WHAT'S MY RESEARCH ABOUT?

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1. History

1.1. One area of my research is *support varieties*. Although they were only defined 30 years ago, philosophically the study of support varieties began with René Descartes and his publication of *La Géométrie* in 1637.

Descartes's revolutionary observation was that if one has an algebraic equation, then one can look at the *graph* of the equation, which is a geometric object. Conversely, if one has a geometric object (e.g. a circle), then one can look at the algebraic equation which has that geometric object as its graph. In this way one can translate algebra into geometry and vice versa. Of course the hope is that after translation a problem becomes easier! I am mainly interested in turning algebra problems into geometry and using geometry to solve them[1].

Figure 1: Descartes

How does this work? For example, imagine we have a polynomial $f(x)$:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Here n is some postive integer and $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers. Such polynomials are one of the main topics of algebra. Say we don't know $f(x)$ but somehow we know what its graph looks like (see Figure 2).

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[1]Descarte was mainly interested in solving geometry problems by translating them into algebra. But Descarte knew it's a two way street and wouldn't be surprised to see it used in the reverse direction.

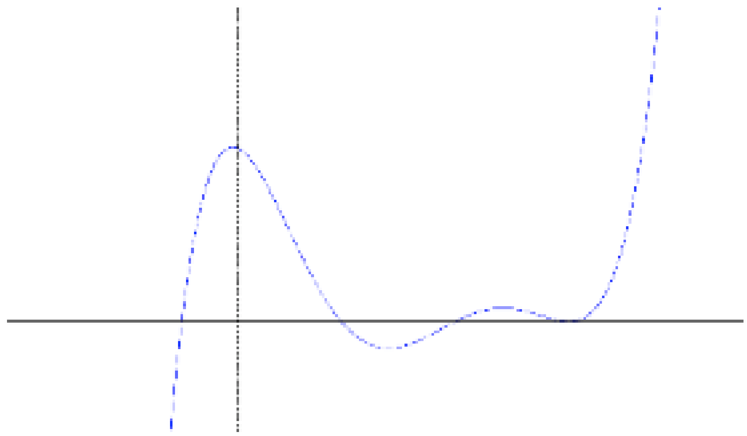

Figure 2: What is the polynomial $f(x)$?

Thus we have an algebra problem (find the polynomial $f(x)$) and have turned to the corresponding geometric object (the graph of $f(x)$) to look for answers. What can we discover about $f(x)$ from its graph?

- Since the two ends look like $\downarrow \cdots \uparrow$, we know the degree n of the polynomial is odd.
- Since the two ends look like $\downarrow \cdots \uparrow$ and not $\uparrow \cdots \downarrow$, we know the leading coefficent a_n must be greater than zero.
- Counting the number of times the graph touches the x-axis, we know the degree of the polynomial is 5 or more[2].
- We can also read off information about local maximums and minums, inflection points, etc., etc.

Incidently, we also discovered a major theme in this approach of turning algebra problems into geometry: when you look at the geometry it becomes very easy to learn *qualitative* information about your original problem. In this example, we still can't identify $f(x)$ but we've learned valuable, albeit incomplete, information about $f(x)$. For example, we still don't know the degree of $f(x)$ but we have narrowed it down to an odd number at least as big as five. That's good progress!

1.2. Let's jump forward a few hundred years. Say that G is a group[3] and k is a field[4]. A *module* for G is a finite dimensional k-vector space M such that G acts on M by invertible

[2]Since the graph touches the x-axis 4 times, we know the degree must be greater than or equal to 4. But we also know the degree is odd so it can't be 4. So it must be 5 or more.

[3]That is, a set with a multiplication operation which satisfies certain rules.

[4]A set with a multiplication and addition which satisfies certain rules. The real numbers and the complex numbers are examples of fields.

linear transformations[5]. For over 100 years people have intensively studied modules of finite groups starting with Frobenius in 1887. *Representation theory* is the study of the modules of some algebraic object like the group G.

Figure 3: Schur

If k is the complex numbers, $\mathbb{C}$, then the representation theory of finite groups is relatively well developed. For example, starting with the work of Frobenius and his Ph.D. student Isaai Schur[6] the modules for the *Symmetric Group* over the complex numbers are quite well understood. All modules are easily constructed from simple modules (ie. modules which have no smaller modules inside them), and pretty much anything you want to known about the simple modules of the symmetric group is known (e.g. their dimensions as vector spaces).

On the other hand one can ask about what happens if k is a field of positive characteristic[7], say p. This was introduced by Schur's Ph.D. student Brauer in 1935. However, despite 70 years of diligent work, the representation theory of finite groups over a field of positive characteristic remains a great mystery. For example, for the symmetric group we still don't even have a guess at the dimensions of the simple modules.

1.3. In an attempt to understand the representation theory of a finite group in characteristic p, one is led to study the *cohomology ring* of G: $\mathrm{H}^\bullet(G,k)$. It would lead us too far astray to define what this is, so let it suffice to say that by starting with the group G and the field k one can construct the ring[8] $\mathrm{H}^\bullet(G,k)$ in a natural way. The fundamental result that gets everything going is the following theorem proved independentally by Golod, Evens, and Venkov around 1960.

Theorem 1. *Let G be a finite group and k a field of positive characteristic, then the ring $\mathrm{H}^\bullet(G,k)$ is a finitely generated commutative ring. That is, it is isomorphic to a polynomial ring modulo some ideal.*

Furthermore, if M is a module for G, then $\mathrm{H}^\bullet(G, M\otimes M^*)$ is a finitely generated module for the ring $\mathrm{H}^\bullet(G,k)$. Here $M\otimes M^*$ is the G-module obtained from M by tensoring M with its dual M^*.

It should be mentioned that there are still many open questions about the ring $\mathrm{H}^\bullet(G,k)$. For example, it can be explicitly described for only the most elementary groups. However, the above result does let us define the support variety of a G-module M. Let I_M be the ideal defined by

$$I_M = \{r \in \mathrm{H}^\bullet(G,k) \mid rx = 0 \text{ for all } x \in \mathrm{H}^\bullet(G, M\otimes M^*)\}.$$

Let the *support variety* of M be the variety

$$\mathcal{V}_G(M) = \{\mathfrak{m} \subseteq \mathrm{H}^\bullet(G,k) \mid \mathfrak{m} \text{ is a maximal ideal of } \mathrm{H}^\bullet(G,k) \text{ and } I_M \subseteq \mathfrak{m}\}.$$

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[5]To say the same thing in a more precise fashion, M is a G-module if there is a group homomorphism $G \to GL(M)$.

[6]Schur's Ph.D. thesis in 1901 introduced what is now known as Schur-Weyl duality. This continues to be a significant part of representation theory – my first paper crucially depends on this 100 year old tool!

[7]For example $k = \mathbb{Z}/p\mathbb{Z}$, the integers modulo a prime number p, is such a field.

[8]A ring is a set with both a multiplication and an addition which satisfies certain properties. It is almost but not quite a field. For example the integers are a ring but not a field.

The definition looks (and is) technical. The point is that we started with an algebraic problem (understand the G-module M) and turned it into a geometric object (the variety $\mathcal{V}_G(M)$). As with Descartes, it turns out that we can discover lots of qualitative information about M by studying the geometry of $\mathcal{V}_G(M)$. One nice example of this is the following result. Recall that p was the characteristic of our field k.

Theorem 2. *Let $d = \operatorname{codim} \mathcal{V}_G(M)$ denote the codimension of $\mathcal{V}_G(M)$. Then p^d divides the dimension of M.*

That is, we might like to know the dimension of M as a vector space over k and the geometric structure of $\mathcal{V}_G(M)$ gives us the partial information that a certain power of p divides the dimension. This is very similar to how the graph of $f(x)$ told us partial information about the degree of $f(x)$!

1.4. Support varieties for finite groups were first introduced by Jon Carlson in the late 1970s and continue to be a very active area of research. Theorem 2 is only one example of the many, many nice results in this area. In fact, because of the great success support varieties have had in studying modules of finite groups, people expanded the use of support varieties to study modules of restricted Lie algebras, finite group schemes, quantum groups, etc. There seems to be no end in sight to their usefulness in representation theory.

2. My Research

2.1. We now come to my research into support varieties for Lie superalgebras. This is joint work with Brian Boe and Daniel Nakano. From now on we will assume $k = \mathbb{C}$ is the complex numbers. A *Lie superalgebra* $\mathfrak{g}$ is a complex vector space with a $\mathbb{Z}/2\mathbb{Z}$-grading,

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}},$$

and a non-associative multiplication called the *bracket*[9]. A *module* for $\mathfrak{g}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded finite dimensional complex vector space $M = M_{\bar{0}} \oplus M_1$ such that $\mathfrak{g}$ acts on M by linear transformations. Lie superalgebras and their modules have played an important role in both mathematics and physics[10] for over 30 years. Despite this they are not well understood. One interesting phenomenon is that, even though the complex numbers have characteristic zero, in many ways the representation theory of Lie superalgebras acts like it is over a field of characteristic $p = 2$ (as we will soon see!).

2.2. Inspired by the success of support varieties in studying modules for finite groups, we decided to try something similar for Lie superalgebras. The first step is to consider the cohomology ring for $\mathfrak{g}$: $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$. We proved the following analogue of Theorem 1.

Theorem 3. *Let $\mathfrak{g}$ be a Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, then*

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong S(\mathfrak{g}_{\bar{1}}^*)^{\mathfrak{g}_{\bar{0}}}.$$

Furthermore, $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ is a finitely generated commutative ring and if M is a $\mathfrak{g}$-module, then $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; M \otimes M^*)$ is a finitely generated $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$-module.

[9] Which, of course, has to satisfy certain rules

[10] In fact, the adjective "super" comes from physics. Lie superalgebras encode both symmetry and anti-symmetry. Taken together they are *supersymmetries*.

Let me briefly explain that $S(\mathfrak{g}_{\bar{1}}^*)$ denotes the ring of polynomials defined on the vector space $\mathfrak{g}_{\bar{1}}$ and this is naturally a $\mathfrak{g}_{\bar{0}}$-module. Then

$$S(\mathfrak{g}_{\bar{1}}^*)^{\mathfrak{g}_{\bar{0}}} = \left\{ f \in S(\mathfrak{g}_{\bar{1}}^*) \mid af = 0 \text{ for all } a \in \mathfrak{g}_{\bar{0}} \right\}$$

is the ring of $\mathfrak{g}_{\bar{0}}$ invariant polynomials. The fact that this is a finitely generated commutative ring uses invariant theory results which go back to Hilbert (see Figure 4) from around 1900. I must admit I was pleased we had the chance to use such a famous result in mathematics!

We are already in better shape than in the finite group situation. As I mentioned before, it is very, very hard to calculate the ring $\mathrm{H}^\bullet(G,k)$. However, using the above theorem and 100+ years of hard work by invariant theory researchers we can calculate the ring $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C})$ in every example we've considered. In fact, using these calculations I can state the following surprising theorem.

Figure 4: Hilbert

Theorem 4. *If $\mathfrak{g}$ is a simple Lie superalgebra with $\mathfrak{g}_{\bar{0}}$ a reductive Lie algebra, then the ring $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C})$ is a polynomial ring.*

2.3. Now that we have these theorems in hand we can define support varieties just as for finite groups. Namely, let I_M be the ideal defined by

$$I_M = \{ r \in \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C}) \mid rx = 0 \text{ for all } x \in \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}}; M \otimes M^*) \}.$$

Let the *support variety* of M be the variety

$$\mathcal{V}_{\mathfrak{g}}(M) = \{ \mathfrak{m} \subseteq \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C}) \mid \mathfrak{m} \text{ is a maximal ideal of } \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C}) \text{ and } I_M \subseteq \mathfrak{m} \}.$$

Again the idea is to turn the algebra problem (understand the $\mathfrak{g}$-module M) into a geometric problem (study the geometry of the variety $\mathcal{V}_{\mathfrak{g}}(M)$).

Although the situation is definitely different than for finite groups, we still been able to develop a very interesting theory. As one example, we have the following result.

Theorem 5. *Let $d = \operatorname{codim} \mathcal{V}_{\mathfrak{g}}(M)$, the codimension of $\mathcal{V}_{\mathfrak{g}}(M)$. Then $2^{\tilde{d}}$ divides the dimension of M.*

Here $\tilde{d} = \lfloor d/2 \rfloor$ is the largest integer smaller than or equal to $d/2$.

Once again, studying the geometry yields interesting qualitative information about our original algebraic problem. Also, compare this result to Theorem 2 (and imagine that $p = 2$ in that theorem) and you see that superalgebras have the flavor of finite groups in characteristic 2!

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