Introduction to Differential Equations: Exam 3

Name Key

ID#

1. a. Show that the Laplace transform of the derivative of a function defined on $[0,+\infty)$ and that satisfies $|f(t)| \le M$ satisfies $L(f') = sL(f) - f(0)$.

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$$L(f') = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{b\to\infty} e^{-st} f(t) \Big|_0^b + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{b\to\infty} \left[e^{-sb} f(b) - f(0) \right] + s\,L(f)$$

$$L(f') = s\,L(f) - f(0)$$

b. Solve the following initial value problem using the Laplace transform

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$$x'' + 4x = 0, x(0) = 1, x'(0) = 0.$$

$$L(x'') = s\,L(x') - x'(0)$$
$$= s\,L(x')$$
$$= s\,(s\,L(x) - x(0)) = s^2 X - s$$

$$s^2 X - s + 4X = 0$$
$$(s^2+4)X = s$$
$$X = \frac{s}{s^2+4}$$
$$x(t) = \cos(2t)$$

2. Find eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix}$.

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$$\det \begin{bmatrix} -4-\lambda & -2 \\ 1 & -1-\lambda \end{bmatrix} = 0$$

$$(\lambda+1)(\lambda+4)+2 = 0$$

$$\lambda^2+5\lambda+6=0$$

$$(\lambda+3)(\lambda+2)=0$$

$$\lambda=-3,\ \lambda=-2$$

$\lambda=-3$: $\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x+2y=0$$
$$x=-2y$$

$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda=-2$

$\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x+y=0$$
$$y=-x$$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3.a. Show that the system $x_1' = 4x_1 - 2x_2$, $x_2' = x_1 + x_2$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ with eigenvectors $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, respectively.

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$$x_1' = 4x_1 - 2x_2$$
$$x_2' = x_1 + x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det \begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\lambda-1)(\lambda-4)+2=0$$

$$\lambda^2-5\lambda+6=0$$

$$(\lambda-2)(\lambda-3)=0$$

$$\lambda=2,\ \lambda=3$$

$\lambda=2$

$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y=x$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda=3$: $\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x=2y$ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

b. Find a general solution of the system.

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$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x_1(t) = c_1 e^{2t} + 2c_2 e^{3t}$$
$$x_2(t) = c_1 e^{2t} + c_2 e^{3t}$$

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c. Find the solution of the initial value problem with $x_1(0) = 1$ and $x_2(0) = 0$.

$X_1(0) = C_1 + 2C_2 = 1$

$x_2(0) = C_1 + C_2 = 0$

$C_2 = -C_1$

$C_1 - 2C_1 = 1$

$C_1 = -1$

$C_2 = 1$

$x_1(t) = -e^{2t} + 2e^{3t}$

$x_2(t) = -e^{2t} + e^{3t}$

$x_1' = x_1 + x_2, \quad x_2' = -4x_1 + x_2$

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4. a. Show that the system $x_1' = x_2$, $x_2' = -4x_1$ has an eigenvalue $\lambda = 1 + 2i$ with eigenvector $u = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\det \begin{bmatrix} 1-\lambda & 1 \\ -4 & 1-\lambda \end{bmatrix} = 0$

$(\lambda - 1)^2 + 4 = 0$

$\lambda = 1 \pm 2i$

$\begin{bmatrix} 1-1-2i & 1 \\ -4 & 1-1-2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2ix + y = 0$

$y = 2ix$

$\vec{u} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$

10 b. Find a general solution of the system.

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{t} e^{2it}$$

$$= e^{t} \begin{bmatrix} 1 \\ 2i \end{bmatrix} (\cos 2t + i \sin 2t)$$

$$= e^{t} \begin{bmatrix} \cos 2t + i \sin 2t \\ 2i \cos 2t - 2 \sin 2t \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{t} \begin{bmatrix} \cos 2t \\ -2 \sin 2t \end{bmatrix} + c_2 e^{t} \begin{bmatrix} \sin 2t \\ 2 \cos 2t \end{bmatrix}$$

10 c. Find the solution of the initial value problem with $x_1(0) = 1$ and $x_2(0) = 0$.

$$x_1(t) = c_1 e^{t} \cos 2t + c_2 e^{t} \sin 2t$$

$$x_2(t) = -2 c_1 e^{t} \sin 2t + 2 c_2 e^{t} \cos 2t$$

$x_1(0) = 1 = c_1$

$x_2(0) = 0 = c_2$

$x_1(t) = e^{t} \cos 2t$

$x_2(t) = -2 e^{t} \sin 2t$

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5. a. Write the equation $x'' + 2x' + x = 0$ as a system of first order equations.

$x_1 = x$

$x_2 = x_1' = x'$

$x_2' + x_2 + x_1 = 0$

$x_1' = x_2$

$x_2' = -x_1 - x_2$

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b. Write the system $x_1' = 2x_1 - x_2$, $x_2' = x_1 + x_2$ as a linear second order equation for x_1.

$x_1'' = 2x_1' - x_2'$

$-x_1' + 2x_1 = x_2$

$x_2' = x_1 + (-x_1' + 2x_1)$

$= -x_1' + 3x_1$

$x_1'' = 2x_1 - (-x_1' + 3x_1)$

$x_1'' - x_1' + x_1 = 0$

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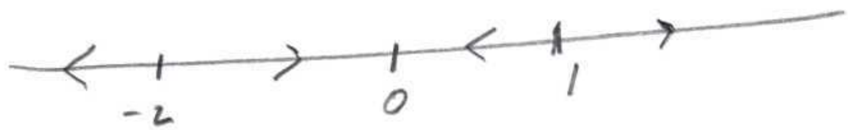
6. Sketch the phase portrait for the equation $x' = F(x) = x(x-1)(x+2)$. Label the equilibrium points (where $F(x) = 0$), indicate the flow directions, and which equilibrium points are stable and unstable.



Equilibrium points: $x = 0, x = 1, x = -2$

unstable: $x = -2, x = 1$

stable: $x = 0$