i) for all v , $B_{\pi_{D_1},v}(f_{1,v}) \neq 0$; (ii) for almost all $v \notin S$, $f_{1,v} = \Xi_v$; (iii) f_{1,v_1} is a supercusp form; and (iv) $f_{1,v} \in C_c^\infty(G_{D_1}(F_v)^{\mathrm{ell}})$ for $v \in S$. There is a matching f_2 by taking $f_{2,v} = f_{1,v}$ for all $v \in S^c$ and using Corollary 3.13 for the remaining v . \square

Remark 6.2. *Note Theorem 6.3 remains valid if the only archimedean assumption one makes is $D_1(F_v) \cong D_2(F_v)$ for each $v|\infty$, i.e., we need not assume E/F is split at each infinite place.*

6.2. Local results. Here we deduce some local consequences of our global results.

Let K/k be a quadratic extension of nonarchimedean local fields of characteristic 0, and $\eta_{K/k}$ the associated quadratic character of k^\times . Then we may choose our quadratic extension of number fields E/F such that, for a fixed place v_0 of F , one has $F_{v_0} \cong k$, $E_{v_0} \cong K$, and E/F is split at each archimedean place and each even place except possibly v_0 . We will also fix an odd split place v_1 and assume F has a split even place $v_2 \neq v_0$ such that $F_{v_2} \cong \mathbf{Q}_2$. Identify $k = F_{v_0}$ and $K = E_{v_0}$.

Take $D \in X(E:F)$, and G, G', H, H' as before. Let τ and τ' be irreducible admissible representations of $G(k)$ and $G'(k)$. Recall τ is $H(k)$ -distinguished if $\mathrm{Hom}_{H(k)}(\tau, \mathbf{C}) \neq 0$. Similarly, τ' is $H'(k)$ - (resp. $(H'(k), \eta_{K/k})$ -) distinguished if $\mathrm{Hom}_{H'(k)}(\tau', \mathbf{C})$ (resp. $\mathrm{Hom}_{H'(k)}(\tau', \eta_{K/k})$) is nonzero.

Theorem 6.4. *Let τ be a supercuspidal representation of $G(k)$, and τ' be its Jacquet–Langlands transfer to $G'(k)$. If τ is $H(k)$ distinguished, then τ' is both $H'(k)$ - and $(H'(k), \eta_{K/k})$ -distinguished.*

Proof. Let $\pi_{v_0} = \tau$. By [HM02b], there exists an $H(F_{v_1})$ -distinguished (tame) supercuspidal representation π_{v_1} of $G(F_{v_1})$. (Murnaghan pointed out to us that one can also

deduce this fact from [Mur11].) By Proposition 4.4, there also exists an $H(F_{v_2})$ -elliptic (simple) supercuspidal representation π_{v_2} of $G(F_{v_2})$. An argument of Hakim and Murnaghan [HM02a] (see [PSP08, Theorem 4.1] for a more general form) shows that π_{v_0} , π_{v_1} and π_{v_2} can be simultaneously globalized to an H -distinguished representation π of $G(\mathbf{A})$. Then, by Theorem 6.1, $\pi' = \text{JL}(\pi)$ is H' - and (H', η) -distinguished. In particular $\pi'_{v_0} \cong \tau'$ is locally $H'(F_{v_0})$ - and $(H'(F_{v_0}), \eta_{v_0})$ -distinguished. \square

Theorem 6.5. *Let $D_1(k)$ and $D_2(k)$ be the two quaternion algebras over k , in some order. Suppose n is even, and τ_{D_1} and τ_{D_2} are irreducible admissible representations of $G_{D_1}(k)$ and $G_{D_2}(k)$ which correspond via Jacquet-Langlands. Assume τ_{D_1} is $H(k)$ -elliptic and supercuspidal. Then if τ_{D_1} is $H(k)$ -distinguished, so is τ_{D_2} .*

Proof. The proof is similar to the previous proof, with the following modifications: We globalize k to a number field F as above such that $F_w \cong F_{v_0} \cong k$ for some place $w \neq v_0$. Globalize D_1 and D_2 so that one is split everywhere and one is ramified only at v_0 and w . Globalize τ_{D_1} to π_{D_1} such that $\pi_{D_1, v_0} \cong \pi_{D_1, w} \cong \tau_{D_1}$. Then argue as above, applying Theorem 6.3 at the end. \square

Now we prove our final local result.

Proof of Theorem 1.7. The globalization result used above ([PSP08, Theorem 4.1]) in fact tells us that there exists a cuspidal, globally $\text{GL}_n(\mathbf{A}_E)$ -distinguished representation π of $\text{GL}_{2n}(\mathbf{A})$ such that, $\pi_{v_0} \cong \tau$, π_{v_1} is an $H(F_{v_1})$ -distinguished supercuspidal, π_{v_2} is an $H(F_{v_2})$ -elliptic supercuspidal, and π_v is unramified for all finite $v \notin \{v_0, v_1, v_2\}$. By Theorem 6.1, we know π must also be H' - and (H', η) -distinguished. Hence π is symplectic and $L(1/2, \pi_E) \neq 0$. In particular, π has a nonzero global Shalika period by [JS90]. Therefore τ has a nonzero local Shalika period, which means τ is symplectic by [JNQ08].

Now $L(1/2, \pi_E) \neq 0$ implies the global root number $\epsilon(1/2, \pi_E) = +1$. The global root number factors into a product of local root numbers $\epsilon(1/2, \pi_{E,v})$ (independent of choice of local additive character by self-duality). Thus to show $\epsilon(1/2, \pi_{E,v_0}) = \epsilon(1/2, \tau_K) = 1$, it suffices to show $\epsilon(1/2, \pi_{E,v}) = +1$ for all $v \neq v_0$.

Because each π_v and $\pi_{E,v}$ is self-dual, we have that each $\epsilon(1/2, \pi_v)$ and $\epsilon(1/2, \pi_{E,v})$ is ± 1 . Since we are working in even dimension, $\epsilon(1/2, \pi_{E,v}) = \epsilon(1/2, \pi_v)\epsilon(1/2, \pi_v \otimes \eta_v)$.

If E/F is split at v , then $\epsilon(1/2, \pi_{E,v}) = \epsilon(1/2, \pi_v)^2 = +1$.

If E/F is inert at $v \neq v_0$, then π_v and $\pi_v \otimes \eta_v$ are both unramified and thus have local root number $+1$, whence $\epsilon(1/2, \pi_{E,v}) = +1$.

Last, suppose E/F is ramified at v and $v \neq v_0$. Then $\pi_v \cong \pi(\chi_1, \dots, \chi_{2n})$ is an unramified principal series, so $\epsilon(1/2, \pi_v) = 1$. It follows that

$$\epsilon(1/2, \pi_v \otimes \eta_v) = \prod \epsilon(1/2, \chi_i \eta_v) = \epsilon(1/2, \eta_v)^{2n} \prod \chi_i(\varpi_v),$$

where ϖ_v is a uniformizer. But this is $+1$ since $\epsilon(1/2, \eta_v) = \pm 1$ and $\prod \chi_i = 1$ as π has trivial central character. \square

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BROOKE FEIGON: DEPARTMENT OF MATHEMATICS, THE CITY COLLEGE OF NEW YORK, CUNY,
NAC 8/133, NEW YORK, NY 10031

E-mail address: bfeigon@ccny.cuny.edu

KIMBALL MARTIN: UNIVERSITY OF OKLAHOMA, DEPARTMENT OF MATHEMATICS, NORMAN, OK
73019

E-mail address: kmartin@math.ou.edu

DAVID WHITEHOUSE