Summary of representations of $G = \mathrm{GL}(2, q)$

We first summarize the results we proved in detail on non-cuspidal representations of $G = \mathrm{GL}(2, q)$ (q a power of a prime p), mostly following Piatetski-Shapiro's *Complex representations of* $\mathrm{GL}(2, K)$ *for finite fields* K. Proposition numbers, etc. here correspond to those is Piatetski-Shapiro, though my formulations are not always exactly the same as Piatetski-Shapiro's. Then we give a brief description of the cuspidal representations.

Notation: $B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ is the standard Borel subgroup of G, $U = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ is its unipotent radical, $D = \begin{pmatrix} * & \\ & * \end{pmatrix}$ is the diagonal subgroup, $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Then U is normalized by D, so $B = U \rtimes D$, and we have the Bruhat decomposition $G = B \sqcup BwB$. The two B-double cosets B and BwB are called Bruhat cells.

It is easy to see that $|B| = (q-1)^2 q$, $[G : B] = q + 1$ and $|G| = (q-1)^2 q(q+1)$.

Proposition 5.1. The conjugacy classes of G consists of the following 4 families:

1. the $q - 1$ classes represented by $c_1(\alpha) = \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}$, $\alpha \in \mathbb{F}_q^\times$;
2. the $q - 1$ classes represented by $c_2(\alpha) = \begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix}$, $\alpha \in \mathbb{F}_q^\times$;
3. the $\frac{1}{2}(q-1)(q-2)$ classes represented by $c_3(\alpha, \beta) = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$, $\alpha, \beta \in \mathbb{F}_q^\times$, $\alpha \neq \beta$ (here $c_3(\alpha, \beta) \sim c_3(\beta, \alpha)$); and
4. the $\frac{1}{2}(q^2 - q)$ classes represented by $c_4(\alpha) = \begin{pmatrix} & -N(\alpha) \\ 1 & \mathrm{tr}\alpha \end{pmatrix}$, where $\alpha \in \mathbb{F}_{q^2} - \mathbb{F}_q$.

Theorem 7.1. The irreducible representations of B are classified as

1. $(q-1)^2$ 1-dimensional representations of the form $\mu(\begin{pmatrix} a & b \\ & d \end{pmatrix}) = \mu_1(a)\mu_2(d)$, where μ_1, μ_2 are 1-dimensional characters of $\mathbb{F}_q^\times$. We will also write $\mu = (\mu_1, \mu_2)$ for this character.
2. $(q-1)$ representation of dimension $q - 1$, which are monomial, induced from 1-dimensions of the subgroup ZU following the construction in Serre, Section 8.2.

Note if ρ is a $(q-1)$-dimensional representation of B, then its induction to G has dimension $(q-1)(q+1) > \sqrt{|G|}$, so the induction cannot be irreducible. However, we can construct many irreducibles of G by inducing 1-dimensionals of B.

A **principal series** representation of G is a representation of the form $\hat{\mu} = \pi(\mu_1, \mu_2) := \mathrm{Ind}_B^G \mu$ for a 1-dimensional representation $\mu = (\mu_1, \mu_2)$ of B. Necessarily $\dim \hat{\mu} = q + 1$.

For a representation ρ of G, we defined the **Jacquet module** $J(\rho) = \rho^U = \{v \in \rho : \rho(u)v = u \, \forall u \in U\}$. Using the fact that $\dim J(\hat{\mu}) = 2$ and $\dim J(\rho) > 0$ if and only if $\rho|_B$ contains some 1-dimensional μ we proved the following. An irreducible representation ρ of G is a component of some principal series $\hat{\mu}$ if and only if $J(\rho) \neq 0$ (Corollary 8.4). Further if $\hat{\mu}$ is reducible, then it has precisely 2 irreducible components, a 1-dimensional and a q-dimensional (Corollary 8.5, Lemma 8.9(a)).

Theorem 8.12. A principal series $\hat{\mu} = \pi(\mu_1, \mu_2)$ is irreducible if and only if $\mu_1 \neq \mu_2$. Moreover, two principal series $\pi(\mu_1, \mu_2), \pi(\mu_1', \mu_2')$ are isomorphic if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$.

We call a representation ρ of G **cuspidal** if $J(\rho) = 0$. Note $\pi(1,1) \simeq 1 \oplus St$ where St is an irreducible q-dimensional representation. We call St the **Steinberg** representation of G.

Exercise 1. For χ a character of $\mathbb{F}_q^\times$, show $\pi(\mu_1, \mu_2) \otimes (\chi \circ \det) = \pi(\mu_1\chi, \mu_2\chi)$.

Theorem 8.13. The irreducible representations of G are as follows:

1. the $(q-1)$ 1-dimensional representations $\chi \circ \det$, $\chi : \mathbb{F}_q^\times \to \mathbb{C}$;
2. the $(q-1)$ q-dimensional representations $St \otimes (\chi \circ \det)$, $\chi : \mathbb{F}_q^\times \to \mathbb{C}$;
3. the $\frac{1}{2}(q-1)(q-2)$ $(q+1)$-dimensional irreducible principal series $\pi(\mu_1, \mu_2)$ where μ_1, μ_2 is an unordered pair of distinct charaters of $\mathbb{F}_q^\times$;
4. the $\frac{1}{2}(q^2 - q)$ irreducible cuspidal representations of G.

Recall that the 1-dimensional representations of a finite group G are precisely the representations that factor through the derived (i.e., commutator) subgroup $G' = [G, G]$. For $q > 2$, all 1-dimensionals factor through the normal subgroup $\mathrm{SL}(2, q)$ (which is the kernel of det), $\mathrm{GL}(2, q)/\mathrm{SL}(2, q) \simeq \mathbb{F}_q^\times$ has order $q - 1$, this means $\mathrm{SL}(2, q)$ is the commutator subgroup of $\mathrm{GL}(2, q)$ (Corollary 8.14). (When $q = 2$, $\mathrm{GL}(2, 2) = \mathrm{SL}(2, 2) \simeq S_3$, and the commutator subgroup is C_3. Here there is 1 irreducible cuspidal representation, which is also 1-dimensional.)

Proposition 10.2. Every irreducible cuspidal representation ρ of G has dimension $q - 1$.

Proof. Since $J(\rho) = 0$, the restriction of ρ to B cannot contain any 1-dimensionals. Theorem 7.1 implies that $\dim \rho$ is a multiple of $(q-1)$. But counting dimensions implies each $\dim \rho = q - 1$. $\square$

For brevity, we deviate from Piatetski-Shapiro's treatment and describe the cuspidal representations of G following Chapter 6 of Bushnell–Henniart's *The local Langlands conjecture for* GL(2).

We may view $\mathbb{F}_{q^2}$ as a 2-dimensional $\mathbb{F}_q$-vector space. Left multiplication by $\mathbb{F}_{q^2}^\times$ thus gives a 2-dimensional $\mathbb{F}_q$-representation, i.e., an embedding of $\mathbb{F}_{q^2}$ as a subgroup T of $\mathrm{GL}(2,q)$. To be explicit, for any quadratic extension of fields E/F with $E = F[\sqrt{\delta}]$ for some $\delta \in F^\times$, we may embed E into $M_2(F)$ via

$$a + b\sqrt{\delta} \mapsto \begin{pmatrix} a & \delta b \\ b & a \end{pmatrix}, \quad a, b \in F.$$

Thus we may regard

$$T = \left\{ \begin{pmatrix} a & \delta b \\ b & a \end{pmatrix} : (a,b) \in F \times F - \{(0,0)\} \right\},$$

where $\delta \in \mathbb{F}_q^\times$ is a non-square.

Let $\theta : T \to \mathbb{C}^\times$ be a 1-dimensional representation of $T \simeq \mathbb{F}_{q^2}^\times$. We say θ is **regular** if $\overline{\theta} \neq \theta$, where $x \mapsto \overline{x}$ is the Galois involution for $\mathbb{F}_{q^2}/\mathbb{F}_q$, i.e., $\overline{x} = x^q$. Hence θ is regular if $\theta^q \neq \theta$. The group of characters of T is isomorphic to $\mathbb{F}_{q^2}^\times \simeq C_{q^2-1}$, and a character θ will be regular if and only $\theta^{q-1} = 1$, i.e., if and only if it has order dividing $q-1$, i.e., if and only if it factors through a character of the cyclic quotient C_{q-1} of C_{q^2-1}. Thus there are $(q^2-1)-(q-1) = q^2 - q$ regular characters, and they occur in pairs $(\theta, \overline{\theta})$. In other words, there are $\frac{1}{2}(q^2-q)$ Galois orbits of regular characters of T. These will parametrize the cuspidal representations of G.

Fix a non-trivial character ψ of U. Consider the character $\theta_\psi : ZU \to \mathbb{C}^\times$ given by $\theta_\psi(\begin{pmatrix} a & \\ & a \end{pmatrix} u) = \theta(a)\psi(u)$ for $a \in \mathbb{F}_q^\times$, $u \in U$. (So θ_ψ only depends on the restriction of θ to $Z \simeq \mathbb{F}_q^\times$.)

Theorem A.

1. For a regular character θ of T, we have $\mathrm{Ind}_{ZU}^G \theta_\psi = \mathrm{Ind}_T^G \theta \oplus \pi_\theta$ for an irreducible $(q-1)$-dimensional cuspidal representation π_θ of G;

2. For two regular characters θ, θ' of T, $\pi_\theta \simeq \pi_{\theta'}$ if and only if $\theta' \in \{\theta, \overline{\theta}\}$; and

3. The representations $\{\pi_\theta\}$ for θ a regular character of T exhaust the irreducible cuspidal representations of G.

Proof. The last part follows from the first 2 parts, which one can show by computing the characters of $\mathrm{Ind}_{ZU}^G \theta_\psi$, $\mathrm{Ind}_T^G \theta$ and letting χ_θ be the difference. Then one checks that $(\chi_\theta, \chi_\theta) = 1$, which means χ_θ is the character of an irreducible representation of G which we call π_θ, and checking $\chi_\theta(1) = q-1$ means that π_θ has dimension $q-1$ and therefore is cuspidal. For the second part, one calculates that $(\chi_\theta, \chi_{\theta'}) = 1$ only if $\theta' \in \{\theta, \overline{\theta}\}$. □

Notice that this description does not directly construct cuspidal representations, and for more general groups (e.g., $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{SO}(n)$...) the cuspidal representations (e.g., for $\mathrm{GL}(n)$, those which are not induced from some block upper-triangular subgroup) are not easy to construct explicitly. For $\mathrm{GL}(n)$, the irreducible cuspidal representations will be parametrized by Galois orbits of regular characters of $\mathbb{F}_{q^n}^\times$. However, at least for $\mathrm{GL}(2,q)$ one can construct the cuspidal representations in a more explicit way—see Piatetski-Shapiro's book, or Section 4.1 of Bump's *Automorphic forms and representations* for a construction using the *Weil representation*.

Exercise 2. Show the the character formulas:

$$\begin{aligned}
\chi_\theta(c_1(\alpha)) &= (q-1)\theta(\alpha) \\
\chi_\theta(c_2(\alpha)) &= -\theta(\alpha) \\
\chi_\theta(c_3(\alpha,\beta)) &= 0 \\
\chi_\theta(c_4(\alpha)) &= -(\theta(\alpha) + \theta(\overline{\alpha})).
\end{aligned}$$

Exercise 3. Using the above exercise, complete the details of the proof of Theorem A.

Exercise 4. Describe the full character table for $\mathrm{GL}(2,q)$ by computing the characters of the non-cuspidal irreducible representations of G.

Now you might notice the families of irreducible representations of G in parallel the families of conjugacy classes of G. One might wonder if there is a "natural" way to elicit a bijection between these two sets. I will discuss this more in lecture.