Radar Placement Based on a Geometric Uncertainty Multiplier Reduction Criterion

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Abstract

In this paper the problem of retrieving wind field information from Doppler radar data motivates the formulation of a method to design radar network configurations. The problem of estimating wind velocities from radar data is posed and used to construct a certain retrieval operator. This operator contains a factor that may be interpreted as an uncertainty multiplier. It depends on the geometry of the configuration of the radar network. The uncertainty multiplier is shown to vary continuously with perturbations of the network configuration. It is also shown to be a generalization of the Doppler angle condition used in meteorology. Numerical examples are presented to determine a network of five radars minimizing the uncertainty multiplier for the problem. Also, a configuration of sites is determined that maximizes the area of the Doppler region.

1. Introduction and Main Results.

The objective of this study is to determine locations of radars within a network of sites situated in a set Ω_o that are in some sense "optimal" with respect to their scanning coverage of a given domain Ω. It is assumed that there are n radars each designed with a scanning range of radius R. Given a point $\mathbf{x}$ in Ω within the range R, a radar may make measurements of the wind field consisting of the radial component of wind velocity with respect to that radar. Hence, if a radar can observe the point $\mathbf{x}$, the components of the wind field may be estimated through a variational retrieval process [11, 12, 2] in which wind components are estimated that match observed data subject to certain constraints.

Using an algebraic approach, if two radars observe a point, see Figure 1, then, under the assumption that the vertical velocity is zero, a set of equations can be obtained estimating the horizontal components in terms of the angle between the radar beams and the observed radial components. It has been noted [1, 11, 2] that inversion of this data is facilitated if the angle between radar beams is between 40 and 140 degrees. For beam angles within this range, the procedures are less sensitive to errors. Thus, in determining optimal locations a second constraint, in addition to the range constraint, is imposed to determine locations such that angle constraints are satisfied. Points satisfying both the range and the angular constraints comprise the region referred to as the Doppler region.

An objective of the National Science Foundation Engineering Center CASA (Collaborative Adaptive Sensing of the Atmosphere) is to determine optimal networks of radars to predict tornadic activity. In the CASA concept, multiple low cost radar of relatively short range, approximately 30 km, are to situated in a region to track atmospheric phenomena. The region for the test network is approximately 150km $\times$ 150km. Determination of radar sites to maximize the Doppler region is the approach used in [2] to test the location of sites within the CASA network that is currently being implemented. Formulations have also been given to determine placement of radars in a network to maximize the Doppler region [14] for networks with arbitrary numbers of radars.

In other work, locations are determined to minimize retrieval errors over a class of prescribed events. In [14] a formulation is given defining a retrieval operator that is a mapping from the space of possible wind fields to the space of retrieved wind fields. In this approach mass continuity is used to supplement radar measurements to estimate wind fields in regions where there is little or no radar data. Using this operator, a retrieval error operator is defined measuring errors between wind fields and their retrieved counterparts. The error operator depends on the location of the observing radars. Defining a class of three dimensional test wind fields that are of interest, the configuration of radar sites within the network is determined to be those locations minimizing the retrieval error over that class.

Even though the two methods described differ, intuitively it would seem that locating the radars so that the the Doppler area is maximized should produce retrieval errors that are generally close to optimal over the domain Ω. One of the objectives of this work is to investigate the relation between the two methods. A second objective is to determine a more general method to assess the placement of radars that takes into account physical constraints in addition to the data. Our approach is to reformulate the Doppler area criteria to obtain one involving an operator that is a factor in the retrieval operator that was obtained in [14]. This operator relates the effect of the radar network configuration geometry on the propagation of errors from radial wind velocity data to the retrieved wind fields. The angle criterion can be related to the norm of this geometric retrieval operator. We refer to the norm of this matrix as the geometric uncertainty multiplier (GUM). The GUM is a function of the radar locations, thus, it provides a tool to capture the properties of the network that can be generalized.

It is shown in this work that the GUM depends continuously on the location of the radar sites within a network. With the formulation given here, the GUM can be given for a network with any number of radar and for any region. The GUM is used as a criterion to be minimized to determine the location of radar sites within a network. An alternative approach maximizes the measure of sets for which the GUM is below a prescribed value. It is demonstrated that the GUM is a generalization of the Doppler angle condition. We present an example with five radars situated within a region of dimensions comparable to the CASA test bed to illustrate the ideas presented. We use five radars sites simply to test a collection of sites that is challenging but not overly complicated.

In Section 2 we formulate the basic retrieval problem. This is covered in detail in [14]. In the present work, however, we include sufficient detail to apply to our considerations. In Section 3 we obtain the GUM from the retrieval formulation. In Section 4 we discuss Doppler region measure using the Doppler area and its continuous dependence on the location of the radar sites. Further, we relate the Doppler area and the general geometric uncertainty multiplier. We present results of numerical experiments in which 5 radar sites are to be located within a planar rectangular domain Ω_o measuring 150 km by 150 km with the objective to observe a region consisting of a rectangular solid Ω with base Ω_o. Both minimizing the GUM and maximizing the Doppler area are considered.

2. Retrieval of Wind Fields from Radar Data.

The retrieval of wind fields from radar data may be posed as a problem that minimizes a retrieval functional to find wind fields matching data while weakly satisfying constraining models. This approach is formulated in [14]. To pose the problem of estimating wind field information from radar data thus requires the specification of a retrieval functional. The retrieval functional includes terms involving the data model, a physics-based model, and a regularization. The data model describes the relation between observed radar data and the vector-valued function constituting the actual wind field. It defines a mapping whose output is associated with the radar data resulting from that wind field. Furthermore, it depends on the characteristics of the measurement process. The physics-based model is used to constrain the estimated wind field. It aids in the interpolation of estimated wind fields between radar sites in a physically reasonable way. Finally, a regularization term is included in the retrieval functional to assure that the associated minimization problem has a unique solution.

To formulate the retrieval problem, let Ω denote a volume of interest that, for ease, is a rectangular volume of points $\mathbf{x} = (x, y, z)^T$ in $\Re^3$ such that

$$\Omega = \{\mathbf{x} : 0 < x < L_x, 0 < y < L_y, 0 < z < L_z\}.$$

In general, when discussing vectors, we consider them to be column vectors unless indicated otherwise. A superscript 'T' denotes vector or matrix transposition. Let

$$\Omega_0 = \{(x, y, 0)^T : 0 < x < L_x, 0 < y < L_y\}$$

denote the set in which radar sites may be located. Assume there are n radar site locations $\mathbf{x}_1, ..., \mathbf{x}_n$ in Ω_0. These vectors are organized as a $3 \times n$ matrix

$$q \;=\; [\mathbf{x}_1 ... \mathbf{x}_n]$$

the columns of which are radar site locations. Since it is assumed in this work that the terrain is flat, the z-component of the coordinates in Ω_0 is zero. Let Q denote the collection of all $3 \times n$ matrices of real numbers.

A norm that is useful in discussing convergence of the matrices is given as follows, see [6]. Let $\mathcal{H} = \mathcal{H}(n, m)$ denote the Hilbert space consisting of $n \times m$ matrices of real numbers with inner product

$$(A|B) = trace[A^T B]$$

and the norm given by

$$|A|_{\mathcal{H}} = (A|A)^{1/2}.$$

The set Q is a Hilbert space and the norm coincides with the Frobenius norm [13]. The admissible set Q_{ad} of site locations is taken to be a compact subset of Q. Define the following (column) vector-valued functions from Ω into $\Re^3$ that are used to describe the wind field within the set Ω.

$$\mathbf{v}_s(\mathbf{x}) \;=\; \text{velocity of scattering particles in the sample volume } \Omega$$

$$\mathbf{v}(\mathbf{x}) \;=\; \text{air velocity } : \; \mathbf{v}(\mathbf{x}) \;=\; v_1(\mathbf{x})\mathbf{i} + v_2(\mathbf{x})\mathbf{j} + v_3(\mathbf{x})\mathbf{k}.$$

Let

$$\mathbf{v}_t(\mathbf{x}) \;=\; \text{terminal velocity of the scatterers } : \; \mathbf{v}_t(\mathbf{x}) = W_t(\mathbf{x})\mathbf{k}$$

where $W_t \geq 0$

$$\mathbf{v}_s(\mathbf{x}) \;=\; \mathbf{v}(\mathbf{x}) - \mathbf{v}_t(\mathbf{x}). \tag{2.1}$$

Describe the unit vector pointing in the direction from the ith radar location $\mathbf{x}_i$ to the point $\mathbf{x} \in \Omega$ by means of the vector-valued function

$$\mathbf{r}_i(\mathbf{x}) = \mathbf{r}(\mathbf{x}, \mathbf{x}_i) \;=\; \frac{\mathbf{x} - \mathbf{x}_i}{|\mathbf{x} - \mathbf{x}_i|} \text{ if } \mathbf{x} \;\neq\; \mathbf{x}_i \text{ and } \mathbf{0} \text{ if } \mathbf{x} = \mathbf{x}_i. \tag{2.2}$$

See Figure 1 illustrating the set up for two radars. The radial velocity observed at the ith radar is then expressed by the product

$$v_r(\mathbf{x}, \mathbf{x}_i) = \mathbf{r}_i(\mathbf{x})^T \mathbf{v}_s. \tag{2.3}$$

The function

$$(\mathbf{x}, \mathbf{x}_i) \mapsto v_r(\mathbf{x}, \mathbf{x}_i)$$

is the expression for radial velocity corresponding to that observed at the point $\mathbf{x}$ (assuming the point is within the coverage set associated with the ith radar) from a radar located at the point $\mathbf{x}_i$. In this case the function $v_r(\mathbf{x}, \mathbf{x}_i)$ is to be compared with an observation, from the ith radar, $v_{r\,i}(\mathbf{x})$.

To model the location and coverage of the ith radar, we designate the pair

$$(\mathbf{x}_i, \varphi_i)$$

where $\mathbf{x}_i$ is the location of the ith radar as indicated above and φ_i is a function modelling the coverage of that radar. For the purposes of this work, we use the simplest coverage function. The function φ_i is defined in terms of a characteristic function

$$\Xi_{\widetilde{\Omega}}(\mathbf{x}) \;=\; 1 \text{ if } \mathbf{x} \in \widetilde{\Omega} \text{ and } \;=\; 0 \text{ otherwise.} \tag{2.4}$$

Of particular interest is the set

$$\widetilde{\Omega} = \{\mathbf{x} : |\mathbf{x}| \leq R\}. \tag{2.5}$$

so that

$$\varphi_i(\mathbf{x}) = \Xi_{\widetilde{\Omega}}(\mathbf{x} - \mathbf{x}_i) \tag{2.6}$$

A Hilbert space formulation is given in [14]. Towards this end, introduce the Hilbert spaces

$$\mathbf{H} \;=\; L^2(\Omega, \Re^3)$$

with the inner product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}^T \mathbf{v} d\mathbf{x}$$

and norm

$$\|\mathbf{u}\|_H \;=\; (\mathbf{u}, \mathbf{u})^{\frac{1}{2}}$$

and

$$\mathbf{V} = H^1(\Omega, \Re^3) \tag{2.7}$$

with inner product

$$(2.8)(i) \qquad (\mathbf{u}, \mathbf{v})_{\mathbf{V}} = \int_\Omega \{\nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2 + \nabla u_3 \cdot \nabla v_3 + \mathbf{u}^T \mathbf{v}\} d\mathbf{x}.$$

and norm

$$(2.8)(ii) \qquad \|\mathbf{v}\|_{\mathbf{V}} = (\mathbf{v}, \mathbf{v})_{\mathbf{V}}^{\frac{1}{2}}.$$

To include the divergence free condition for the continuity equation, define the bilinear form on $\mathbf{V}$

$$(\mathbf{u}, \mathbf{v})_1 = \int_\Omega [\nabla^T \mathbf{u}][\nabla^T \mathbf{v}] d\mathbf{x}. \tag{2.9}$$

Note that a norm equivalent to (2.8)(ii), see [7], on $\mathbf{V}$ may be obtained using the bilinear form

$$((\mathbf{u}, \mathbf{v})) = \int_\Omega \{\nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2 + \nabla u_3 \cdot \nabla v_3\} d\mathbf{x}$$

along with a functional associated with a radar located at a site $\mathbf{x}_i$ is defined by

$$\int_\Omega \mathbf{u}^T [\varphi_i^2 \mathbf{r}_i \mathbf{r}_i^T] \mathbf{v} d\mathbf{x}.$$

The subscript "i" is an index for the point $\mathbf{x}_i$ that is the site of a radar from which measurements are made. Define

$$\|\mathbf{v}\|_{\widetilde{V}} = \{((\mathbf{v}, \mathbf{v})) + K \int_\Omega \mathbf{v}^T [\varphi_i^2 \mathbf{r}_i \mathbf{r}_i^T] \mathbf{v} d\mathbf{x}\}^{\frac{1}{2}} \tag{2.10}$$

where K is a positive number. There exist constants C_1 and C_2 that depend on the location $\mathbf{x}_0$ of the radar such that

$$C_1 \|\mathbf{v}\|_{\mathbf{V}}^2 \le \|\mathbf{v}\|_{\widetilde{V}}^2 \le C_2 \|\mathbf{v}\|_{\mathbf{V}}^2. \tag{2.11}$$

We use the subscript $\mathbf{V}$ to denote the appropriate norms and inner products on $\mathbf{V}$. We use H and V to denote the spaces of real-valued functions $L^2(\Omega)$ and $H^1(\Omega)$, respectively.

The weak formulation of the retrieval problem is posed as a minimization problem over the space $\mathbf{V}$. The objective functional is given as

$$\mathcal{V}(\mathbf{v}) = \frac{\epsilon}{2}((\mathbf{v}, \mathbf{v})) + \frac{K_0}{2}(\mathbf{v}, \mathbf{v})_1 +$$

$$(2.12) \qquad +\frac{K}{2}\int_\Omega \{\sum_{i=1}^{n} \varphi_i^2(\mathbf{x})[v_r(\mathbf{x},\mathbf{x}_i) - v_{ri}(\mathbf{x})]^2\} d\mathbf{x}$$

and is defined over the space of functions $\mathbf{V}$ where $\epsilon, K_0,$ and K are positive constants. The retrieval problem is posed as

$$(2.13) \qquad \text{Find } \mathbf{u} \in \mathbf{V} \text{ such that } \mathcal{V}(\mathbf{u}) \;=\; \text{infimum } \{\mathcal{V}(\mathbf{v}) : \mathbf{v} \in \mathbf{V}\}.$$

Remark 2.1. In the formulation given here, time is not explicitly included. The retrieval problem is solved over a sequence of times. It is assumed that the radial velocity is known at each point $\mathbf{x}$ within a given radar's range at each time.

Our interest here concerns the dependence of solutions of the retrieval problem on the collection of radar locations. Hence, we view the $3 \times n$ matrix

$$q = [\mathbf{x}_1 \ \mathbf{x}_2 \ . \ . \ . \ \mathbf{x}_n] \in Q_{ad}$$

as a parameter to be determined. Define the matrix-valued function

$$(2.14) \qquad \Phi(q)(\mathbf{x}) \;=\; \sum_{i=1}^{n} \varphi_i(\mathbf{x})^2 \mathbf{r}_i(\mathbf{x})\mathbf{r}_i^T(\mathbf{x}),$$

the vector-valued functions

$$(2.15) \qquad \mathbf{F}(q)(\mathbf{x}) \;=\; \sum_{i=1}^{n} \varphi_i^2(\mathbf{r}_i^T \mathbf{v}_t + v_{r_i})\mathbf{r}_i,$$

and the constants

$$(2.16) \qquad C(q) = \int_\Omega \{\sum_{i=1}^{n} \varphi_i^2(\mathbf{r}_i^T \mathbf{v}_t + v_{r_i})^2 d\mathbf{x}.$$

It is also convenient to define the bilinear forms on $\mathbf{H}$

$$(2.17) \qquad (\mathbf{u}, \mathbf{v})_{\Phi(q)} = \int_\Omega \mathbf{u}(\mathbf{x})^T \Phi(q)(\mathbf{x})\mathbf{v}(\mathbf{x}) d\mathbf{x}.$$

Thus, from (2.10) the expression

$$\{((\mathbf{v}, \mathbf{v})) + K(\mathbf{v}, \mathbf{v})_{\Phi(q)}\}^{\frac{1}{2}}$$

gives a norm on $\mathbf{V}$ equivalent to $\|\mathbf{v}\|$. With these definitions, we may write the criterion $\mathcal{V}$ as

$$(2.18) \qquad \mathcal{V}(\mathbf{v}) = \frac{\epsilon}{2}((\mathbf{v}, \mathbf{v})) + \frac{K_0}{2}(\mathbf{v}, \mathbf{v})_1 + \frac{K}{2}(\mathbf{v}, \mathbf{v})_{\Phi(q)} \; -$$

$$-K(\mathbf{F}(q),\mathbf{v})+\frac{K}{2}C(q).$$

The existence of a unique minimizer follows from classical Hilbert space theory [6, 14].

Proposition 2.2. There exists a unique solution to the minimization problem (2.13).

The solution of the minimization problem (2.13) is characterized by the optimality conditions.

Proposition 2.3. The derivative of $\mathcal{V}$ is given by

$$D\mathcal{V}(\mathbf{u})\mathbf{v}=\epsilon((\mathbf{u},\mathbf{v}))+K_0(\mathbf{u},\mathbf{v})_1+K(\mathbf{u},\mathbf{v})_{\Phi(q)}-K(\mathbf{F}(q),\mathbf{v})$$

and the solution of the minimization problem (2.13) satisfies the equation

$$\epsilon((\mathbf{u},\mathbf{v}))+K_0(\mathbf{u},\mathbf{v})_1+K(\mathbf{u},\mathbf{v})_{\Phi(q)}= \\ =K(\mathbf{F}(q),\mathbf{v}) \tag{2.19}$$

for all $\mathbf{v}\in\mathbf{V}$.

Remark 2.4. The coverage functions φ belong to H so that from elliptic regularity the retrieved wind fields belong to $H^2(\Omega,\Re^3)$ cf. [4].

The finite element approximation of the retrieval problem follows classical arguments [10]. Approximations of subspaces of V may be based on finite elements obtained as tensor products of piecewise linear splines defined on partitions of the intervals $(0,L_x),\ (0,L_y),$ and $(0,L_z)$ into $n_x,\ n_y,$ and n_z subintervals, respectively. Setting $m_x=n_x+1, m_y=n_y+1, m_z=n_z+1$ to represent the number of $x,\ y,$ and z elements, respectively. The number of basis elements for the three dimensional problem is given by $m=m_x\times m_y\times m_z$. We denote the basis elements as

$$b_1(\mathbf{x}),\ \ldots,\ b_m(\mathbf{x})$$

spanning the subspace V^m of V. Define the column m vector-valued function on Ω by

$$\mathbf{x}\mapsto\widehat{b}(\mathbf{x})=[b_1(\mathbf{x}),...,b_m(\mathbf{x})]^T$$

and the $3\times 3m$ matrix-valued function on Ω by

$$\mathbf{x}\mapsto B(\mathbf{x})=\begin{bmatrix}\widehat{b}(\mathbf{x})^T & 0 & 0\\ 0 & \widehat{b}(\mathbf{x})^T & 0\\ 0 & 0 & \widehat{b}(\mathbf{x})^T\end{bmatrix}$$

where 0 represents an m-row vector of zeros. We also define the column m-vectors $\underline{c}_1,\ \underline{c}_2,$ and $\underline{c}_3$ as well as the 3m-column vector $\widetilde{c}=[\underline{c}_1^T,\underline{c}_2^T,\underline{c}_3^T]^T$.

With the above definitions, the components of the wind velocity may be represented as

$$v_{1m}(\mathbf{x}) = \widehat{b}(\mathbf{x})^T \underline{c}_1$$

$$v_{2m}(\mathbf{x}) = \widehat{b}(\mathbf{x})^T \underline{c}_2$$

$$v_{3m}(\mathbf{x}) = \widehat{b}(\mathbf{x})^T \underline{c}_3,$$

and express the approximating wind velocity vector

$$\mathbf{v}^m(\mathbf{x}) = [v_{1m}(\mathbf{x})\ v_{2m}(\mathbf{x})\ v_{3m}(\mathbf{x})]^T = B(\mathbf{x})\widetilde{c}.$$

To approximate the objective functional, define the $3m \times 3m$ matrices

$$G_1 = \int_\Omega [\nabla^T B(\mathbf{x})]^T [\nabla^T B(\mathbf{x})] d\mathbf{x}.$$

Define the $m \times m$ matrix $[g_2]$ by setting entries

$$[g_2]_{ij} = \int_\Omega [\nabla b_i(\mathbf{x})]^T [\nabla b_j(\mathbf{x})] d\mathbf{x}$$

for $i, j = 1, ..., m$. Let the $3m \times 3m$ matrix be given by

$$G_2 = \begin{bmatrix} g_2 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_2 \end{bmatrix}.$$

Further, define the $3m \times 3m$ matrix

$$G(q) = \int_\Omega B(\mathbf{x})^T \Phi(q) B(\mathbf{x}) d\mathbf{x}.$$

Also, define the $3m$ column vector

$$\widetilde{F}(q) \ = \ \{\int_\Omega \mathbf{F}(q)(\mathbf{x})^T B(\mathbf{x}) d\mathbf{x}\}^T.$$

With these definitions the objective functional evaluated at the finite element approximations of the wind velocity is given by

$$\mathcal{V}(\widetilde{c}) = \mathcal{V}(\mathbf{v}^m) = \frac{1}{2}\widetilde{c}^T[\epsilon G_2 + K_0 G_1 + K G(q)]\widetilde{c} \ - \ K_0 \widetilde{F}(q)^T \widetilde{c} + C(q). \tag{2.20}$$

Remark 2.5. The finite dimensional minimization problem looks for a vector $\widetilde{c}$ or a function of the form

$$\mathbf{u}^m(\mathbf{x}) \ = \ B(\mathbf{x})\widetilde{c}$$

in $\mathbf{V}^m$ minimizing the functional $\mathcal{V}$ over $\mathbf{V}^m$.

The derivative of this functional gives the system

$$\epsilon((\mathbf{u}^m, \mathbf{v})) \ + \ K_0(\mathbf{u}^m, \mathbf{v})_1 \ + \ K(\mathbf{u}^m, \mathbf{v})_{\Phi(q)} = K(\mathbf{F}(q), \mathbf{v}) \tag{2.21}$$

for all $\mathbf{v} \in \mathbf{V}^m$. In terms of matrices the solution vector $\widetilde{c}$ satisfies

$$[\epsilon G_2 \ + \ K_0 G_1 + \ KG(q)]\widetilde{c} \ = \ K\widetilde{F}(q). \tag{2.22}$$

Define the matrix-valued function $q \mapsto H(q)$ by

$$H(q) = \epsilon G_2 \ + \ K_0 G_1 + \ KG(q). \tag{2.23}$$

We assume that the numbers $\epsilon, K_0,$ and K are chosen so that the matrix $H(q)$ is invertible for each $q \in Q_{ad}$.

Remark 2.6. Convergence under grid refinement is established by standard techniques using Cea's lemma and regularity [3].

3. The Retrieval Operator and the Geometric Uncertainty Multiplier Reduction.

To study the geometric scaling of uncertainty, the dependence of $\widetilde{F}(q)$ on q in equation (2.22) is ignored [1, 12]. Instead, attention is focused on the matrix $H(q)$ and the equation

$$H(q)\widetilde{c}(q) = \widetilde{F}. \tag{3.1}$$

Note that under the formulation of the retrieval problem the matrix $H(q)$ is symmetric and invertible for each admissible configuration q. Thus,

$$\widetilde{c}(q) = H(q)^{-1}\widetilde{F},$$

and the approximating retrieved wind field is given by

$$\mathbf{u}^m(q)(\mathbf{x}) \ = \ B(\mathbf{x})\widetilde{c}(q). \tag{3.2}$$

Multiplying by the transpose, from (3.2), we obtain

$$|\mathbf{u}^m(q)(\mathbf{x})|^2 = \widetilde{c}(q)^T B(\mathbf{x})^T B(\mathbf{x})\widetilde{c}(q) =$$

$$= \widetilde{F}^T H(q)^{-1} B(\mathbf{x})^T B(\mathbf{x}) H(q)^{-1}\widetilde{F}. \tag{3.3}$$

Because of linearity, the variation $\delta\mathbf{u}^m$of $\mathbf{u}^m$ with respect to $\widetilde{F}$, $\delta\widetilde{F}$, satisfies

$$|\delta\mathbf{u}^m(q)(\mathbf{x})|^2 = [\delta\widetilde{F}]^T H(q)^{-1} B(\mathbf{x})^T B(\mathbf{x}) H(q)^{-1}[\delta\widetilde{F}]. \tag{3.4}$$

Hence, the matrix

$$R(q)(\mathbf{x}) = H(q)^{-1}B(\mathbf{x})^T B(\mathbf{x})H(q)^{-1} \tag{3.5}$$

may be interpreted as a scaling factor that determines the extent to which the uncertainty in $\widetilde{F}$ effects the approximating retrieved wind field $\mathbf{u}^m(q)(\mathbf{x})$. We may consider pointwise and global scaling in terms of the norm of the matrix $R(q)(\mathbf{x})$ and the matrix

$$R_0(q) = H(q)^{-1}G_0 H(q)^{-1} \tag{3.6}$$

where equation (3.4) has been integrated to obtain

$$\|\delta\mathbf{u}^m\|^2_{\mathbf{H}} = [\delta\widetilde{F}]^T H(q)^{-1}G_0 H(q)^{-1}[\delta\widetilde{F}]. \tag{3.7}$$

The pointwise uncertainty factor is estimated as the norm of the matrix $R(q)(\mathbf{x})$, and the global uncertainty factor may be estimated by the norm of $R_0(q)$. Since both $R(q)(\mathbf{x})$ and $R_0(q)$ are symmetric and nonnegative definite, it follows that their norms as operators on the underlying Hilbert space coincide with the largest eigenvalue. We will be interested in a global measure of uncertainty over Ω as well as a pointwise measure. For the global measure define the geometric uncertainty multiplier (GUM) as the nonnegative function of q given by the matrix norm

$$\Lambda(q) = |R_0(q)|. \tag{3.8}$$

This quantity provides a general indicator of the sensitivity over the entire domain. Clearly, it is possible to focus on any subdomain by integrating $R(q)(\mathbf{x})$ over that subdomain. For a pointwise indicator, we define the pointwise uncertainty factor as

$$\Lambda(\mathbf{x}, q) = |R(q)(\mathbf{x})|. \tag{3.9}$$

We consider two criteria for selection of the radar location matrix q. The first minimizes the global general indicator $\Lambda(q)$ over Q_{ad}. This determines a configuration of radar sites that globally reduces the sensitivity to errors in the data vector. Clearly, it is possible to determine measures over other such prescribed subsets of Ω as well.

To formulate a second criterion, an upper bound ϵ is specified for the pointwise GUM and a subset $\mathcal{A}(q)$ of Ω is defined by

$$\mathcal{A}(q) = \{\mathbf{x} \in \Omega : \Lambda(\mathbf{x}, q) < \epsilon\}.$$

We define the functional

$$J(q) = \int_{\mathcal{A}(q)} d\mathbf{x}$$

over the admissible set Q_{ad} of network configurations, and we seek radar configurations to maximize $J(q)$ over Q_{ad}.

Existence of optimizers in either case depends on the continuity properties of the mappings $q \mapsto R(q)(\mathbf{x})$ and $q \mapsto R_0(q)$ of Q_{ad} into $3m \times 3m$ matrices. Thus, we briefly discuss the continuous dependence on the parameter q of radar site locations. The following are easy to establish.

Lemma 3.1. Let $\mathbf{r}_i$ for $i = 1, 2$ be given by (2.2) and suppose that $\mathbf{x}$ is not equal to $\mathbf{x}_i$. Then

$$|\mathbf{r}_1(\mathbf{x}) - \mathbf{r}_2(\mathbf{x})| \leq \frac{2}{|\mathbf{x} - \mathbf{x}_2|} |\mathbf{x}_2 - \mathbf{x}_1|.$$

Lemma 3.2 Let $\mathbf{x}_i$ be a sequence of points with functions φ_i defined as in (2.4)-(2.6) for $i = 0, 1, 2, ...$ Suppose that $\mathbf{x}_i \to \mathbf{x}_0$ as $i \to \infty$. Then

$$\varphi_i(\mathbf{x}) \to \varphi_0(\mathbf{x})$$

at every point $\mathbf{x}$ with the exception of the set

$$A \ = \ \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| = R\}.$$

Define the matrix valued functions for $i = 0, 1, 2, ...$

$$\Phi_i(\mathbf{x}) = \varphi_i(\mathbf{x}) \mathbf{r}_i(\mathbf{x}) \mathbf{r}_i(\mathbf{x})^T \tag{3.10}$$

and the matrices

$$G_i = \int_\Omega B(\mathbf{x})^T \Phi_i(\mathbf{x}) B(\mathbf{x}) dx. \tag{3.11}$$

Consider the Hilbert space $\mathcal{H} = \mathcal{H}(n, n)$ consisting of $n \times n$ matrices of real numbers with an inner product

$$(A|B) = trace[A^T B]$$

and the norm given by

$$|A|_{\mathcal{H}} = (A|A)^{1/2}.$$

We note that the $|\cdot|_{\mathcal{H}}-$ norm of the matrix $\Phi_0(\mathbf{x}) = \varphi_0(\mathbf{x}) \mathbf{r}_0(\mathbf{x}) \mathbf{r}_0(\mathbf{x})^T$ associated with the point $\mathbf{x}_0$ satisfies

$$\begin{aligned} |\Phi_0(\mathbf{x})|^2_{\mathcal{H}} \ &= \ \varphi_0(\mathbf{x})^2 trace(\Phi_0(\mathbf{x})^T \Phi_0(\mathbf{x})) \\ &= \ \varphi_0(\mathbf{x})^2 trace[(\mathbf{r}_0(\mathbf{x}) \mathbf{r}_0(\mathbf{x})^T)^T (\mathbf{r}_0(\mathbf{x}) \mathbf{r}_0(\mathbf{x})^T)] \end{aligned}$$

$$
\begin{aligned}
&= \varphi_0(\mathbf{x})^2 trace[\mathbf{r}_0(\mathbf{x})\mathbf{r}_0(\mathbf{x})^T] \\
&= \varphi_0(\mathbf{x})^2 \mathbf{r}_0(\mathbf{x})^T \mathbf{r}_0(\mathbf{x}) \\
&= \varphi_0(\mathbf{x})^2
\end{aligned}
$$

We now have the following estimate as a result of a straight forward calculation.

Lemma 3.3. Let Φ_1 and Φ_0 be defined as in (3.10) for points $\mathbf{x_1}$ and $\mathbf{x_0}$, respectively. Then for any $\mathbf{x}$ not equal to $\mathbf{x}_1$ or $\mathbf{x}_0$

$$
|\Phi_1(\mathbf{x}) - \Phi_0(\mathbf{x})|_{\mathcal{H}} \ \leq 2|\varphi_1(\mathbf{x}) - \varphi_0(\mathbf{x})|[2|\mathbf{r}_1(\mathbf{x}) - \mathbf{r}_0(\mathbf{x})| \ + \ |\varphi_1(\mathbf{x}) - \varphi_0(\mathbf{x})|].
$$

It follows from Lemmas 3.1-3.3 that the matrices converge pointwise at every point in the complement of the set A. Thus, we have the following results.

Proposition 3.4 If $\mathbf{x}_i \to \mathbf{x}_0$ as $i \to \infty$, the matrix-valued functions

$$
\Phi_i(\mathbf{x}) \to \Phi_0(\mathbf{x}) \text{ in } \mathcal{H}
$$

as $i \to \infty$ for each $\mathbf{x}$ in the complement of the set A.

Proposition 3.5 If $\mathbf{x}_i \to \mathbf{x}_0$ as $i \to \infty$, then

$$
G_i \to G_0 \text{ in } \mathcal{H}.
$$

Thus, the function $q \mapsto H(q)$ is continuous as is $q \mapsto H(q)^{-1}$.

Proof. The set A is of measure zero. The components of the matrices Φ_i are bounded functions, and the set Ω has finite measure. Furthermore, $\Phi_i(\mathbf{x}) \to \Phi(\mathbf{x})$ in $\mathcal{H}$ for every $\mathbf{x}$ in the complement of A. Convergence in $\mathcal{H}$ implies almost everywhere convergence of the entries of Φ_i. From (3.11) the Lebesgue dominated convergence theorem then implies convergence

$$
G_i \to G_0
$$

as $i \to \infty$.

Applying the above results to the matrices $H(q)$, we see that the result holds since $H(q)^{-1}$ exists for every q.

Summarizing continuity properties, we have the following result.

Proposition 3.6 For each q the function $\mathbf{x} \mapsto R(q)(\mathbf{x})$ defined by (3.5) is a continuous matrix valued function on Ω. Furthermore, if $q_i \to q$, then the matrix functions $R(q_i)$ converges uniformly to $R(q)$ on Ω. The function $q \mapsto R_0(q)$ defined by (3.6) and the function $q \mapsto \Lambda(q)$ is continuous on Q. For each q the function $\mathbf{x} \mapsto \Lambda(\mathbf{x}, q)$ defined by (3.9) is continuous.

The continuous dependence of the area functional $q \mapsto J(q)$ depends on the convergence of the functions

$$\Xi_{\mathcal{A}(q)} = 1 \text{ if } \mathbf{x} \in \mathcal{A}(q) \text{ and } 0, \text{ otherwise}$$

as $q_i \to q$ as $i \to \infty$. We introduce the assumption

$$\text{meas}\{\mathbf{x} : \Lambda(\mathbf{x}, q) = 0\} \;=\; 0. \tag{3.12}$$

The following result is easily established.

Proposition 3.7. Suppose (3.12) holds and $q_i \to q$ as $i \to \infty$. Then

$$\Xi_{\mathcal{A}(q_i)} \to \Xi_{\mathcal{A}(q)}$$

almost everywhere in Ω.

Continuity of the mapping $q \mapsto J(q)$ follows from the Lebesgue dominated convergence theorem.

Corollary 3.8. Suppose (3.12) holds and $q_i \to q$ as $i \to \infty$. Then

$$J(q_i) \to J(q)$$

as $i \to \infty$.

From Proposition 3.6, the mapping

$$q \mapsto \Lambda(q)$$

is a continuous real-valued function on Q. Hence, as an immediate consequence of the continuity results, we have the underlying existence results.

Theorem 3.9. Suppose that Q_{ad} is compact. Then there exists a $q_0 \in Q_{ad}$ such that

$$\Lambda(q_0) \;=\; \inf\{\Lambda(q) : q \in Q_{ad}\}.$$

Theorem 3.10. Suppose that Q_{ad} is compact. Then there exists a $q_1 \in Q_{ad}$ such that

$$J(q_1) \;=\; \max\{J(q) : q \in Q_{ad}\}.$$

4. Relation Between GUM and the Doppler Angle Criterion.

In this Section, we relate the considerations of the geometric uncertainty multiplier (GUM) in Section 3 to those associated with the Doppler region discussed in [1, 12]. In [2] the area of the Doppler region is used as a criterion to distinguish between configurations of networks of radars. We show the angle condition is related to the size of the tolerance used in specifying the set $\mathcal{A}(q)$. Let $\mathbf{x}$ be a point in Ω and suppose $\mathbf{x}$ is a point within range of a radar located at $\mathbf{x}_i$. From (2.3) we have for the ith radar at a point $\mathbf{x}$

$$\mathbf{r}_i \cdot \mathbf{v}_s = v_r(\mathbf{x}, \mathbf{x}_i). \tag{4.1}$$

Denote the azimuthal and elevational angles by θ and ϕ, respectively. See Figure 1. Setting the vector $\mathbf{v}_s = (u, v, w)$ and using the angles ϕ_i and θ_i, we have an equation of the form

$$cos(\phi_i)cos(\theta_i)u + cos(\phi_i)sin(\theta_i)v + cos(\theta_i)w = v_{ri} \tag{4.2}$$

for each i that is within range of the location $\mathbf{x}$. For $\mathbf{x}$ to be in the Doppler region, there are must be at least two such equations. We assume that $i = 1, 2$. Typically, it is assumed that the vertical component of the wind velocity $w = 0$ and the elevation angles ϕ_i are small. Defining the matrix

$$A = \begin{bmatrix} cos(\phi_1)cos(\theta_1) & cos(\phi_1)sin(\theta_1) \\ cos(\phi_2)cos(\theta_2) & cos(\phi_2)sin(\theta_2) \end{bmatrix}$$

equation (4.2) becomes

$$A \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v_{r1} \\ v_{r2} \end{bmatrix}. \tag{4.3}$$

The equation (4.3) may be solved to obtain

$$u = \frac{-cos(\phi_2)sin(\theta_2)v_{r1} + cos(\phi_1)sin(\theta_1)v_{r2}}{cos(\phi_1)cos(\phi_2)sin(\theta_1 - \theta_2)} \tag{4.4)(i}$$

$$v = \frac{cos(\phi_2)cos(\theta_2)v_{r1} - cos(\phi_1)cos(\theta_1)v_{r2}}{cos(\phi_1)cos(\phi_2)sin(\theta_1 - \theta_2)}. \tag{4.4)(ii}$$

Hence, the inverse of the matrix A is given by

$$A^{-1} = \tag{4.5}$$

$$= \frac{1}{cos(\phi_1)cos(\phi_2)sin(\theta_2 - \theta_1)} \begin{bmatrix} -cos(\phi_2)sin(\theta_2) & cos(\phi_1)sin(\theta_1) \\ cos(\phi_2)cos(\theta_2) & -cos(\phi_1)cos(\theta_1) \end{bmatrix}.$$

The matrix A^{-1} expresses a geometric factor that can be viewed as an uncertainty multiplier

$$\begin{bmatrix} \delta u \\ \delta v \end{bmatrix} = A^{-1} \begin{bmatrix} \delta v_{r1} \\ \delta v_{r2} \end{bmatrix} \tag{4.6}$$

that relates the effect of perturbations of the data on the retrieved horizontal wind fields. The geometric uncertainty multiplier corresponds to the norm of

A^{-1}. Using the vector space $\mathcal{H} = \mathcal{H}(2,2)$ for 2×2 matrices introduced previously, we have

$$|A^{-1}| \;=\; trace[(A^{-1})^T A^{-1}]^{\frac{1}{2}}$$
$$= \frac{(cos^2(\phi_2) + cos^2(\phi_1))^{\frac{1}{2}}}{cos(\phi_1)cos(\phi_2)|sin(\theta_2 - \theta_1)|}.$$

Thus, for small elevational angles, we have

$$|A^{-1}| \approx \frac{\sqrt{2}}{|sin(\theta_2 - \theta_1)|}. \tag{4.7}$$

Remark 4.1. From the estimate (4.7), it is clear that the closer to 90 degrees that the difference

$$\theta_2 - \theta_1$$

is, the smaller the uncertainty factor. The bound on the difference, $|\theta_2 - \theta_1|$, is typically between 40 degrees and 140 degrees which gives an uncertainty factor of approximately 2.

Remark 4.2. In [1, 2, 12] the angle criterion is used for setting constraints in determining Doppler regions. From the presentation here it is seen that the geometric uncertainty multiplier is a generalization of the Doppler angle condition. The angle condition utilizes assumptions on the size of elevation angles and the resulting approximations. Also, the vertical velocity is assumed to be zero. The GUM makes no such assumptions.

To illustrate the implementation of the Doppler criterion, we give a simplified 2 dimensional problem in which the objective is to maximize the area of the region consisting of points at which the doppler angle between radar beams is between a prescribed lower and upper bound and that are within the radar range. A configuration of radar sites is prescribed. At a given point $\mathbf{x} \in \Omega_0$, where Ω_0 is a region of interest in $\Re^2$, if that angle is between certain bounds θ_{min} and $\theta_{max} = \pi - \theta_{min}$ and the point is within the range of the radars, then the point is considered to belong to the Doppler region. The angle θ_{min} is taken to be 40 degrees. We suppose that there are n radar locations $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$. Recall the functions $\mathbf{r}(\mathbf{x}, \mathbf{x}_i)$ defined in (2.2) and the coverage functions $\varphi_i(\mathbf{x})$ defined in (2.6). Denote by θ_{ij} the angle between the vectors with tail at the point $\mathbf{x}_i$ and $\mathbf{x}_j$ and head at the point $\mathbf{x}$. There are $\frac{(n-1)n}{2}$ such angles and

$$\cos(\theta_{ij}) = \mathbf{r}(\mathbf{x}, \mathbf{x}_i) \cdot \mathbf{r}(\mathbf{x}, \mathbf{x}_j).$$

It is convenient to consider the condition

$$|\cos(\theta)| < \cos(\theta_{min}) \;=\; \gamma$$

as a constraint quantifying those $\theta \in [\theta_{min}, \pi - \theta_{min}]$.

We define the following functions

$$C(\mathbf{x}) = \min|(\cos(\theta_{ij}))|$$

where the minimum is taken of the set of angles between rays from the various radar locations and the point $\mathbf{x}$. To impose the Doppler angle cutoff condition, we define the function

$$\widehat{C}(\mathbf{x}) = C(\mathbf{x}) \text{ if } C(\mathbf{x}) \leq \ \gamma \text{ and } 0 \text{ otherwise.}$$

The radar range constraints are imposed through the functions ϕ_i. We define the function

$$\widehat{\Delta}(\mathbf{x}) = \phi_i(\mathbf{x})\phi_j(\mathbf{x})\widehat{C}(\mathbf{x})$$

to describe the intensity of the Doppler coverage. Finally, the Doppler area function is defined by

$$A(x) = 1 \text{ if } \widehat{\Delta}(x) > 0 \text{ and } 0 \text{ otherwise.}$$

Designating a configuration of radar locations by $q = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$, we denote the dependence of the intensity and the area functions on q by $\widehat{\Delta}(\cdot, q)$ and $A(\cdot, q)$. We then define an objective function expressing the area of the region satisfying the range condition and the Doppler angle condition associated with a configuration q by

$$J_d(q) = \int_\Omega A(\mathbf{x})d\mathbf{x}.$$

Remark 4.3. Results on the existence of configurations maximizing the Doppler area are similar to those mentioned in the treatment of the GUM. Uniqueness does not hold. In fact it is easy to see that if a configuration is obtained then subsequent configurations that result from the application of area-preserving transformations will also be solutions.

5. Numerical Examples.

We present the results of numerical experiments in which there are five radar locations to be determined within a region $\Omega_0 = [0, 150] \times [0, 150]$ to scan the domain $\Omega = (0, 150) \times (0, 150) \times (0, 3)$. We give results based on minimization of the GUM criterion and also give an example maximizing the Doppler area. It is assumed that one of the sites is fixed at coordinates $(75, 75)$. To formulate the approximation equations for retrieval, we use tensor products of piecewise linear splines [10]. The basis elements are defined on a uniform mesh constructed with 10 equal subintervals in the x and y dimensions and 2 subdivisions in the z dimension. The finite dimensional retrieval operator associated with a given matrix q of radar sites is determined from (2.23) and (3.5) and the associated norm mapping $\Lambda(q)$ given in (3.6) and (3.8) is calculated. The minimization procedure presented here is a simple brute force minimization in which different (1000 in this computation) configurations of sites are randomly generated and the associated $\Lambda(q)$ is calculated. In Figure 2, values of $\Lambda(q)$ are portrayed for 100 randomly generated network configurations containing 5 radars in which 4 sites are randomly chosen and one site is fixed at (75,75). Determining the locations q_o minimizing the general GUM criterion (actually over the collection of simulated test site configurations), a function $\Lambda(x, q_o)$ is computed for $x \in \Omega$ that gives the value of norm of pointwise GUM. The level contours of this function are portrayed in the first graph in Figure 3. The second graph shows the coverage functions corresponding to the radar sites. This is simply a function that is 1 at a point if it is within the coverage radius of some radar and zero otherwise. Finally, Figure 4 shows the results of a Doppler angle area computation maximizing the functional J_d described in Section 4. This computation is based on 100,000 randomly chosen radar locations in which, again, one location is fixed at (75,75). The contour plot corresponds to the level curves of the function $\widehat{C}(\mathbf{x})$ based on a 100×100 grid. These level curves consist of those points satisfying the same angle condition. We note that the maximizing Doppler angle area depicted in Figure 4 is more conservative than that obtained from GUM in the sense that it is smaller than that obtained from the minimization of the GUM.

Finally, the brute force simulation that is carried out obtains maxima with respect to the ensemble of generated configurations. In Table 1, the maximum Doppler angle area is shown as a function of the number of simulations. Also, the relative change is shown in the third column. We note that the maximum Doppler area stabilizes as the number of simulated network configurations increases.

Table 1. Maximum Doppler angle area dependence on the number of radar network simulations for a five radar network.

Simulations	Area	Relative change
1000	3791	—
10,000	3874	0.022
100,000	4052	0.0460
200,000	4091	0.0072
500,000	4097	0.0015

6. Conclusions.

In this work, we have focused on the so-called geometric uncertainty multiplier (GUM) that is associated with wind retrieval from radar data. The GUM depends continuously on the location of radar sites within a network. Hence, locations of sites within the network may be determined that minimize the GUM associated with the network over the whole domain, or a portion thereof. Alternatively, by considering the GUM as a function defined over the domain, problems may be formulated to maximize the measure of the subsets in which the GUM is below a certain threshold. We also demonstrate that the GUM is a generalization of the Doppler angle condition used to determine Doppler regions associated with radar. Numerical examples are presented using a basic brute force random search method to minimize the global GUM for a network of five radars. A simple example maximizing the area of the Doppler for five radar is also presented. The region obtained by maximizing the area of the Doppler angle region is smaller than that obtained by minimizing the GUM.

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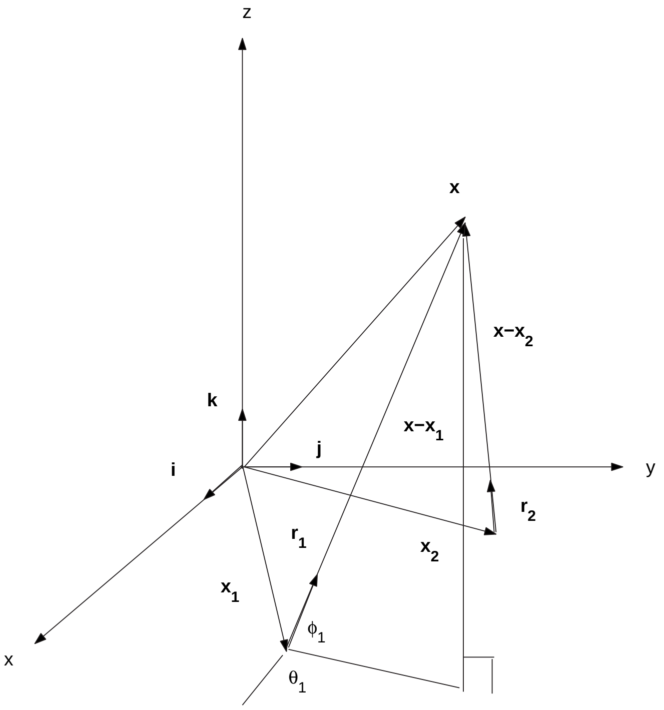

Figure 1: Three Dimensional Set Up for Two Radar

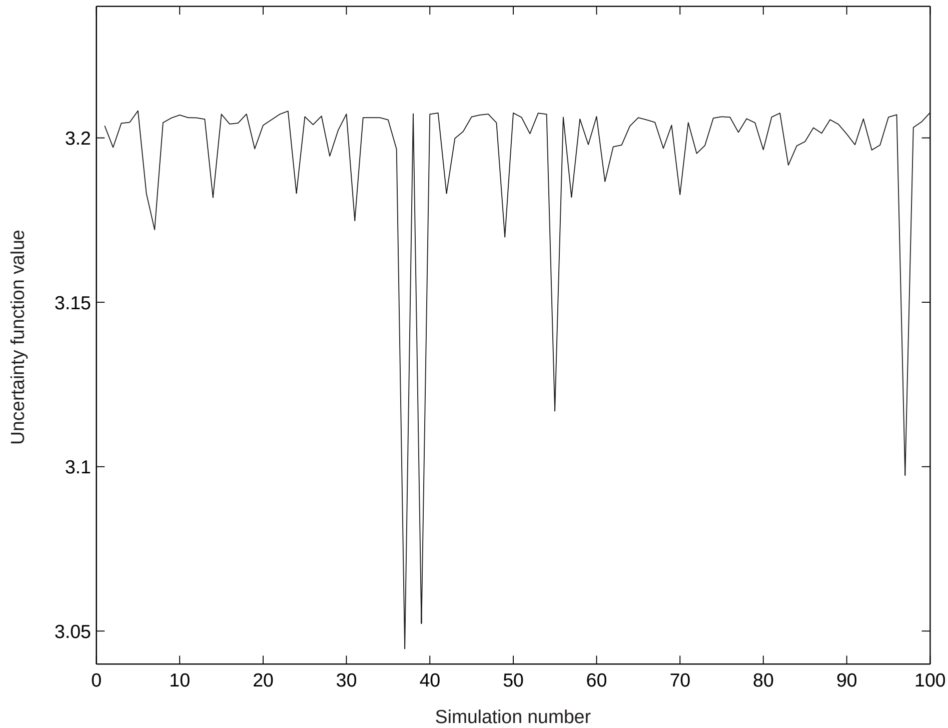

Figure 2: Global Uncertainty Multiplier Function for 100 Radar Locations

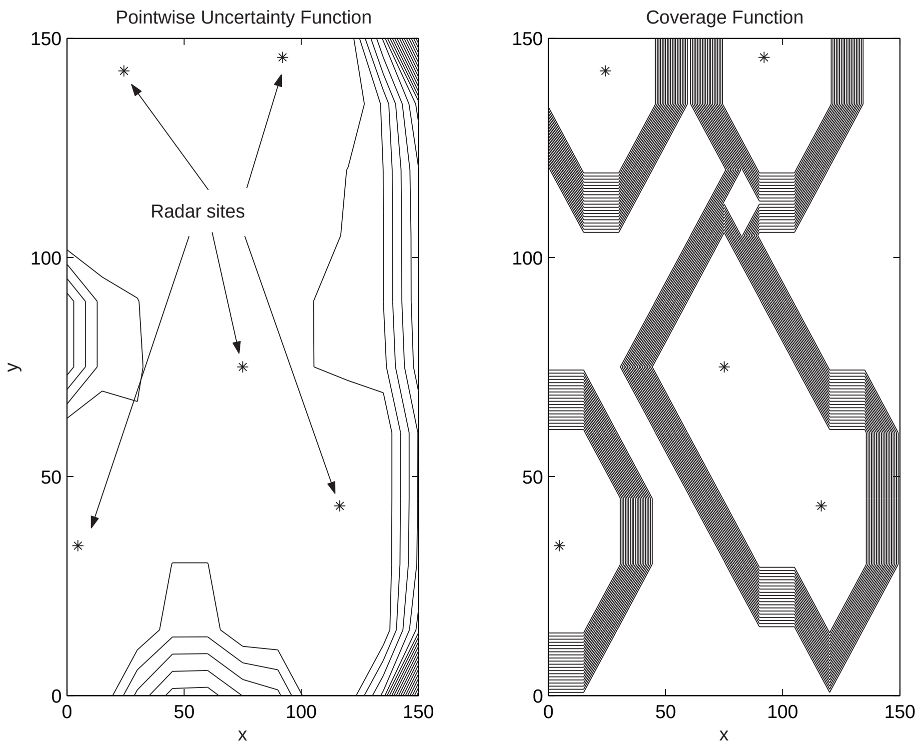

Figure 3: Comparison of Contours for Optimal Pointwise Uncertainty Factor and Coverage Function

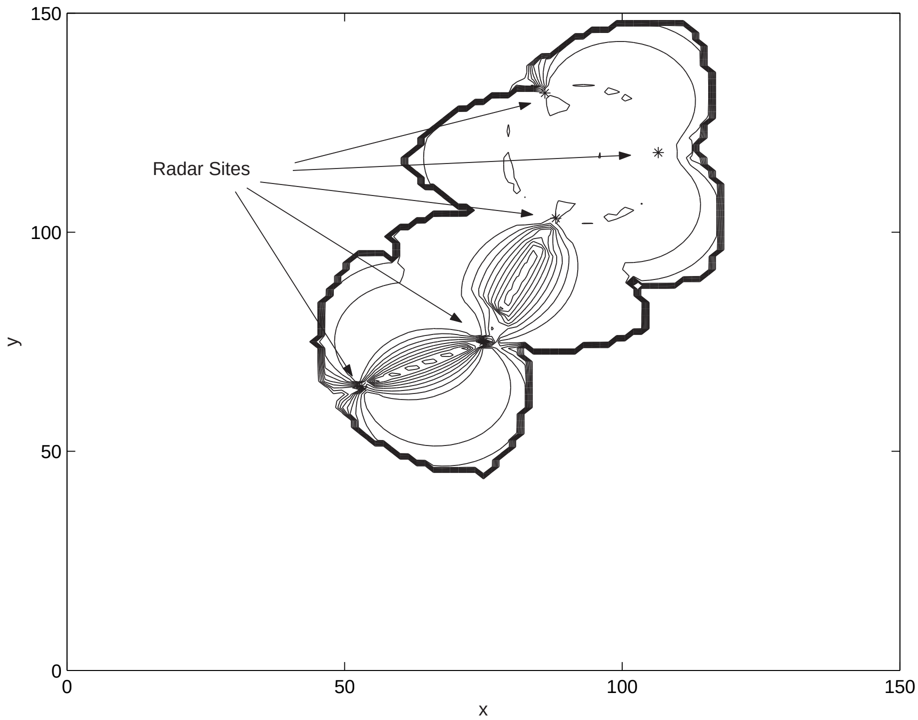

Figure 4: Contours for Maximized Doppler Area