CLOSED ORBITS OF SEMISIMPLE GROUP ACTIONS AND THE REAL HILBERT-MUMFORD FUNCTION

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Abstract The action of a noncompact semisimple Lie group G on a finite dimensional real vector space V is said to be *stable* if there exists a nonempty Zariski open subset O of V such that the orbit G(v) is closed in V for all v $\in$ O. We study a Hilbert-Mumford numerical function M : V $\to \mathbb{R}$ defined by A. Marian that extends the corresponding function in the complex setting defined by D. Mumford and studied further by G. Kempf and L. Ness. The G-action may be stable on V if M ≥ 0 on V, as in the adjoint action of G on its Lie algebra $\mathfrak{G}$. However, we show that the G-action on V is always stable if M(v) < 0 for some v $\in$ V. We show that M(v) $< 0 \Leftrightarrow$ the orbit G(v) is closed in V and the stability subgroup G_v is compact. The subset of V where M is negative is open in the vector space topology of V but not necessarily open in the Zariski topology of V. We give criteria for M to be negative on a nonempty Zariski open subset of V, and we consider several examples.

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INTRODUCTION

Let G be a semisimple algebraic group in GL(V), where V is a finite dimensional real vector space. We study the closed orbits of G in V, primarily through a function M : V $\to \mathbb{R}$ introduced by Mumford for complex varieties and extended to the real setting by A. Marian [Ma]. The function M is semicontinous, invariant under G and takes on finitely many values. The points v where M(v) is negative are particularly interesting, and these points v occur precisely when G(v) is closed in V and the stability group G_v is compact. The set of vectors v where M(v) is negative is open in the vector space topology but not necessarily Zariski open as we show for the adjoint representation of a noncompact

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semisimple Lie group. In this case, M is negative somewhere on the Lie algebra $\mathfrak{G} \Leftrightarrow$ some maximal compact subgroup of G contains a maximal abelian subgroup of G. Equivalently, for an element X of $\mathfrak{G}$, M(X) is negative $\Leftrightarrow$ the stability group G_X is compact. In particular ad X : $\mathfrak{G} \to \mathfrak{G}$ has purely imaginary eigenvalues, so M can never be negative on a nonempty Zariski open subset of $\mathfrak{G}$. Moreover, the stability groups G_X have positive dimension for all X $\in \mathfrak{G}$. By contrast, in the complex setting M(v) is negative $\Leftrightarrow$ G(v) is closed and G$_v$ is discrete, and these two conditions hold on a nonempty Zariski open subset .

We say that v $\in$ V is a $stable$ point of the G action if M(v) < 0. In addition to implying that G(v) is closed the condition M(v) < 0 also implies that H(v) is closed for any closed subgroup H of G. This property does not hold in general if G(v) is closed and M(v) $= 0$ as we show by example at the end of section 3.

We say that G acts $stably$ on V if there is a nonempty Zariski open subset O of V such that the orbit G(v) is closed in V for all v $\in$ O. It is well known that G acts stably on its Lie algebra $\mathfrak{G}$ in the adjoint representation. If M is negative somewhere on V, then G acts stably on V, and there is a nonempty subset O of V, open in the vector space topology, such that M(v) < 0 and the stability group G$_v$ is compact for all v $\in$ O. Conversely, if one stability group G_v is compact, then M is negative somewhere on V. If one stability group G_v is discrete, then G acts stably on V, and M is negative on a nonempty Zariski open subset of V.

$Remark$ The problem of stability for reductive subgroups has also been considered in Theorem 4 of [Vin]. There it is shown that if a G-action is stable for a reductive group G, then the H-action of any reductive subgroup H is also stable.

There are other distinctions between the complex and real settings for linear actions that are captured by the function M : V $\to \mathbb{R}$. In the complex setting the stability groups for linear actions are conjugate on a nonempty Zariski open set. In the real case the stability groups may be quite different topologically although their Lie algebras have the same complexification on a nonempty Zariski open set. This is illustrated by the adjoint representation. If O is the nonempty Zariski open subset of $\mathfrak{G}$ consisting of those vectors X such that $\mathfrak{G}_X$has minimum dimension, then G(X) is closed for all X $\in$ O. Moreover, for X $\in$ O either M(X) $= 0$ and G_X is noncompact or M(X) < 0 and G_X is compact. The first case always occurs, but the second case occurs only under the conditions discussed above. In the simplest case, where G $= SL(2,\mathbb{R})$, we have the following possibilities for X $\in$ O : a) det X > 0, M(X) < 0 and the stability group G_X is a circle or b) det X < 0, M(X) $= 0$ and the stability group G_X is a homeomorphic to a line.

For the adjoint representation there is a further stratification of the vectors X in O for which M(X) $= 0$. Let $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ denote the Cartan decomposition of $\mathfrak{G}$ into the $+1$ and -1 eigenspaces of a Cartan involution θ of $\mathfrak{G}$. Let rank $\mathfrak{P}$ denote the dimension of a maximal abelian subspace of $\mathfrak{P}$, and let rank $\mathfrak{G}$ denote the dimension of a Cartan subalgebra of $\mathfrak{G}$ (i.e. a maximal abelian subalgebra whose elements are ad semisimple). For every integer r with $1 \leq r \leq rank$ $\mathfrak{P}$, there exists a subset O$_r$ of O such that O$_r$ is open in the vector space topology of $\mathfrak{G}$ and for every X $\in O_r$ it follows that M(X) $= 0$ and $(G_X)_0$ is homeomorphic to $\mathbb{R}^r$ x $T^{(rank\ \mathfrak{G} - r)}$. Here T^p denotes the p-torus for any positive integer p.

In studying the closed orbits of G acting on V we make use of the notion of $minimal\ vector$ for the G-action, which is discussed by Ness in the complex setting in [KN] and [Nes] and is extended to the real setting by Richardson and Slodowy in [RS]. An orbit G(v) is closed in V $\Leftrightarrow$ G(v) intersects the set $\mathfrak{M}$ of minimal vectors, and in this case G(v) $\cap\ \mathfrak{M}$ is a single K orbit, where K is a maximal compact subgroup of G.

In the course of this article we develop sufficient conditions for M to be negative on V, including negative on a nonempty Zariski open subset of V. We study the M function for several examples in addition to the adjoint representation.

1. THE MOMENT MAP AND MINIMAL VECTORS

1.1. **Definitions and basic properties.** In this article we consider the closed orbits of a semisimple group G acting on a finite dimensional real vector space V. More precisely let $G^{\mathbb{C}}$ denote a semisimple algebraic subgroup of GL(n,$\mathbb{C}$) defined over $\mathbb{R}$, and let $G^{\mathbb{C}}(\mathbb{R})^0$ denote the identity component in the classical topology of the real Lie group $G^{\mathbb{C}}(\mathbb{R}) = G^{\mathbb{C}} \cap GL(n, \mathbb{R})$. In the sequel G will denote a closed subgroup of $G^{\mathbb{C}}(\mathbb{R})$ that contains $G^{\mathbb{C}}(\mathbb{R})^0$ and is Zariski dense in $G^{\mathbb{C}}$. These are the hypotheses of Richardson-Slodowy [RS]. This article is an outgrowth of [RS] and [Ma], and these two works are extensions to the real case of the work of G. Kempf and L. Ness ([KN],[Nes]) and D.Mumford ([Mu]).

Remark If G is a semisimple subgroup of GL(n,$\mathbb{R}$) with finitely many connected components, then G satisfies the conditions stated above.

We show this first in the case that G is connected. Since $\mathfrak{G}$ is semisimple it is algebraic in the sense of Chevalley ; that is, there exists a real algebraic group H $\subset$ GL(n,$\mathbb{R}$) whose Lie algebra is $\mathfrak{G}$. (See pp. 171-185 of [C] or pp. 105-110 of [Bor] for further details.) If H^0 and H_0 denote respectively the Hausdorff and Zariski components of H that contain the identity, then $G = H^0 \subset H_0$ since G is connected in both the Hausdorff and Zariski topologies. Let $G^{\mathbb{C}}$ denote the Zariski closure of H_0 in GL(n,$\mathbb{C}$), and let $\mathfrak{G}^{\mathbb{C}}$ denote the complexification of $\mathfrak{G}$. Then $G^{\mathbb{C}}$ is defined over $\mathbb{R}$, and $L(G^{\mathbb{C}}) = \mathfrak{G}^{\mathbb{C}}$ by Proposition 2 of [C, Chapter II, section 8]. If $\overline{G}$ denotes the Zariski closure of G in GL(n,$\mathbb{C}$), then $\overline{G} \subset G^{\mathbb{C}}$, and $\overline{G}$ is a connected algebraic group defined over $\mathbb{R}$ (cf. [Bor, Chapter I, section 2.1]). Moreover, $L(\overline{G}) = \mathfrak{G}^{\mathbb{C}}$ since $\mathfrak{G}^{\mathbb{C}} \subset L(\overline{G}) \subset L(\overline{H_0}) = \mathfrak{G}^{\mathbb{C}}$. Hence $\overline{G} = G^{\mathbb{C}}$ since both groups are Zariski connected, defined over $\mathbb{R}$ and have Lie algebra $\mathfrak{G}^{\mathbb{C}}$ (cf. [Bor, Chapter II, section 7.1]). Finally, if $G^{\mathbb{C}}(\mathbb{R})$ denotes $G^{\mathbb{C}} \cap GL(n, \mathbb{R})$, then $L(G^{\mathbb{C}}(\mathbb{R})) = L(G^{\mathbb{C}}) \cap L(GL(n, \mathbb{R})) = \mathfrak{G}$ by [Bor, Chapter II, section 7.1]. We conclude that $G = G^{\mathbb{C}}(\mathbb{R})^0$ since both groups are Hausdorff connected with Lie algebra $\mathfrak{G}$.

Next, suppose that $G = \bigcup_{\alpha\in A} g_\alpha G^0$, where A is a finite set, and let $G^{\mathbb{C}} = \overline{G} = \bigcup_{\alpha\in A} g_\alpha H$, where $H = \overline{G^0} \subset GL(n, \mathbb{C})$. Hence $H = G_0^{\mathbb{C}}$ since H is Zariski connected, and $L(G^{\mathbb{C}}) = L(H) = \mathfrak{G}^{\mathbb{C}}$ by the discussion above. Clearly $G^0 \subset G^{\mathbb{C}}(\mathbb{R})^0$ and equality holds since both connected Lie groups have the same Lie algebra $\mathfrak{G}$. Hence $G^{\mathbb{C}}(\mathbb{R})^0 = G^0 \subset G \subset G^{\mathbb{C}}$. This completes the remark.

Now, let $G^{\mathbb{C}}(\mathbb{R}) \subset GL(n, \mathbb{R})$ satisfy the basic conditions stated above. By a result from section 7 of [Mo2] there exists an inner product $\langle , \rangle_0$ on $\mathbb{R}^n$ such that $G^{\mathbb{C}}(\mathbb{R})$ is self adjoint, that is, invariant under the involution $\theta_0 : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ given by $\theta_0(g) = (g^t)^{-1}$, where g^t denotes the metric transpose of g. If $\mathfrak{G}$ denotes the Lie algebra of $G^{\mathbb{C}}(\mathbb{R})$, which is also the Lie algebra of G, then θ_0 defines a Lie algebra automorphism of $\mathfrak{G}$, also denoted by θ_0, which is called a *Cartan involution* of $\mathfrak{G}$. Let $\mathfrak{K}_0, \mathfrak{P}_0$ denote respectively the $+1$ and -1 eigenspaces of $\theta_0 : \mathfrak{G} \to \mathfrak{G}$. It is easy to see that the elements of $\mathfrak{K}_0$ and $\mathfrak{P}_0$ are skew symmetric and symmetric elements respectively of End($\mathbb{R}^n$). It follows that $\mathfrak{K}_0$ is the Lie algebra of the maximal compact subgroup $K = Fix(\theta_0) = G^{\mathbb{C}}(\mathbb{R}) \cap O(n, \mathbb{R})$, and $K_0 \subseteq K \cap G$. See (2.2) of [RS]. Let $\langle , \rangle_{\mathfrak{G}}$ be any Ad K invariant inner product on $\mathfrak{G}$; for example, let $\langle X, Y \rangle_{\mathfrak{G}} = -B(\theta_0(X), Y)$, where B is the Killing form of $\mathfrak{G}$.

If G $\subset$ GL(n,$\mathbb{R}$) is a real algebraic group, then a representation $\rho : G \to GL(V)$ is said to be *rational* if $f \circ \rho$ is a polynomial function with real coefficients on GL(n,$\mathbb{R}$) whenever f is a polymonial function with real coefficients on GL(V). Let V be a finite dimensional real vector space, and let $\rho : G^{\mathbb{C}}(\mathbb{R}) \to GL(V)$ be a rational representation. Then $\rho(G^{\mathbb{C}}(\mathbb{R}))$ is an algebraic group in GL(V) and $\rho(G)$ satisfies the hypotheses above. The remarks of the previous paragraph now extend to $\rho(G)$ equipped with an inner product $\langle , \rangle$ and corresponding involution $\theta : GL(V) \to GL(V)$ such that $\rho(G^{\mathbb{C}}(\mathbb{R}))$ is θ-stable and $\theta \circ \rho = \rho \circ \theta_0 : G^{\mathbb{C}}(\mathbb{R}) \to GL(V)$. The existence of $\langle , \rangle$ and θ follows from section 7 of [Mo2] and (2.3) of [RS]. If we let θ, ρ and θ_0 also denote the differentials of these homomorphisms, then $\theta \circ \rho = \rho \circ \theta_0 : \mathfrak{G} \to End(V)$, where $\mathfrak{G}$ is the Lie algebra of G and $G^{\mathbb{C}}(\mathbb{R})$. If $\mathfrak{K}$ and $\mathfrak{P}$ denote the +1 and -1 eigenspaces of θ on $\rho(\mathfrak{G})$, then $\rho(\mathfrak{K}_0) = \mathfrak{K}$ and $\rho(\mathfrak{P}_0) = \mathfrak{P}$. As above, the elements of $\mathfrak{K}$ and $\mathfrak{P}$ act on V by skew symmetric and symmetric linear maps respectively.

In the sequel, by abuse of notation, we shall assume the framework above and we shall identify G and $G^{\mathbb{C}}(\mathbb{R})$ with their images $\rho(G)$ and $\rho(G^{\mathbb{C}}(\mathbb{R}))$ in GL(V).

The moment map

If $X \in \mathfrak{K}$ and v $\in$ V, then $\langle X(v), v\rangle = 0$ by the skew symmetry of X. If v $\in$ V is fixed, then for X $\in \mathfrak{P}$ the map $X \to \langle X(v), v\rangle$ is an element of $\mathfrak{P}^*$, which may be identified with $\mathfrak{P}$ by means of the inner product $\langle , \rangle$. We obtain a map $m : V \to \mathfrak{P}$ defined by the condition $\langle m(v), X\rangle_{\mathfrak{G}} = \langle X(v), v\rangle$ for v $\in$ V and $X \in \mathfrak{P}$. The map m is called the *moment map*. See [Ma] for a justification of this terminology. It follows from the definitions that m is a homogeneous polynomial function of degree two such that m(kv) = Ad(k)(m(v)) for all v $\in$ V and all k $\in$ K.

Remark

Let G be a self adjoint subgroup of GL(V) that is a direct product G_1 x G_2 of self adjoint subgroups. If $\mathfrak{P}_1, \mathfrak{P}_2$ and $\mathfrak{P}$ are the -1 eigenspaces of θ in $\mathfrak{G}_1, \mathfrak{G}_2$ and $\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$ respectively, then $\mathfrak{P} = \mathfrak{P}_1 \oplus \mathfrak{P}_2$. Moreover, it follows from the definitions that $m(v) = m_1(v) + m_2(v)$ for v $\in$ V, where $m : V \to \mathfrak{P}, m_1 : V \to \mathfrak{P}_1$ and $m_2 : V \to \mathfrak{P}_2$ are the moment maps for G, G_1 and G_2 respectively.

Examples of moment maps

Example 1. Let G = $SL(q, \mathbb{R})$ and V = $\mathfrak{so}(q, \mathbb{R})^p := \mathfrak{so}(q, \mathbb{R}) \oplus \ \dots \ \oplus \mathfrak{so}(q, \mathbb{R})$ (p times). Let G act diagonally on V by $g(C^1, \ \dots \ , C^p) = (gC^1g^t, \ \dots \ gC^pg^t)$. The Lie algebra $\mathfrak{G}$ acts on V by $X(C^1, \ \dots \ C^p) = (XC^1 + C^1X^t, \ \dots \ , XC^p + C^pX^t)$. On V we define the inner product $\langle (C^1 \ \dots \ C^p), (D^1 \ \dots \ D^p)\rangle = -\sum_{i=1}^p trace \ C^iD^i$. It is easy to check that G is self adjoint with respect to this inner product on V. Moreover, $\mathfrak{K} = \mathfrak{so}(q, \mathbb{R})$ and $\mathfrak{P} = \{X \in \mathfrak{G} : X = X^t\}$.

Assertion If C = $(C^1 \ \dots \ C^p) \in V$, then m(C) = $-2\sum_{i=1}^p (C^i)^2 - \lambda(C) \ Id$,where $\lambda(C) = \frac{2\,|C|^2}{q}$.

Let X $\in \mathfrak{P}$ and C $\in$ V be given. Extend the inner product $\langle , \rangle$ on $\mathfrak{so}(q, \mathbb{R})$ to $\mathfrak{G}$ by $\langle \zeta, \eta\rangle = trace(\zeta\eta^t)$ for all $\zeta, \eta \in \mathfrak{G}$. Then $\langle m(C), X\rangle = \langle X(C), C\rangle = -\sum_{r=1}^p trace \ (XC^r + C^rX)(C^r) = -2\sum_{r=1}^p trace \ X(C^r)^2 = \langle X, -2 \ \sum_{r=1}^p (C^r)^2\rangle = \langle X, -2 \ \sum_{r=1}^p (C^r)^2 - \lambda(C) \ Id\rangle$. This proves the assertion since $-2\sum_{i=1}^p (C^i)^2 - \lambda(C) \ Id$ is symmetric with trace zero and hence belongs to $\mathfrak{P}$.

Example 2. Let V = $\mathfrak{so}(q, \mathbb{R})^p$ as in the first example, and observe that V is isomorphic to $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ under the map $C = (C^1, \ \dots \ , C^p) \to \sum_{i=1}^p C^i \otimes e_i$,where $\{e_i\}$ is the standard basis of $\mathbb{R}^p$. Let G = G_1 x G_2, where $G_1 = SL(q, \mathbb{R})$ and $G_2 = SL(p, \mathbb{R})$, and let G act on V by $(g_1, g_2)(\sum_{i=1}^p C^i \otimes e_i) = \sum_{i=1}^p (g_1C^ig_1^t) \otimes g_2(e_i)$. Here G_2 acts on $\mathbb{R}^p$ in the standard fashion. The previously defined inner product $\langle , \rangle$ on V = $\mathfrak{so}(q, \mathbb{R})^p$ now becomes the unique inner product on V = $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ such that $\langle C \otimes v, D \otimes w\rangle = \langle C, D\rangle\langle v, w\rangle$ for C,D $\in \mathfrak{so}(q, \mathbb{R})$ and v,w $\in \mathbb{R}^p$. Here $\langle C, D\rangle = -trace(CD)$ and $\langle , \rangle$ is the standard inner product on $\mathbb{R}^p$ for which the standard basis $\{e_i\}$ is orthonormal.

Note that $\mathfrak{P} = \mathfrak{P}_1 \oplus \mathfrak{P}_2$ and the moment map m : $V \to \mathfrak{P}$ becomes m(C) = (m_1(C),m_2(C)), where $m_i \to \mathfrak{P}_i$ is the moment map for G_i for i= $1, 2$.

$Assertion$ For C $= (C^1 \ldots C^p) \in V$, let $\lambda(C) = \frac{2\,|C|^2}{q}$ and let $\mu(C) = \frac{|C|^2}{p}$. Let $m_2^*(C)$ be the element of $\mathfrak{P}_2$ such that $m_2^*(C)_{ij} = \langle C^i, C^j\rangle$. Then $m_1(C) = -2\sum_{i=1}^{p}(C^i)^2 - \lambda(C)\ Id$, and $m_2(C) = m_2^*(C) - \mu(C)\ Id$.

The statement for $m_1(C)$ was proved above in the discussion of the first example. If Y $\in \mathfrak{P}_2$ and $C = \sum_{i=1}^{p} C^i \otimes e_i \in$ V are given, then $\langle m_2(C), Y\rangle = \langle Y(C), C\rangle = \langle \sum_{i=1}^{p} C^i \otimes Y(e_i),$
$\sum_{j=1}^{p} C^j \otimes e_j\rangle = \sum_{i,j=1}^{p} \langle C^i, C^j\rangle\langle Y(e_i), e_j\rangle = trace\ m_2^*(C)Y = trace\ (m_2^*(C) - \mu(C)\ Id)Y = \langle (m_2^*(C) - \mu(C)\ Id), Y\rangle$. The assertion for m_2(C) follows since $m_2^*(C) - \mu(C)\ Id$ has trace zero and hence belongs to $\mathfrak{P}_2$.

Example 3. Let V $= M(n, \mathbb{R})$, the n x n matrices with real entries, and let G $= SL(n, \mathbb{R})$ act on V by conjugation.

$Assertion$ For C $\in$ V, m(C) $= CC^t - C^tC$.

The action of $\mathfrak{G}$ on V is given by X(C) $= XC - CX$ for X $\in \mathfrak{G}$ and C $\in$ V. For X $\in \mathfrak{P}$ and C $\in$ V we compute $\langle m(C), X\rangle = \langle X(C), C\rangle = trace(XC - CX)C^t = traceX(CC^t - C^tC) = \langle X, CC^t - C^tC\rangle$. The assertion follows since $CC^t - C^tC$ is symmetric with trace zero and hence belongs to $\mathfrak{P}$.

$Minimal\ vectors$

A vector v of V is called $minimal$ if $m(v) = 0$. We denote the set of minimal vectors in V by $\mathfrak{M}$. Note that $\mathfrak{M}$ is invariant under K by the Ad K equivariance of the moment map m. We recall some results from [RS]. The next two results are restatements of Theorem 4.3 of [RS].

Proposition 1.1. *The following conditions are equivalent for a vector v of V :*

1) v is minimal

2) The identity $1 \in G$ *is a critical point of the function* $F_v : G \to V$ *given by* $F_v(g) = |g(v)|^2$ *for all g* $\in$ *G.*

3) The identity $1 \in G$ *is a minimum point of the function* $F_v : G \to V$.

If v $\in$ *V is minimal, then* G_v *is self adjoint. In particular* $\mathfrak{G}_v = \mathfrak{P}_v \oplus \mathfrak{K}_v$, *where* $\mathfrak{G}_v$ *denotes the Lie algebra of* G_v, $\mathfrak{K}_v = \mathfrak{G}_v \cap \mathfrak{K}$ *and* $\mathfrak{P}_v = \mathfrak{G}_v \cap \mathfrak{P}$.

Proposition 1.2. *For v* $\in$ *V the orbit G(v) is closed in V* $\Leftrightarrow$ *G(v) contains a minimal vector. If* $w \in G(v) \cap \mathfrak{M}$ *for some v* $\in$ *V, then* $G(v) \cap \mathfrak{M} = K(w)$.

$Remark$ It may be the case that $\{0\}$ is the only minimal vector.

Corollary 1.3. *There is a bijection between the closed orbits of G in V and the space* $\mathfrak{M}/K$.

Proof. Given a closed orbit G(v) for some v in V we associate the point (G(v) $\cap\, \mathfrak{M})/K \in \mathfrak{M}/K$. This map is a well defined bijection by the preceding result. ☐

Corollary 1.4. *Let G(v) be closed for v* $\in$ *V, v* $\neq 0$*. Then* G_v *is completely reducible.*

Proof. By (1.2) there exists g $\in$ G such that w = g(v) is minimal. By (1.1) $G_w = g\, G_v\, g^{-1}$ is self adjoint, hence reductive. It suffices to show that G_w is completely reducible since G_v is conjugate to G_w. To show that G_w is completely reducible it suffices by Theorem 4 in section 6.5 of [Bou] to show that if X $\in \mathfrak{Z}_w$, the center of $\mathfrak{G}_w$, then X : V $\to$ V is semisimple. Note that $\mathfrak{Z}_w$ is θ-invariant since $\mathfrak{G}_w$ is θ-invariant. Let X $\in \mathfrak{Z}_w$ be given, and write X = K + P, where $K = (1/2)(X + \theta(X)) \in \mathfrak{K} \cap \mathfrak{Z}_w$ and $P = (1/2)(X - \theta(X)) \in \mathfrak{P} \cap \mathfrak{Z}_w$. The elements K and P are respectively skew symmetric and symmetric on V, and as elements of $\mathfrak{Z}_w$ they commute. Hence X = K + P is semisimple on V. ☐

The next result is stated in section (7.2) of [RS]

Corollary 1.5. *If G(v) is not closed in V for some v $\in$ V, then $\overline{G(v)}$ contains a unique closed orbit of G.*

The next result is Lemma 3.3 of [RS]

Proposition 1.6. *Let v $\in$ V and assume that G(v) is not closed. Then there exists X $\in \mathfrak{P}$ and $v_0 \in$ V such that $e^{tX}(v) \to v_0$ as t $\to \infty$ and the orbit G(v_0) is closed.*

Rank of the moment map

For $\xi, v \in$ V let $\xi_v \in T_vV$ denote $\alpha'(0)$, where $\alpha(t) = v + t\xi$. Similarly for X $\in \mathfrak{P}$ we define $X_{m(v)} \in T_{m(v)}\mathfrak{P}$.

Proposition 1.7. *Let X $\in \mathfrak{P}$ be given. Then $X_{m(v)}$ is orthogonal to $m_*(T_vV) \Leftrightarrow X(v) = 0$. In particular,*

a) The rank of m at v = $dim\ \mathfrak{P} - dim\ \mathfrak{P}_v$.

b) The moment map m : V $\to \mathfrak{P}$ fails to have maximal rank at a point v of V $\Leftrightarrow X(v) = 0$ for some nonzero element X $\in \mathfrak{P}$.

Proof. Fix v $\in$ V. For $\xi \in$ V and X $\in \mathfrak{P}$ we compute $\langle m_*(\xi_v), X_{m(v)}\rangle = \frac{d}{dt}|_{t=0}\langle m(v + t\xi), X\rangle = \frac{d}{dt}|_{t=0}\langle X(v + t\xi), v + t\xi\rangle = \langle X(v), \xi\rangle + \langle X(\xi), v\rangle = 2\langle X(v), \xi\rangle$. The result follows since $\xi \in$ V is arbitrary. □

Corollary 1.8. *Suppose that G_v is a compact subgroup of G for some v $\in$ V. Then there exists a nonempty Zariski open subset O of V such that m : V $\to \mathfrak{P}$ has maximal rank at every v $\in$ O.*

Proof. If O $= \{x \in V : m \text{ has maximal rank at x}\}$, then O is a Zariski open subset of V. Let G_v be compact for some nonzero v $\in$ V. We show that v $\in$ O by showing that $\mathfrak{P}_v = \{0\}$ and applying (1.7). Let X(v) = 0 for some X $\in \mathfrak{P}$. The eigenvalues of elements of G_v have modulus 1 since G_v leaves invariant some inner product on V. However, X is symmetric on V with real eigenvalues λ, and the eigenvalues of $exp(tX) \subset G_v$ have the form $e^{t\lambda}$, which have modulus 1 for all t only if $\lambda = 0$. Hence $\mathfrak{P}_v = 0$. □

Proper maps

For a nonzero element v $\in$ V let $f_v : G \to V$ be the C^∞ map given by $f_v(g) = g(v)$ for g $\in$ G and v $\in$ V.

Proposition 1.9. *Let G be a closed subgroup of GL(V), and let v be a nonzero element of V. Then $f_v : G \to V$ is a proper map $\Leftrightarrow$ G(v) is closed in V and the stability group G_v is compact.*

Remark See Proposition 3.9 and the remarks that follow for an extension of this result.

Proof. If $f_v : G \to V$ is a proper map, then it is routine to prove that G(v) is closed and G_v is compact. To prove the converse we make a preliminary observation.

Lemma Let v $\neq 0 \in$ V be given. If the map $f_v : G \to V$ fails to be proper, then there exists a nonzero element Y of $\mathfrak{P}$ and an element $v_0 \in$ V such that $Y(v_0) = 0$ and $exp(tY)(v) \to v_0$ as t $\to \infty$. In particular G_{v_0} is noncompact.

Proof of the lemma If f_v is not proper, then there exists an unbounded sequence $\{g_n\} \subset G$ such that $\{g_n(v)\}$ is a bounded sequence in V. By the selfadjointness of G we may write $g_n = k_n exp(X_n)$, where $k_n \in K, X_n \in \mathfrak{P}$ and $|X_n| \to \infty$ as n $\to \infty$. Since K is compact it follows that $exp(X_n)(v) \to w \in V$ by passing to a subsequence if necessary.

Let $Y_n = X_n/|X_n|, t_n = |X_n|$ and let $Y_n \to Y \in \mathfrak{P}$, where $|Y| = 1$, by passing to a subsequence if necessary. If $f_n(t) = |exp(tY_n)(v)|^2$ and $f(t) = |exp(tY)(v)|^2$, then

$f_n(t) \to f(t)$ for all t as n $\to \infty$. It is proved in Lemma 3.1 of [RS] that the functions $f_n(t)$ and $f(t)$ are convex; that is , $f''_n(t) \geq 0$ for all n and all t $\in \mathbb{R}$, and $f''(t) \geq 0$ for all t $\in \mathbb{R}$. By hypothesis $f_n(t_n) \to |w|^2$ as n $\to \infty$. By the convexity of $f_n(t)$ we conclude that $f_n(t) \leq \ max\{f_n(0), f_n(t_n)\} \leq |v|^2 + |w|^2 + 1$ if $0 \leq t \leq t_n$ and n is sufficiently large. Hence f(t) $\leq |v|^2 + |w|^2 + 1$ for t ≥ 0, and it follows by convexity that f(t) is nonincreasing on $\mathbb{R}$.

Let Λ denote the set of nonzero eigenvalues of Y and let $V = V_0 \oplus \sum_{\lambda \in \Lambda} V_\lambda$ be the direct sum decomposition of V into orthogonal eigenspaces of Y $\in \mathfrak{P}$, where $Y \equiv 0$ on V_0 and $Y \equiv \lambda \ Id$ on V_λ for all $\lambda \in \Lambda$. Write $v = v_0 + \sum_{\lambda \in \Lambda} v_\lambda$, where $v_0 \in V_0$ and $v_\lambda \in V_\lambda$ for all $\lambda \in \Lambda$. Then $exp(tY)(v) = v_0 + \sum_{\lambda \in \Lambda} e^{t\lambda} v_\lambda$ and f(t) $= |exp(tY)(v)|^2 = |v_0|^2 + \sum_{\lambda \in \Lambda} e^{2t\lambda}|v_\lambda|^2$. By the previous paragraph $lim_{t \to \infty} f(t)$ exists, and it follows that $\lambda \in \Lambda$ is negative if $v_\lambda \neq 0$. We conclude that $exp(tY)(v) \to v_0$ as $t \to \infty$. Moreover, $Y(v_0) = 0$ since $v_0 \in V_0$. The eigenvalues of $e^{tY} \in G_{v_0}$ are unbounded in t since Y $\neq 0$ and hence G_{v_0} is noncompact. This completes the proof of the lemma.

We complete the proof of the proposition. Suppose that for some v $\in$ V the orbit G(v) is closed in V and G_v is a compact subgroup of G. If $f_v : G \to V$ is not a proper map, then by the lemma above there exists an element $v_0 \in \overline{G(v)} = G(v)$ such that G_{v_0} is noncompact. Choose g $\in$ G such that $g(v) = v_0$. Then G_{v_0} is compact since G_v is compact and $gG_vg^{-1} = G_{g(v)} = G_{v_0}$. This contradiction shows that $f_v : G \to V$ is a proper map. □

Proposition 1.10. *The map m : V $\to \mathfrak{P}$ is a proper map $\Leftrightarrow \mathfrak{M} = \{0\}$. Moreover, if $\mathfrak{M} = \{0\}$, then for every nonzero v $\in$ V there exists a nonzero X $\in \mathfrak{P}$ such that $e^{tX}(v) \to 0$ as t $\to +\infty$.*

Proof. Let $\mathfrak{M} = \{0\}$ and suppose that m : V $\to \mathfrak{P}$ is not a proper map. Then there exists an unbounded sequence $\{v_n\}$ in V such that $m(v_n) \to X$ for some X $\in \mathfrak{P}$. Let $w_n = v_n/|v_n|$ and let w $\in$ V be a unit vector that is an accumulation point of $\{w_n\}$. Since m : V $\to \mathfrak{P}$ is a homogeneous polynomial function of degree two it follows that $m(w) = lim_{n \to \infty} m(w_n) = lim_{n \to \infty} \frac{1}{|v_n|^2} m(v_n) = 0$. This contradicts the hypothesis that $\mathfrak{M} = \{0\}$. Hence m : V $\to \mathfrak{P}$ is proper.

Next suppose that m : V $\to \mathfrak{P}$ is a proper map. If v is a nonzero element of $\mathfrak{M}$, then $m(tv) = t^2 m(v) = 0$ for all t $\in \mathbb{R}$, which contradicts the properness of m. Hence $\mathfrak{M} = \{0\}$ if m is proper.

The final assertion of the proposition follows immediately from (1.2) and (1.6). □

The deformation retraction

We recall some results of Neeman [Nee] and G.Schwarz [S]. See also [RS] for a brief discussion.

Proposition 1.11. *Assume that $\mathfrak{M} \neq \{0\}$. Let h : V $\to \mathbb{R}$ be given by h(v) $= |m(v)|^2$. Then*

1) (grad h)(v) $= 4m(v)(v)$ (same vector components) and grad h is nonzero on V $- \mathfrak{M}$.

2) Let $\{\psi_t\}$ denote the flow of $- grad(h)$, and let $\rho_t = \psi_{tan(t\pi/2)}$. Then ρ_t is defined for $0 \leq t \leq 1$. The map $\rho : V \ x \ [0,1] \to \mathfrak{M}$ given by $\rho(v,t) = \rho_t(v)$ is a deformation retraction of V onto $\mathfrak{M} = \rho_1(V)$ such that $\rho(kv,t) = k\rho(v,t)$ for all k $\in$ K and all t $\in$ [0,1]. In particular the map $\pi : V \to \mathfrak{M}$ given by $\pi(v) = \rho_1(v)$ is a continuous retraction of V onto $\mathfrak{M}$ such that $\pi \circ k = k \circ \pi$ for all k $\in$ K.

3) The map $\overline{\rho_t} : V/K \to V/K$ given by $\overline{\rho_t}(K(v)) = K(\rho_t(v))$ is a well defined deformation retraction of V/K onto $\mathfrak{M}/K = \overline{\rho_1}(V/K)$.

Proof. The assertions in 2) are proved in [S] and [RS]. We note that the K-equivariance of ρ follows from the fact that h $\circ$ k = h for all k $\in$ K. In particular, k_* grad h = grad h and k permutes the integral curves of $-$ grad h for all k $\in$ K.

The assertion in 3) follows from 2) and the K-equivariance of the retraction $\rho : V\ x\ [0,1] \to \mathfrak{M}$.

We prove 1). We recall from the proof of (1.7) that $\langle m_*(\xi_v), X\rangle = 2\langle X(v), \xi\rangle$ for all $\xi \in$ V and all $X \in \mathfrak{P}$. Now $\langle \xi_v, (grad\ h)(v)\rangle = dh(\xi_v) = \xi_v(h) = (h \circ \alpha)'(0)$, where $\alpha(t) = v + t\xi$. By definition $(h \circ \alpha)(t) = \langle m(v + t\xi), m(v + t\xi)\rangle$, and we conclude that $(h \circ \alpha)'(0) = 2\langle m_*(\xi_v), m(v)\rangle = 4\langle m(v)(v), \xi\rangle$. This proves the first assertion in 1) since $\xi \in$ V was arbitrary.

If v $\in V - \mathfrak{M}$, then $\langle grad\ h(v), v\rangle = 4\ \langle m(v), m(v)\rangle > 0$, which completes the proof of 1). ☐

$Remark$ We recall the observation of [S] and [RS] that the deformation retraction ρ : $V\ x\ [0,1] \to \mathfrak{M}$ of 2) above has the property that $\rho(v,t) \in G(v)$ for all $(v,t) \in V\ x\ [0,1)$. This is a consequence of the fact that the vector field $-$ grad(h) is tangent to the immersed submanifolds G(v) for all v $\in$ V.

2. The set of vectors with closed G-orbits

Let G,V be as above. We note that if an orbit G(v) is closed in V for some vector v $\in$ V, then G(v) is an imbedded submanifold of V. For a proof, see for example Theorem 2.9.7 of [Va].

Proposition 2.1. *Let G,V be as above, and let V ′ = $\{v \in V : G(v)\ is\ closed\ in\ V\ and\ dim\ G(v)\ is\ maximal\}$. If V ′ is nonempty, then V ′ is a G-invariant Zariski open subset of V.*

Proof. This result is already known in the complex setting; that is, for $G^{\mathbb{C}}$ and $V^{\mathbb{C}}$. See for example Proposition 3.8 of [New]. We indicate how to extend the result to the real setting. We note that V′ is clearly G-invariant.

Let G and $G^{\mathbb{C}}$ be as above. Then $G^{\mathbb{C}}$ has a natural induced representation on the complexification $V^{\mathbb{C}}$ of V.

Lemma 2.2. *Let v $\in$ V. Then the orbit G(v) is closed in V $\Leftrightarrow$ the orbit $G^{\mathbb{C}}$(v) is closed in $V^{\mathbb{C}}$.*

Proof. We suppose first that G(v) is closed in V. Then w = g(v) is minimal for some g $\in G \subset G^{\mathbb{C}}$ by (1.2). By Lemma 8.1 of [RS] the vector w is minimal for the action of $G^{\mathbb{C}}$ on $V^{\mathbb{C}}$. Hence $G^{\mathbb{C}}$(w) = $G^{\mathbb{C}}$(v) is closed in $V^{\mathbb{C}}$. Conversely, suppose that $G^{\mathbb{C}}$(v) is closed in $V^{\mathbb{C}}$. By Proposition 2.3 of [BH] the set $G^{\mathbb{C}}$(v) $\cap$ V is the union of finitely many orbits of $G^{\mathbb{C}}(\mathbb{R})^0$, and each of these orbits is closed. Since $G^{\mathbb{C}}(\mathbb{R})^0$ has finite index in G it follows that G(v) is closed in V. ☐

The next observation will be useful, but we omit the proof, which is routine.

Lemma 2.3. *If O is a nonempty Zariski open subset of $V^{\mathbb{C}}$, then O $\cap$ V is a nonempty Zariski open subset of V.*

We now complete the proof of the proposition. By definition $V' = \{v \in V : \mathrm{G(v)\ is\ closed\ in\ V\ and\ dim\ G(v)\ is\ maximal}\}$, and similarly we define $(V^{\mathbb{C}})' = \{v \in V^{\mathbb{C}} : \mathrm{G}^{\mathbb{C}}\mathrm{(v)\ is\ closed\ in\ V}^{\mathbb{C}}\ \mathrm{and\ dim\ G}^{\mathbb{C}}\mathrm{(v)\ is\ maximal}\}$. For v $\in$ V we note that $dim_{\mathbb{R}} G_v = dim_{\mathbb{C}} G_v^{\mathbb{C}}$ since $\mathfrak{G}_v^{\mathbb{C}} = (\mathfrak{G}_v)^{\mathbb{C}}$. Hence $dim_{\mathbb{R}} G(v) = dim_{\mathbb{C}} G^{\mathbb{C}}(v)$ since G(v) and $G^{\mathbb{C}}$(v) are diffeomorphic to the coset spaces G / G_v and $G^{\mathbb{C}}$ / $G_v^{\mathbb{C}}$ respectively. By (2.2) it follows that $V' = V \cap (V^{\mathbb{C}})'$. Since $(V^{\mathbb{C}})'$ is known to be Zariski open in $V^{\mathbb{C}}$ it follows immediately from (2.3) that V′ is Zariski open in V. ☐

$Stability\ of\ the\ G - action$

Let G,V be as above. We say that the action of G on V is $stable$ or G acts $stably$ on V if there exists a nonempty Zariski open subset O of V such that G(v) has maximal dimension and is closed in V for all v ∈ O. It follows from (2.1) that G acts stably on V if there is a single nonzero vector v ∈ V such that G(v) has maximal dimension and is closed in V. This observation has simple but useful consequences.

Proposition 2.4. *Let G_i, V_i be as above for i $= 1, 2$. Let $G = G_1$ x G_2 and let $V = V_1 \oplus V_2$. Then G acts stably on V ⇔ G_i acts stably on V_i for $i = 1, 2$.*

Proof. Let $v = (v_1, v_2) \in V_1 \oplus V_2$. Then $G(v) = (G_1(v_1), G_2(v_2))$ has maximum dimension and is closed in V ⇔ $G_i(v_i)$ has maximal dimension and is closed in V_i for $i = 1, 2$. The assertion now follows immediately from (2.1). □

$Remark$ Let G,V be as above, and let X be the union of all closed G-orbits in V. If G does not act stably on V, then X has empty interior in the vector space topology of V.

If X contained a subset U of V that is open in the vector space topology of V, then the stability group G_v would have minimal dimension for some v ∈ U since G_v has minimal dimension for a nonempty Zariski open subset of V. It would follow that G(v) has maximal dimension and is closed in V, which by (2.1) would imply that G acts stably on V.

$Example$ Let G_1, V_1 be arbitrary, as above. Let $V_2 = \mathbb{R}^n$ and let $G_2 = SL(n, \mathbb{R})$ act on V_2 in the standard way. Let X_1 be the union of all closed G_1 orbits in V_1. Since $\{0\}$ is the only closed G_2 orbit in V_2 it follows that $X = X_1$ x $\{0\} \subset V_1$ x $\{0\}$ is the union of all closed G orbits in V.

The next result shows that G acts stably on V if a single stabilizer G_v is discrete for some v ∈ V. This result is strengthened later in Corollary 3.12. We note that if G_v is discrete, then G_v is finite.

Corollary 2.5. *Suppose that $G_{v'}$ is discrete for some nonzero v′ in V. Then there exists a nonzero G-invariant Zariski open subset O of V such that G(v) is closed and G_v is finite for all v ∈ O.*

Proof. We recall that $G^{\mathbb{C}}(\mathbb{R})^0 \subseteq G \subset G^{\mathbb{C}}$, where $G^{\mathbb{C}}$ is a semisimple algebraic group defined over $\mathbb{R}$. Since $\mathfrak{G}_v^{\mathbb{C}} = (\mathfrak{G}_v)^{\mathbb{C}}$ it follows that $G_v^{\mathbb{C}}$ is discrete. If U $= \{v \in V^{\mathbb{C}} : G_v^{\mathbb{C}} \text{ is discrete}\}$, then U is a nonempty $G^{\mathbb{C}}$-invariant Zariski open subset of $V^{\mathbb{C}}$. For v ∈ U the stability group $G_v^{\mathbb{C}}$ is finite and hence reductive since $G^{\mathbb{C}}$ is algebraic. Note that the subgroup G_v is also finite for v ∈ U. It follows from a result of V. Popov [P] that there exists a $G^{\mathbb{C}}$-invariant Zariski open subset U′ of $V^{\mathbb{C}}$ such that $G^{\mathbb{C}}(v)$ is closed and has maximal dimension dim G. An orbit G(v) has dimension dim G ⇔ $G_v^{\mathbb{C}}$ is discrete, and hence $U' \subseteq U$. If $O = U' \cap V$, then by (2.2) O is a G-invariant nonempty Zariski open subset of V, and G(v) is closed with G_v finite for all v ∈ O. □

$Remark$ If G_v is discrete it is not necessarily true that G(v) is closed in V. For example, let V be the 4-dimensional real vector space of homogeneous polynomials of degree 3 in the variables x,y. Let G $= SL(2, \mathbb{R})$ act on V by (gf)(x,y) = f((x,y)g). If f(x,y) $= x^2y$, then it is easy to compute that $G_f = \{Id\}$. On the other hand G(f) is not closed since if $g(t) = diag(e^{-t}, e^t)$, then $g(t)(f) = e^{-t}f \to 0$ as t → ∞.

We extend the previous result to show that G acts stably on V if a single stabilizer G_v is compact for some v ∈ V. This result will also be strengthened later in (3.13).

Proposition 2.6. *Suppose that G_v is compact for some nonzero v in V. Then*

1) There exists an open neighborhood U of v in V such that G_w is compact for all w $\in$ U.

2) G acts stably on V.

Proof. 1) Let d be a complete Riemannian metric on End(V), and let R > 0 be chosen so that d(e,g) $\leq R$ for all g $\in G_v$. We assert that for every R$'$ $>$ R there exists an open neighborhood U of v such that d(e,h) $\leq R'$ for all h $\in (G_w)_0$ and all w $\in$ U. Suppose this is false for some R$'$ $>$ R, and let $\{v_n\} \subset$ V and $\{h_n\} \subset (G_{v_n})_0$ be sequences such that $v_n \to$ v and $d(e,h_n) > R'$ for all n. Since $(G_{v_n})_0$ is arc connected there exists a sequence $\{g_n\} \subset (G_{v_n})_0$ such that $d(e,g_n) = R'$ for all n. By the completeness of d there exists a cluster point g of $\{g_n\}$, and by continuity we see that g $\in G_v$ and d(e,g) $= R' > R$. This contradicts the choice of R.

The argument above and the completeness of d show that $(G_w)_0$ is compact for all w in some neighborhood U of v. It follows that G$_w$ is compact for all w in U since $(G_w)_0$ has finite index in G$_w$.

2) It is known that there exists a nonempty Zariski open subset A of V$^{\mathbb{C}}$ such that the stabilizers $\{G_v^{\mathbb{C}}, v \in A\}$ are conjugate in $G^{\mathbb{C}}$. See for example section 7 of [PV]. If U is the open set discussed in 1), then G$_v$ is compact for all v $\in$ U. It follows that $\mathfrak{G}_v$ is reductive and the center of $\mathfrak{G}_v$ consists of semisimple automorphisms of V. The same is true for $(\mathfrak{G}^{\mathbb{C}})_v = (\mathfrak{G}_v)^{\mathbb{C}}$ for all v $\in$ U, where $\mathfrak{G}^{\mathbb{C}}$ is the Lie algebra of $G^{\mathbb{C}}$. Hence $(\mathfrak{G}^{\mathbb{C}})_v$ is completely reducible in $V^{\mathbb{C}}$ for all v $\in$ U by Theorem 4 in section 6.5 of [Bou]. Since A $\cap$ V is Zariski open in V we see that A $\cap$ V $\cap$ U is nonempty. In particular the generic stabilizer $(\mathfrak{G}^{\mathbb{C}})_w$, w $\in$ A, is completely reducible in $V^{\mathbb{C}}$. By Theorem 1 of [P] there exists a nonempty Zariski open subset B of $V^{\mathbb{C}}$ such that $G^{\mathbb{C}}(v)$ has maximal dimension and is closed in $V^{\mathbb{C}}$ for all v $\in$ B. If v $\in O = B \cap V$, a nonempty Zariski open subset of V, then G(v) has maximal dimension, and by (2.2) G(v) is closed in V. ☐

Connected components of the space of closed orbits

We consider the case that there exists a nonempty Zariski open subset O of V such that G(v) is closed for all v $\in$ V. Since G has stabilizers of minimal dimension on a nonempty Zariski open subset of V we shall also assume, without loss of generality, that G has a stabilizer of minimal dimension at every point v of O.

We consider the connected components of O. It is well known that a Zariski open set O has only finitely many connected components. See for example Theorem 4 of [W].

Let $\mathfrak{M}' = \mathfrak{M} \cap O$. We first describe a decomposition of the set $\mathfrak{M}'$.

We recall from (1.11) that there is a continuous retraction $\pi : V \to \mathfrak{M}$ such that $\pi \circ k = k \circ \pi$ all k $\in$ K, and $\pi(v) \in \overline{G(v)}$ by the remark following (1.11). Given v $\in$ O there exists g $\in$ G such that $\pi(v) = g(v)$ since G(v) is closed in V. Hence $\pi(v) \in G(O) = O$, and it follows that the map π restricts to a continuous retraction $\pi : O \to \mathfrak{M}'$.

Proposition 2.7. *Let $O_1, \ldots O_r$ denote the connected components of O. For $1 \leq \alpha \leq r$ let $\mathfrak{M}_\alpha = O_\alpha \cap \mathfrak{M}$. Then*

1) The sets $\{\mathfrak{M}_\alpha : 1 \leq \alpha \leq r\}$ are disjoint arc connected subsets of $\mathfrak{M}'$, and $\mathfrak{M}' = \bigcup_{\alpha=1}^r \mathfrak{M}_\alpha$.

2) $G_0(\mathfrak{M}_\alpha) = O_\alpha$ for all α, where G_0 denotes the identity component of G.

Proof. 1) Note that $\pi(O_\alpha) \subseteq \mathfrak{M}_\alpha \cap O_\alpha = \mathfrak{M}_\alpha$ for all α since $\pi : V \to \mathfrak{M}$ is defined by a deformation retraction and O_α is both open and closed in O. The set inclusion is an equality since π is the identity on $\mathfrak{M}$. The sets $\{\mathfrak{M}_\alpha : 1 \leq \alpha \leq r\}$ are clearly disjoint since they belong to the distinct components $\{O_\alpha\}$ of O, and each set $\mathfrak{M}_\alpha$ is arc connected since the open set O_α is arc connected. Finally, $\mathfrak{M}' = \mathfrak{M} \cap O = \bigcup_{\alpha=1}^r \mathfrak{M} \cap O_\alpha = \bigcup_{\alpha=1}^r \mathfrak{M}_\alpha$.

2) We start with two preliminary results.

Lemma 1 $G_0(\mathfrak{M}\,') = O$.

Proof. Since $\mathfrak{M}\,' \subset O$ it follows that $G_0(\mathfrak{M}\,') \subset G(O) \subset O$. Conversely, let v ∈ O. Then $\pi(v) \in \mathfrak{M}\,'$ and $\pi(v) = g(v)$ for some g ∈ G since G(v) is closed in V. By (2.2) of [RS] we may write $g = k\ exp(X)$ for some k ∈ K and some X ∈ $\mathfrak{P}$. Then $w = k^{-1}\pi(v) = exp(X)(v) \in \mathfrak{M}\,'$ since $\mathfrak{M}\,' = \mathfrak{M} \cap O$ is invariant under K. It follows that $v = exp(-X)(w) \in G_0(\mathfrak{M}\,')$, which proves that $O \subset G_0(\mathfrak{M}\,')$. □

Lemma 2 $G_0(\mathfrak{M}_\alpha) \subseteq \pi^{-1}(\mathfrak{M}_\alpha) = O_\alpha$.

Proof. We note that it follows immediately from the definitions of O_α and $\mathfrak{M}_\alpha = \pi(O_\alpha)$ that $O_\alpha = \pi^{-1}(\mathfrak{M}_\alpha)$.

Let α and v ∈ $\mathfrak{M}_\alpha$ be given. Since $\mathfrak{M}_\alpha \subset O$ and O is G - invariant it follows that $G_0(v) \subset G(v) \subset O$. Since $G_0(v)$ is arc connected, O_α is a connected component of O and v ∈ O_α it follows that $G_0(v) \subset O_\alpha$. The lemma is proved since $v \in \mathfrak{M}_\alpha$ was arbitrary. □

We complete the proof of 2) of the proposition. By Lemmas 1 and 2 and 1) of the proposition we have $O = G_0(\mathfrak{M}\,') = \bigcup_{\alpha=1}^r G_0(\mathfrak{M}_\alpha) \subseteq \bigcup_{\alpha=1}^r O_\alpha = O$. Hence $G_0(\mathfrak{M}_\alpha) = O_\alpha$ for all α by Lemma 2 since the sets $\{O_\alpha\}$ are disjoint. □

Proposition 2.8. *For each $1 \le \alpha \le r$ there exist nonnegative integers k_α, p_α such that*
a) dim $\mathfrak{K}_v = k_\alpha$ for all v ∈ $\mathfrak{M}_\alpha$.
b) dim $\mathfrak{P}_v = p_\alpha$ for all v ∈ $\mathfrak{M}_\alpha$.
c) dim $\mathfrak{M}_\alpha = dim\ V - dim\ \mathfrak{P} + p_\alpha$.

Proof. Assertions a) and b) are contained in the next result.
Lemma 1 For each $1 \le \alpha \le r$ there exist nonnegative integers k_α, p_α such that
a) dim $\mathfrak{K}_v = k_\alpha$ for all v ∈ $\mathfrak{M}_\alpha$.
b) dim $\mathfrak{P}_v = p_\alpha$ for all v ∈ $\mathfrak{M}_\alpha$.

Proof. Let v ∈ $\mathfrak{M}_\alpha \subset O_\alpha \subset O$ be given. By continuity there exists an open set U of $\mathfrak{M}_\alpha$ such that v ∈ U and $dim\ \mathfrak{P}_w \le dim\ \mathfrak{P}_v$ and $dim\ \mathfrak{K}_w \le dim\ \mathfrak{K}_v$ for all w ∈ U. The stability Lie algebras $\{\mathfrak{G}_w\}$ are self adjoint by (1.1) and by hypothesis they have constant dimension for all w ∈ O. Since $dim\ \mathfrak{G}_v = dim\ \mathfrak{K}_v\ +\ dim\ \mathfrak{P}_v$ and $dim\ \mathfrak{G}_w = dim\ \mathfrak{K}_w\ +\ dim\ \mathfrak{P}_w$ it follows that $dim\ \mathfrak{K}_v = dim\ \mathfrak{K}_w$ and $dim\ \mathfrak{P}_v = dim\ \mathfrak{P}_w$ for all w ∈ U. The assertion of Lemma 1 follows since $\mathfrak{M}_\alpha$ is connected. □

We note that the elements of G permute the connected components $\{O_\alpha\}$ of O since O is invariant under G. Similarly, the elements of K permute the connected components $\{\mathfrak{M}_\alpha\}$ of $\mathfrak{M}\,'$ since $\mathfrak{M}\,'$ is invariant under K. For $1 \le \alpha \le r$ let $G_\alpha = \{g \in G : g(O_\alpha) = O_\alpha\}$ and let $K_\alpha = \{k \in K : k(\mathfrak{M}_\alpha) = \mathfrak{M}_\alpha\}$. Note that $G_0 \subseteq G_\alpha \subset G$ and $K_0 \subseteq K_\alpha \subset K$. Moreover, $K_\alpha \subset G_\alpha$ for all α since $\mathfrak{M}_\alpha \subset k(O_\alpha) \cap O_\alpha$ for all k ∈ K_α and all α.

To prove 2) we need some additional preliminary results.

Lemma 2 Let $1 \le \alpha \le r$ and v ∈ O_α be given. Then $G_\alpha(v) \cap \mathfrak{M}_\alpha = K_\alpha(\pi(v))$.

Proof. We show first that (*) $\pi(v) \in G_\alpha(v) \cap \mathfrak{M}_\alpha$ for all v ∈ O_α. Given v ∈ O_α choose g ∈ G such that $\pi(v) = g(v)$. It follows that $\pi(v) \in O_\alpha \cap g(O_\alpha)$ since $\mathfrak{M}_\alpha \subset O_\alpha$ and it follows that g ∈ O_α. This proves (*).

Since $K_\alpha \subset G_\alpha$ it follows from (*) that $K_\alpha(\pi(v)) \subset G_\alpha(v) \cap \mathfrak{M}_\alpha$. Now let w′ ∈ $G_\alpha(v) \cap \mathfrak{M}_\alpha$ be given, and let w = $\pi(v) \in G_\alpha(v) \cap \mathfrak{M}_\alpha$. Then $w' \in G_\alpha(w) \cap \mathfrak{M}_\alpha$, and hence $w' = \varphi(w)$ for some $\varphi \in K$ by (1.2). It follows that $\varphi \in K_\alpha$ since $w' \in$

$\mathfrak{M}_\alpha \cap\ \varphi(\mathfrak{M}_\alpha)$. This proves that $G_\alpha(v) \cap\ \mathfrak{M}_\alpha \subset K_\alpha(w) = K_\alpha(\pi(v))$ and completes the proof of the lemma. □

Lemma 3 Let $\rho : O_\alpha \ / \ G_\alpha \to \mathfrak{M}_\alpha \ / \ K_\alpha$ be given by $\rho(G_\alpha(v)) = K_\alpha(\pi(v))$ for all v $\in O_\alpha$. Then ρ is a continuous bijection with respect to the quotient topologies.

Proof. If $G_\alpha(v) = G_\alpha(w)$ for elements v,w of O_α, then $K_\alpha(\pi(v)) = K_\alpha(\pi(w))$ by Lemma 2. Hence ρ is well defined. Suppose that $\rho(G_\alpha(v)) = \rho(G_\alpha(w))$ for v,w $\in O_\alpha$. Then $K_\alpha(\pi(v)) = K_\alpha(\pi(w))$, which implies that $G_\alpha(v) \ \cap\ \mathfrak{M}_\alpha = G_\alpha(w) \ \cap\ \mathfrak{M}_\alpha$ by Lemma 2. Hence $G_\alpha(v) = G_\alpha(w)$, and we conclude that ρ is injective. Finally, if v $\in \mathfrak{M}_\alpha$, then $\pi(v) = v$ and $\rho(G_\alpha(v)) = K_\alpha(v)$. This shows that ρ is surjective. The continuity of ρ follows routinely from the definitions of ρ and the quotient topologies. □

We now prove c) of the proposition by computing separately the dimensions of $O_\alpha \ / \ G_\alpha$ and $\mathfrak{M}_\alpha \ / \ K_\alpha$ and using Lemma 3.

For v $\in \mathfrak{M}_\alpha$ the stabilizer $(G_\alpha)_v$ has dimension $k_\alpha + p_\alpha$ by a) and b) of the proposition since $G_0 \subset G_\alpha \subset G$ and $\mathfrak{G}_v$ is self adjoint by (1.2). Hence for all v $\in O_\alpha$ the dimension of the stabilizer $(G_\alpha)_v$ is $k_\alpha + p_\alpha$ since G_v has constant dimension for all v $\in O_\alpha$. We conclude that the dimension of the orbit $G_\alpha(v)$ is dim G $-(k_\alpha + p_\alpha)$ for all v $\in O_\alpha$. It follows that the orbit space $O_\alpha \ / \ G_\alpha$ has dimension equal to dim V $-$ dim G $+\ k_\alpha +\ p_\alpha$.

The orbits of K_α in $\mathfrak{M}_\alpha$ all have dimension equal to dim K $-k_\alpha$ by a) of the proposition and the fact that $K_0 \subset K_\alpha \subset K$. Hence the dimension of the orbit space $\mathfrak{M}_\alpha \ / \ K_\alpha$ equals dim $\mathfrak{M}_\alpha -$ dim K + k_α.

By Lemma 3 the dimensions of $O_\alpha \ / \ G_\alpha$ and $\mathfrak{M}_\alpha \ / \ K_\alpha$ are equal. Recall that $dim\ \mathfrak{P} = dim\ \mathfrak{G} -\ dim\ \mathfrak{K} = dim\ G - dim\ K$. The assertion c) now follows from the formulas above for the dimensions of $O_\alpha \ / \ G_\alpha$ and $\mathfrak{M}_\alpha \ / \ K_\alpha$. □

$Example$ We use the adjoint representation to illustrate the results above. We begin with some terminology and basic facts.

Let G be a connected, noncompact, semisimple Lie group whose Lie algebra $\mathfrak{G}$ has no compact factors. Let $V = \mathfrak{G}$ and let Ad : G $\to$ GL(V) denote the adjoint representation. For an element X of $\mathfrak{G}$ we note that the stabilizer Lie algebra $\mathfrak{G}_X$ equals the centralizer Z(X).

Let $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ be a Cartan decomposition of $\mathfrak{G}$ determined by a Cartan involution $\theta : G \to G$ and its differential map $\theta : \mathfrak{G} \to \mathfrak{G}$. If $\mathfrak{B}_1, \mathfrak{B}_2$ are two maximal abelian subspaces of $\mathfrak{P}$, then $\mathfrak{B}_2 = Ad(\varphi)(\mathfrak{B}_1)$ for some element φ of K = Fix(θ). Conversely, if $\mathfrak{B}$ is a maximal abelian subspace of $\mathfrak{P}$, then Ad(φ)($\mathfrak{B}$) is another for all $\varphi \in$ K since Ad K leaves $\mathfrak{P}$ invariant.

We let $rank$ $\mathfrak{P}$ denote the dimension of a maximal abelian subspace of $\mathfrak{P}$. For a nonzero element P $\in \mathfrak{P}$ we let E_P denote the intersection of all maximal abelian subspaces of $\mathfrak{P}$ that contain P.

A $Cartan\ subalgebra$ of $\mathfrak{G}$ is a maximal abelian subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ such that ad Y : $\mathfrak{G} \to \mathfrak{G}$ is semisimple for all Y $\in \mathfrak{A}$.

Recall that $\mathfrak{M}$ denotes the set of minimal vectors in $\mathfrak{G}$ for the action of G.

Proposition 2.9. *Let G and V = $\mathfrak{G}$ be as above. Then*

1) $\mathfrak{M} = \{X \in \mathfrak{G} : \mathfrak{G}_X = Z(X)\ is\ invariant\ under\ \theta\}$.

2) Let O $= \{X \in \mathfrak{G} : \mathfrak{G}_X = Z(X)\ has\ minimal\ dimension\ and\ G(X)\ is\ closed\ in\ \mathfrak{G}\}$. *Then X $\in$ O $\Leftrightarrow \mathfrak{A} = \mathfrak{G}_X$ is a Cartan subalgebra of $\mathfrak{G}$.*

3) Let X $\in \mathfrak{M} \cap O$ and write X = K + P, where $K = (1/2)(X + \theta(X)) \in \mathfrak{K}_X = \mathfrak{G}_X \cap \mathfrak{K}$ *and* $P = (1/2)(X - \theta(X)) \in \mathfrak{P}_X = \mathfrak{G}_X \ \cap\ \mathfrak{P}$. *Then* $\mathfrak{P}_X = E_P$. *Conversely, for every nonzero P $\in \mathfrak{P}$ there exists K $\in \mathfrak{K}$ such that if X = K + P, then X $\in \mathfrak{M} \cap O$ and* $\mathfrak{P}_X = E_P$.

4) Let r be any integer with $1 \leq r \leq rank\ \mathfrak{P}$*. Then there exists X* $\in \mathfrak{M} \cap O$ *such that dim* $\mathfrak{P}_X = r$*.*

Proof. If X $\in \mathfrak{M}$, then $\mathfrak{G}_X$ is θ-invariant by (1.2). Conversely, if $\mathfrak{G}_X = Z(X)$ is θ-invariant for X $\in$ O, then $0 = [X, \theta(X)]$ and it follows that X $\in \mathfrak{M}$ by Lemma 5.3.1 of [RS]. This proves 1). We omit the proofs of 3) and 4) for reasons of space. We prove 2), referring to results that will be proved in section 5. If X $\in$ O, then X is semisimple by (5.5) and $Z(X) = \mathfrak{G}_X$ is a Cartan subalgebra by (5.3). Conversely, if $Z(X) = \mathfrak{G}_X$ is a Cartan subalgebra, then X $\in Z(X)$ is semisimple and X $\in$ O by (5.3) and (5.5). □

3. The M-function

The result (2.5) gives a useful criterion for the existence of a nonempty Zariski open subset O such that G(v) is closed for all v in O. However, it gives no criterion for determining if the G orbit of a given vector v in V is closed in V. In this section we consider a G-invariant function M : V $\to \mathbb{R}$ with finitely many values such that G(v) is closed if M(v) is negative. This result is the real analogue of a result of Mumford. The function M in this context has also been used by A. Marian [Ma].

The $\mu - function$

Let $\langle , \rangle$ be an inner product for which G is self adjoint in its action on V, and let $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ be a Cartan decomposition compatible with $\langle , \rangle$. Let $V', \mathfrak{P}'$ denote the nonzero vectors in V,$\mathfrak{P}$ respectively.

For X $\in \mathfrak{P}'$ let Λ_X be the set of eigenvalues of X, and for $\mu \in \Lambda_X$ let $V_{\mu,X}$ denote the eigenspace in V corresponding to μ. For v $\in$ V$'$ and X $\in \mathfrak{P}'$ let $\mu(X, v)$ denote the smallest eigenvalue μ such that v has a nonzero component in $V_{\mu,X}$.

We collect some properties of the function $\mu : \mathfrak{P}'$ x $V' \to \mathbb{R}$.

Proposition 3.1. *Let (Y,v)* $\in \mathfrak{P}'$ *x* V' *be given.*

1) $\mu(Y, v) = 0 \Leftrightarrow$ *the following two conditions hold*

a) The component v_0 *of v in Ker Y is nonzero*

b) $e^{tY}(v) \to v_0$ *as t* $\to -\infty$*.*

2) $\mu(Y, v) > 0 \Leftrightarrow e^{tY}(v) \to 0$ *as t* $\to -\infty$*.*

Proof. We prove only 1) since the proof of 2) is just a slight modification of the proof of 1). For Y $\in \mathfrak{P}'$ let $\Lambda_Y^{'}$ denote the set of nonzero eigenvalues of Y acting on V. Write $v = v_0 + \sum_{\lambda \in \Lambda_Y^{'}} v_\lambda$, where $v_0 \in$ Ker Y and $v_\lambda \in V_\lambda$. Then

(*) $e^{tY}(v) = v_0 + \sum_{\lambda \in \Lambda_Y^{'}} e^{t\lambda} v_\lambda$.

If $\mu(Y, v) = 0$, then v_0 is nonzero and $\lambda > 0$ whenever v_λ is nonzero. It follows from (*) that $e^{tY}(v) \to v_0$ as t $\to -\infty$. Hence conditions a) and b) of 1) hold. Conversely, if these two conditions hold, then it is easy to see from (*) that $\mu(Y, v) = 0$. □

Next we prove a semicontinuity property of $\mu : \mathfrak{P}'$ x $V' \to \mathbb{R}$.

Proposition 3.2. *Let Y,v be nonzero vectors in* $\mathfrak{P}$*,V respectively. Given* $\epsilon > 0$ *there exist neighborhoods U* $\subseteq$ *V of v and O* $\subseteq \mathfrak{P}$ *of Y such that* $\mu(Y', v') < \mu(Y, v) + \epsilon$ *for all* $(Y', v') \in$ *O x U.*

Proof. Suppose the assertion is false for some nonzero vectors v $\in$ V and Y $\in \mathfrak{P}$. Then there exist $\epsilon > 0$ and sequences $\{v_n\} \subset$ V and $\{Y_n\} \subset \mathfrak{P}$ such that $(Y_n, v_n) \to (Y, v)$ as n $\to \infty$ and $\mu(Y_n, v_n) \geq \mu(Y, v) + \epsilon$ for all n. Using the fact that $Y_n \to Y$ as n $\to \infty$ and by passing to a subsequence we conclude that there exists an integer N > 0 with the following properties :

a) For every n, Y_n has N distinct eigenvalues $\{\lambda_1^{(n)}, \ldots, \lambda_N^{(n)}\}$ and there exist orthogonal subspaces $\{V_1^{(n)}, \ldots, V_N^{(n)}\}$ of V such that $V = V_1^{(n)} \oplus \ldots \oplus V_N^{(n)}$ and $Y_n = \lambda_i^{(n)}\ Id$ on $V_i^{(n)}$ for every n.

b) There exist subspaces $V_1, \ldots, V_N$ of V and real numbers $\lambda_1, \ldots, \lambda_N$ such that for $1 \leq i \leq N$ we have $\lambda_i^{(n)} \to \lambda_i$ as n $\to \infty$ and $V_i^{(n)} \to V_i$ (uniformly on compact subsets) as n $\to \infty$.

c) $V = V_1 \oplus \ldots \oplus V_N$, orthogonal direct sum, and $Y = \lambda_i$ Id on V_i for $1 \leq i \leq N$.

By c) the eigenvalues of Y (possibly with repetition) are $\{\lambda_1, \ldots, \lambda_N\}$. Choose k such that $\mu(Y, v) = \lambda_k$. Then v has a nonzero component in V_k, and by b) we conclude that there exists a positive integer N_0 such that v_n has a nonzero component in $V_k^{(n)}$ for all n $\geq N_0$. Hence for n $\geq N_0$ we have $\lambda_k^{(n)} \geq \mu(Y_n, v_n) \geq \mu(Y, v) + \epsilon$. Since $\lambda_k^{(n)} \to \lambda_k$ as n $\to \infty$ by b) we conclude that $\mu(Y, v) = \lambda_k \geq \mu(Y, v) + \epsilon$, which is impossible. This completes the proof of the lemma. □

$The\ M - function$

We define M : V $\to \mathbb{R}$ by M(v) = $max\{\mu(X, v) : X \in \mathfrak{P},\ |X| = 1\}$.

This definition is closely modeled on the discussion of L. Ness in [Nes]. We recall some results about the M function from [Ma].

Proposition 3.3. *The function M : V $\to \mathbb{R}$ has the following properties.*

1) M is constant on G-orbits

2) M has finitely many values

3) Let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{K}$. Let $\mathfrak{A}$ be a maximal abelian subalgebra of $\mathfrak{P}$, and define $M^{\mathfrak{A}}$: V$\to \mathbb{R}$ by $M^{\mathfrak{A}}(v) = max\{\mu(X, v) : X \in \mathfrak{A}, |X| = 1\}$. Then M(v) = $max\{M^{\mathfrak{A}}(kv) : k \in K\}$.

Proposition 3.4. *Let T be an element of GL(V) that commutes with the elements of G. Then M(T(v)) = M(v) for all nonzero elements v of V.*

Proof. It suffices to show that $\mu(X, v) = \mu(X, T(v))$ for all nonzero v $\in$ V and all nonzero X $\in \mathfrak{P}$. Given a nonzero X in $\mathfrak{P}$ let Λ denote the eigenvalues of X, and for $\lambda \in \Lambda$ let V_λ denote the λ - eigenspace for X. Since T commutes with the elements of G it commutes with the elements of $\mathfrak{G}$, and in particular, T commutes with X. It follows that T leaves invariant each eigenspace V_λ. If v $\in$ V has a nonzero component v_λ in V_λ, then T(v) also has a nonzero component $T(v_\lambda)$ in V_λ since T is invertible. It follows immediately that $\mu(X, v) = \mu(X, T(v))$. □

Corollary 3.5. *Let V be a G-module and let p be an integer with $2 \leq p \leq$ dim V. Let G act diagonally on W = V x ... x V (p times). Let $W_0 = \{v = (v_1, \ldots v_p) \in W$: $\{v_1, \ldots, v_p\}$* is linearly independent in V$\}$. *For $v = (v_1, \ldots v_p) \in W_0$ let span(v) = $span\{v_1, \ldots v_p\} \subset V$. If v,w are elements of W_0 with span(v) = span(w) then M(v) = M(w).*

Proof. Fix the standard basis $\{e_1, \ldots, e_p\}$ of $\mathbb{R}^p$. Then W = V x ... x V (p times) is isomorphic as a vector space to V $\otimes \mathbb{R}^p$ under the map $(v_1, \ldots, v_p) \to \sum_{i=1}^p v_i \otimes e_i$. Let G x GL(p,$\mathbb{R}$) act on V $\otimes \mathbb{R}^p$ by $(g, h)(v \otimes \zeta) = g(v) \otimes h(\zeta)$. Define an action of G x GL(p,$\mathbb{R}$) on W = V x ... x V (p times) by $(g, h)(v_1, \ldots, v_p) = (w_1, \ldots, w_p)$, where $w_j = \sum_{i=1}^p h_{ji} g(v_i)$ and $h(e_i) = \sum_{j=1}^p h_{ji} e_j$. It is routine to check that the isomorphism given above between W = V x ... x V (p times) and V $\otimes \mathbb{R}^p$ preserves the actions of G x GL(p,$\mathbb{R}$). It is obvious that the actions of G and GL(p,$\mathbb{R}$) commute on V $\otimes \mathbb{R}^p$, and hence they also commute on W = V x ... x V (p times).

Now suppose that $v = (v_1, \ldots, v_p)$ and $w = (w_1, \ldots, w_p)$ are elements of W_0 such that span(v) = span(w). Then there exists a unique element h = (h_{ij}) of GL(p,$\mathbb{R}$) such that

$w_j = \sum_{i=1}^{p} h_{ji} v_i$ for $1 \leq i \leq p$. Then $h(v) = w$ and it follows from the preceding result that M(v) = M(w) since h $\in GL(W)$ commutes with G. □

Proposition 3.6. *For every nonzero element v $\in$ V there exists a neighborhood O of v in V such that M(w) $\leq$ M(v) for all w $\in$ O.*

Proof. Suppose the statement of the proposition is false for some nonzero element v in V. Then there exists a sequence $\{v_n\} \subset$ V such that $v_n \to v$ as n $\to \infty$ and $M(v_n) > M(v)$ for all n. Since M has only finitely many values we may assume, by passing to a subsequence, that $M(v_n) = c > M(v)$ for some real number c and for all n. Choose unit vectors $\{\beta_n\} \subset \mathfrak{P}$ such that $c = M(v_n) = \mu(\beta_n, v_n)$ for all n. Passing to a further subsequence let $\{\beta_n\}$ converge to a unit vector $\beta \in \mathfrak{P}$. Choose $\epsilon > 0$ such that $c > M(v) + \epsilon$. By (3.2) above there exists a positive integer N_0 such that $\mu(\beta_n, v_n) < \mu(\beta, v) + \epsilon$ for n $\geq N_0$. Hence $c = M(v_n) = \mu(\beta_n, v_n) < \mu(\beta, v) + \epsilon \leq M(v) + \epsilon < c$, which is impossible. □

Proposition 3.7. *Let V,W be G-modules, and let V$\oplus$W be the induced G-module. Then M(v,w) $\leq min\{M(v), M(w)\}$ for all nonzero vectors v $\in$ V and w $\in$ W.*

Proof. Let X be a unit vector in $\mathfrak{P}$ and let v,w be nonzero vectors in V,W respectively. By the definitions of μ and M it follows that $\mu(X, (v, w)) = min\{\mu(X, v), \mu(X, w)\} \leq min\{M(v), M(w)\}$. The result follows since X is an arbitrary unit vector in $\mathfrak{P}$. □

Null cone

We say that v $\in$ V lies in the *null cone* if $\overline{G(v)}$ contains the zero vector. The next two results are the real analogues of Theorem 3.2 of [Nes].

Proposition 3.8. *For v$\in$ V the following conditions are equivalent :*

1) v lies in the null cone

2) M(v) > 0.

3) There exists X $\in \mathfrak{P}$ such that $e^{tX}(v) \to 0$ as t $\to +\infty$.

Proof. We show that 1) $\Rightarrow$ 3). By (1.6) there exists X $\in \mathfrak{P}$ and $v_0 \in$ V such that $e^{tX}(v) \to v_0$ as t $\to +\infty$ and G(v_0) is closed in V. By 1) $\{0\}$ and G(v_0) are closed orbits in $\overline{G(v)}$, and hence $v_0 = 0$ by (1.5).

We show that 3) $\Rightarrow$ 2). If $e^{tX}(v) \to 0$ as t $\to +\infty$ for some nonzero vector X $\in \mathfrak{P}$, then $\mu(-X, v) > 0$ by (3.1). Without loss of generality we may assume that X is a unit vector, and hence M(v) $\geq \mu(-X, v) > 0$.

We show that 2) $\Rightarrow$ 1). Choose a unit vector Y $\in \mathfrak{P}$ so that M(v) $= \mu(Y, v) > 0$. Then $e^{tY}(v) \to 0$ as t $\to -\infty$ by (3.1). □

Stable vectors

Following [Mu] and [Nes] we call a nonzero vector v$\in$ V *stable* if M(v) < 0. By (3.6) the stable vectors form an open set in the Hausdorff topology of V. We shall see later that the set of stable vectors is not always Zariski open in V. See Example 1 in section 5. In the complex setting for a linear action the stable vectors, where M is negative, are those vectors where G(v) is closed and G_v is discrete, and here the stable vectors form a nonempty Zariski open subset.

Proposition 3.9. *The following conditions are equivalent for a nonzero vector v in V :*

1) M(v) < 0; that is, v is stable.

2) The orbit G(v) is closed and the stability group G_v is compact.

3) The map F_v : G $\to [0, \infty)$ is proper, where $F_v(g) = |g(v)|^2$.

Remarks

1) The inner product $\langle , \rangle$ on V relative to which G is self adjoint is not unique, and the values of the M function depend on the choice of $\langle , \rangle$. However, equivalence 2) of the result above shows that the stable vectors of V are independent of the choice of $\langle , \rangle$.

2) It is easy to see that the map $F_v : G \to [0, \infty)$ is proper $\Leftrightarrow$ the map $f_v : G \to V$ given by $f_v(g) = g(v)$ is proper. Hence the result above extends (1.9).

Proof. We prove 1) $\Rightarrow 2)$. Since G is semisimple G is a closed subgroup of SL(V). (See the main theorem in section 6 of [Mo1]). If G(v) is not closed, then the map $f_v : G \to V$ given by $f_v(g) = g(v)$ is not a proper map by (1.9). By (3.1) and the lemma in the proof of (1.9) it follows that $\mu(Y, v) = 0$ for some nonzero element Y $\in \mathfrak{P}$. Hence M(v) $\geq \mu(Y, v) = 0$, which contradicts 1). Hence G(v) is closed in V. If G_v were noncompact, then it would follow immediately that $f_v : G \to V$ is not a proper map, which would lead to the same contradiction as above. Hence 1) $\Rightarrow$ 2).

We prove 2) $\Rightarrow$ 3). If $F_v : G \to \mathbb{R}$ is not proper, then $f_v : G \to$ V is also not proper, which contradicts (1.9).

We prove 3) $\Rightarrow$ 1). Suppose that M(v) ≥ 0 and choose a unit vector Y $\in \mathfrak{P}$ such that $\mu(Y, v) = M(v) \geq 0$. By (3.1) there exist a nonzero vector Y $\in \mathfrak{P}$ and a vector $v_0 \in$ V such that $e^{tY}(v) \to v_0$ as t $\to -\infty$. Hence F_v : G $\to [0, \infty)$ is not proper since $F_v(e^{tY}) \to |v_0|^2$ as t $\to -\infty$. This contradiction to the hypothesis of 3) shows that 3) $\Rightarrow 1)$. □

In the remainder of this section we derive some useful applications of the result above.

Corollary 3.10. *Suppose M(v′) <0 for some nonzero vector v′ of V. Then G acts stably on V.*

Proof. If $U = \{v \in V : M(v) < 0\}$, then U is open in the Hausdorff topology of V by (3.6). If $U' = \{v \in V : G(v) \text{ has maximal dimension}\}$, then U′ is a nonempty Zariski open subset of V. Since U′ is dense in V relative to the Hausdorff topology it follows that $U \cap U'$ is nonempty. If $v \in U \cap U'$, then G(v) is closed by (3.9) and G(v) has maximal dimension since v $\in U'$. The assertion now follows from (2.1). □

Remark Let G act stably on V, and let $O = \{v \in V : G(v) \text{ is closed and dim G(v) is maximal}\}$ If M(v′) < 0 for some nonzero vector v′ of V, then by (3.6) $\{v \in O : M(v) < 0\}$ is a nonempty open subset of O in the Hausdorff topology of V. However, this subset may not be Zariski open ; in particular it may not be a dense subset of O. See Example 1 in section 5.

Corollary 3.11. *Let v $\in$ V be a nonzero minimal vector. The following conditions are equivalent :*

1) M(v) < 0

2) G(v) is closed and G_v is compact.

3) The moment map m : V $\to \mathfrak{P}$ has maximal rank at v.

4) If X(v) = 0 for some X $\in \mathfrak{P}$, then $X = 0$.

Proof. The conditions 3) and 4) are equivalent by (1.7). Conditions 1) and 2) are equivalent by the preceding result. Since v is minimal the Lie algebra $\mathfrak{G}_v$ of G_v is self adjoint by (1.1),and hence $\mathfrak{G}_v = \mathfrak{K}_v \oplus \mathfrak{P}_v$. It follows that G_v is compact $\Leftrightarrow \mathfrak{P}_v = \{0\}$. Hence 2) $\Rightarrow$ 4). Since v is minimal G(v) is closed by (1.2) and hence 4) $\Rightarrow$ 2). □

Corollary 3.12. *Suppose that $G_{v'}$ is discrete for some nonzero vector v′ $\in$ V. Then there exists a nonempty G - invariant Zariski open subset O of V such that G(v) is closed, G_v is finite and M(v) < 0 for all v $\in$ O.*

Proof. By (2.5) there exists a nonempty G - invariant Zariski open subset O of V such that G(v) is closed and G_v is finite for all v ∈ O. It now follows from (3.9) that M(v) < 0 for all v ∈ O. □

Proposition 3.13. *Suppose that $G_{v'}$ is compact for some nonzero vector v′ ∈ V. Then G acts stably on V, and M(v) < 0 for some nonzero vector v ∈ V.*

Proof. G acts stably on V by 2) of (2.6). Let O be a nonempty Zariski open subset of V such that G(v) has maximal dimension and is closed for all v ∈ O. If U = $\{v \in V : G_v \text{ is compact}\}$, then U is nonempty and open in V by 1) of (2.6). If v ∈ $O \cap U$, then M(v) < 0 by (3.9). □

Remarks

1) Examples 1 and 2 of section 5 illustrate the conditions of (3.13).

2) It is not necessarily true that if G_v is compact then M(v) < 0. The remark following (2.5) gives an example where $G_v = \{Id\}$ and M(v) > 0.

The next application of (3.9) shows that stability of a vector v is, in a certain sense, inherited by closed subgroups H of G.

Corollary 3.14. *Let H be a closed subgroup of G. If M_G(v) < 0, then H(v) is closed and H_v is compact.*

Proof. Let w ∈ $\overline{H(v)}$, and let $\{h_n\} \subset H \subset G$ be a sequence such that $h_n(v) \to w$ as n $\to \infty$. Since M_G(v) < 0 it follows from 3) of (3.9) that $\{h_n\}$ has a subsequence converging to an element h of G, and h ∈ H since H is closed in G. Hence $w = h(v) \in H(v)$, which proves that H(v) is closed in V. By 2) of (3.9) G_v is compact. Since H is closed in G, G_v is compact and $H_v = H \cap G_v$ it follows that H_v is compact. □

Remark The corollary above is false if G(v) is closed but $M_G(v) = 0$.

Example Let $H = SL(2, \mathbb{R})$ act by conjugation on $\mathfrak{H} = \{A \in M(2, \mathbb{R}) : trace\ A = 0.\}$. Let G = H x H act on V = $\mathfrak{H} \oplus \mathfrak{H}$ by $(h_1, h_2)(X, Y) = (h_1 X h_1^{-1}, h_2 X h_2^{-1})$. Define elements v,w in $\mathfrak{H}$ by $v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$. Note that hvh$^{-1} = w$ if h $= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathfrak{H}$, and hence H(v) = H(w). The discussion later in Example 1 of section 5 shows that H(v) = H(w) is closed in $\mathfrak{H}$, and hence G((v,w)) = (H(v),H(w)) is closed in V = $\mathfrak{H} \oplus \mathfrak{H}$. Note that the stability group $G_{(v,w)} = H_v\ x\ H_w$ is noncompact since H_v consists of the diagonal matrices in H and $H_w = hH_v h^{-1}$. It follows that $M_G(v) \geq 0$ by (3.9), and we conclude that $M_G(v) = 0$ by (3.8) since G((v,w)) is closed.

Let $\Delta = \{(h, h) \in G : h \in H\}$. Clearly Δ is a closed subgroup of G, but we show that the orbit Δ((v,w)) is not closed in V. If h(t) = $diag\ (e^{-t}, e^t)$ and g(t) = $(h(t), h(t)) \in \Delta$, then g(t)((v,w)) → (v,v) as t → $+\infty$. Hence (v,v) ∈ $\overline{\Delta((v, w))}$. However, $\Delta_{(v,w)} = H_v \cap H_w = \pm\{Id\}$, while $\Delta_{(v,v)}$ contains g(t) for all t. It follows that (v,v) ∈ $\overline{\Delta((v, w))} - \Delta((v, w))$ since $\Delta_{(v,v)}$ is not conjugate in Δ to $\Delta_{(v,w)}$. We conclude that the orbit $\Delta((v, w))$ is not closed in V.

4. THE INDEX METHOD

Let V be a nontrivial G-module. For a nonzero element X of $\mathfrak{P}$ let $I_G(X)$ denote the largest dimension of a subspace W of V on which X is negative definite. Let $I_G(V) = min\{I_G(X) : 0 \neq X \in \mathfrak{P}\}$. We call $I_G(V)$ the *index* of G acting on V. Note that trace X = 0 for every X ∈ $\mathfrak{P}$ since G is semisimple, which implies that $[\mathfrak{G}, \mathfrak{G}] = \mathfrak{G}$. Hence every nonzero element X of $\mathfrak{P}$ has a negative eigenvalue. This shows that $I_G(V) \geq 1$ since V is a nontrivial G-module.

The index of G apparently depends on the choice of a G-compatible inner product $\langle , \rangle$ on V ; that is, an inner product $\langle , \rangle$ such that G is invariant under the involution $\theta : g \to (g^t)^{-1}$. However, this is not the case.

Proposition 4.1. *The index of G acting on V does not depend on the choice of G-compatible inner product* $\langle , \rangle$.

Proof. Let $\langle , \rangle_1$ and $\langle , \rangle_2$ be two G-compatible inner products on V, and let $\mathfrak{G} = \mathfrak{K}_1 \oplus \mathfrak{P}_1$ and $\mathfrak{G} = \mathfrak{K}_2 \oplus \mathfrak{P}_2$ denote the corresponding Cartan decompositions. It is known that there exists g ∈ G such that $\mathfrak{K}_2 = Ad(g)(\mathfrak{K}_1)$ and $\mathfrak{P}_2 = Ad(g)(\mathfrak{P}_1)$; see for example Theorem 7.2 of Chapter III in [H]. Since X and Ad(g)(X) acting on V have the same eigenvalues for all X ∈ $\mathfrak{P}_1$ if follows that $I^1_X(V) = I^2_{Ad(g)(X)}(V)$. It follows immediately that $I^1_G(V) = I^2_G(V)$. □

Proposition 4.2. *Let K denote a maximal compact subgroup of G. If* $I_G(V) > dim\ K$, *then* $\{v \in V : M(v) < 0\}$ *is an open subset of V with full measure in V.*

Proof. We carry out the proof in several steps

(1) Weight space decomposition of V

Let $\langle , \rangle$ be an inner product on V relative to which G is self adjoint. Let $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ be the Cartan decomposition of $\mathfrak{G}$ defined by the Cartan involution $\theta : g \to (g^t)^{-1}$ that leaves $G^{\mathbb{C}}(\mathbb{R})$ invariant. Fix a maximal abelian subspace $\mathfrak{A}$ of $\mathfrak{P}$. It is well known that every maximal abelian subspace of $\mathfrak{P}$ has the form Ad(k)($\mathfrak{A}$) for some k ∈ K, and every element of $\mathfrak{P}$ lies in some maximal abelian subspace of $\mathfrak{P}$. The elements of $\mathfrak{P}$ are symmetric with respect to $\langle , \rangle$, and hence $\mathfrak{A}$ is a commuting family of symmetric linear maps on V.

For $\lambda \in \mathfrak{A}^*$ let $V_\lambda = \{v \in V : X(v) = \lambda(X)v \text{ for all X } \in \mathfrak{A}\}$. If $\Lambda = \{\lambda \in \mathfrak{A}^* : V_\lambda \neq 0\}$, then Λ is a finite set, called the $weights$ of the representation, and we obtain the $weight\ space\ decomposition$

(*) $V = V_0 \oplus \sum_{\lambda \in \Lambda} V_\lambda$

where $V_0 = \{v \in V : X(v) = 0 \text{ for all X} \in \mathfrak{A}\}$.

(2) The subspaces V^+_X and V^-_X

For a nonzero element X of $\mathfrak{A}$ we let $\Lambda^+_X = \{\lambda \in \Lambda : \lambda(X) > 0\}$ and $\Lambda^-_X = \{\lambda \in \Lambda : \lambda(X) < 0\}$. We define $V^+_X = V_0 \oplus \sum_{\lambda \in \Lambda^+_X} V_\lambda$ and $V^-_X = \sum_{\lambda \in \Lambda^-_X} V_\lambda$. The following assertions follow routinely from the definitions :

a) $\mu(X, v) \geq 0$ for some nonzero X ∈ $\mathfrak{A}$ ⇔ $v \in V^+_X$.

b) $I_G(X) = dim\ V^-_X$.

c) $V = V^+_X \oplus V^-_X$.

(3) There exists a finite set of nonzero vectors $\{X_1, \ldots, X_N\} \subset \mathfrak{A}$ such that for every nonzero X ∈ $\mathfrak{A}$ there exists $1 \leq i \leq N$ such that $V^+_X = V^+_{X_i}$.

Since Λ is a finite set the number of distinct subsets $\{\Lambda^+_X : 0 \neq X \in \mathfrak{A}\}$ is also finite. Choose nonzero elements $\{X_1, \ldots, X_N\} \subset \mathfrak{A}$ such that for every nonzero X ∈ $\mathfrak{P}$ there exists $1 \leq i \leq N$ such that $\Lambda^+_X = \Lambda^+_{X_i}$. This is the desired set.

(4) $\{v \in V : M(v) \geq 0\} = \bigcup_{i=1}^N K(V^+_{X_i})$, where $\{X_1, \ldots, X_N\}$ are chosen as in (3).

By (2) it follows that M(v) ≥ 0 for all v ∈ $V^+_{X_i}$, $1 \leq i \leq N$. From the G-invariance of M we conclude that M(v) ≥ 0 for all v ∈ $\bigcup_{i=1}^N K(V^+_{X_i})$. Conversely, let v be a nonzero vector in V such that M(v) ≥ 0. Let X be a unit vector in $\mathfrak{P}$ such that $\mu(X, v) = M(v) \geq 0$. Choose k ∈ K such that Y = Ad(k)(X) ∈ $\mathfrak{A}$. Then $\mu(Y, k(v)) = \mu(X, v) \geq 0$. By (2) and (3) it follows that k(v) ∈ $V^+_Y = V^+_{X_i}$ for some i, $1 \leq i \leq N$. Hence v ∈ $K(V^+_{X_i}) \subset$

$\bigcup_{i=1}^N K(V_{X_i}^+)$, which completes the proof of (4).

We now complete the proof of the proposition. By hypothesis and (2) we obtain dim K $< I_G(V) \leq I_G(X) = dim\ V_X^- = dim\ V\ -\ dim\ V_X^+$ for all nonzero elements X of $\mathfrak{P}$. For $1 \leq i \leq N$ we define $\varphi_i : K$ x $V_{X_i}^+ \to V$ by $\varphi_i(k,v) = k(v)$. Note that dim $(K$ x $V_{X_i}^+) = dim\ K + dim\ V_{X_i}^+ < dim\ V$ for every i, and hence $K(V_{X_i}^+) = \varphi_i(K$ x $V_{X_i}^+)$ has measure zero in V. Hence $\{v \in V : M(v) \geq 0\}$ has measure zero in V by (4). □

Proposition 4.3. *Let $\{V_1,\ \dots, V_N\}$ be nontrivial G-modules, and let $V = V_1$ x ... x V_N be the corresponding G-module. Then $I_G(V) \geq \sum_{i=1}^N I_G(V_i)$.*

Proof. Let X $\in \mathfrak{A}$ be a nonzero element. Using the notation and discussion of (2) above it is easy to see that $V_X^- = \sum_{i=1}^N (V_i)_X^-$ and $I_G^V(X) = \sum_{i=1}^N I_G^{V_i}(X) \geq \sum_{i=1}^N I_G(V_i)$. If X $\in \mathfrak{P}$ is any nonzero element, then $Y = Ad(k)(X) \in \mathfrak{A}$ for some k $\in$ K. It follows that $I_G^V(X) = I_G^V(Y)$ since X and Y have the same eigenvalues on V. Hence $I_G(V) = min\{I_G^V(X) : 0 \neq X \in \mathfrak{P}\} = min\{I_G^V(X) : 0 \neq X \in \mathfrak{A}\} \geq \sum_{i=1}^N I_G(V_i)$. □

Corollary 4.4. *Let V be a G-module that is the direct sum of p $>$ $dim\ K$ nontrivial submodules. Then $\{v \in V : M(v) < 0\}$ is an open subset of full measure in V.*

Proof. For each of the submodules V_i the index of G is at least 1 by the discussion at the beginning of this section. Hence $I_G(V) \geq p > dim\ K$ by (4.3), and the assertion now follows from (4.2). □

We can strengthen the result above in the case that the G-submodules are all equivalent.

Proposition 4.5. *Let V be a nontrivial G-module of dimension n, and let G act diagonally on $V^p = V \oplus\ \dots\ \oplus V$ (p times), where p is any positive integer. Then*

1) If $p > n$, then there exists a nonempty Zariski open subset O of V such that $M(v) < 0$ for all v∈ O.

2) If $p = n$, then there exists a negative real number c and a nonempty Zariski open subset O of V^p such that M(v) = c for all v ∈ O.

3) If G = SL(V) and $1 \leq p \leq n-1$, then there exists a positive real number c such that M(v) = c for all nonzero v in V^p.

Proof. 1) By (3.12) it suffices to prove that $\mathfrak{G}_v = \{0\}$ for some nonzero v $\in V^p$. Since $p > n$ there exists $v = (v_1,\ \dots, v_p) \in V^p$ such that $V = span\{v_1,\ \dots, v_p\}$. If $X \in \mathfrak{G}_v$, then $0 = X(v) = (X(v_1),\ \dots, X(v_p))$, which implies that $X(v_i) = 0$ for $1 \leq i \leq p$. Hence $X = 0$.

2) Since $p = n$ there exists a nonempty Zariski open subset O of V^p such that $\{v_1,\ \dots, v_n\}$ is a basis of V for all $v = (v_1,\ \dots, v_n) \in$ O. By (3.5) it follows that there exists a real number c such that M(v) = c for all v ∈ O. To show that c is negative it suffices by (3.12) to show that $\mathfrak{G}_v = \{0\}$ for every v ∈ O. This follows as in 1) above.

3) Let $v = (v_1,\ \dots, v_p)$ be a nonzero element of V^p, where $1 \leq p \leq n-1$, and let X $\in \mathfrak{P}$ be an element such that $X = -Id$ on span(v). Then $e^{tX}(v) \to 0$ as t → ∞, and it follows from (3.8) that M(v) > 0. Since G acts transitively on V^p and M is G-invariant we conclude that M is constant on $V^p - \{0\}$. □

Remark If G = SL(V), then by the argument above a generic stabilizer G_v is discrete for G acting on V^n, n = dim V. By (3.11) and the result above a generic orbit G(v) is therefore a closed hypersurface in V^n. It is not difficult to show that $v = (v_1,\ \dots\ v_n) \in V^n$ is minimal for the G action ⇔ there exists a positive constant c such that $\langle v_i, v_j \rangle = c\ \delta_{ij}$. Note that GL(V) acts transitively on $V^n - \{0\}$.

For the index of G on a tensor product we have the following

Proposition 4.6. *Let V,W be G-modules. Then* $I_G(V \otimes W) \geq I_G(V) \cdot I_G W$.

Proof. If $0 \neq X \in \mathfrak{A}$, then X is negative definite on $V_X^- \otimes W_X^-$. Hence $I_G^{V \otimes W}(X) \geq (dim\ V_X^-) \cdot (dim\ W_X^-) = I_G^V(X) \cdot I_G^W(X) \geq I_G(V) \cdot I_G(W)$. If $0 \neq X \in \mathfrak{P}$ and $Y = Ad(k)(X) \in \mathfrak{A}$ for some k $\in$ K, then $I_G^{V \otimes W}(X) = I_G^{V \otimes W}(Y) \geq I_G(V) \cdot I_G(W)$. $\square$

We now apply the results above to the representations of G $= SL(2, \mathbb{R})$.

Proposition 4.7. *Let G* $= SL(2, \mathbb{R})$*, and let V be a G-module with dim V* ≥ 4*. If V has no trivial G-submodules, then* $\{v \in V : M(v) < 0\}$ *is a nonempty open subset of full measure in V.*

Proof. Let $\rho : G \to GL(V)$ be a rational representation. Let $\langle , \rangle_0$ be the standard inner product on $\mathbb{R}^2$, and let $\theta_0, \mathfrak{K}_0, \mathfrak{P}_0, \langle , \rangle, \theta, \mathfrak{K}$ and $\mathfrak{P}$ be defined as in the beginning of section (1.1). The elements of $\mathfrak{K}_0$ and $\mathfrak{P}_0$ are skew symmetric and symmetric 2x2 matrices respectively. Relative to $\langle , \rangle$ the elements of $\mathfrak{K} = \rho(\mathfrak{K}_0)$ and $\mathfrak{P} = \rho(\mathfrak{P}_0)$ are symmetric and skew symmetric linear transformations on V respectively. The maximal compact subgroup $\rho(K)$ of $\rho(G)$ is 1-dimensional, and $\mathfrak{P}$ is 2-dimensional.

If V is not irreducible, then the result follows by (4.4). Suppose now that V is irreducible. We need a preliminary result.

Lemma Let H_0 be any nonzero element of $\mathfrak{P}_0$. Then there exist c > 0, and X,Y $\in \mathfrak{G}$ such that if H′ = cH_0, then [H′,X] =2X, [H′,Y] = − 2Y and [X,Y] = H′.

Proof. If $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then $\{H_0, X, Y\}$ satisfies the conditions of the Lemma with c $= 1$. Hence $\{Ad(\varphi)H_0, Ad(\varphi)X, Ad(\varphi)Y\}$ also satisfies the conditions of the lemma for all $\varphi \in$ K. The group Ad K acts transitively on the lines through the origin in $\mathfrak{P}_0$ since dim $\mathfrak{P}_0 = 2$. This completes the proof. $\square$

We complete the proof of the proposition by showing that $I_G(H) \geq 2$ for all nonzero H $\in \mathfrak{P} = \rho(\mathfrak{P}_0)$. By the lemma above, for any nonzero element H_0 of $\mathfrak{P}_0$ there exist c > 0 and elements X,Y of $\mathfrak{G}$ such that H′ = cH_0, X and Y satisfy the conditions of the lemma. It suffices to prove that $I_G(\rho(H')) \geq 2$ since $I_G(H) = I_G(cH)$ for all positive real numbers c and all H $\in \mathfrak{P}$. By the representation theory of $\mathfrak{G} = \mathfrak{sl}(2, \mathbb{R})$ it is well known that the eigenvalues of $\rho(H')$ decrease from dim V -1 to $1-$ dim V in jumps of two. Since dim V ≥ 4 it follows that $\rho(H')$ has at least two distinct negative eigenvalues. Hence $I_G(H) = I_G(H') \geq 2$ for all nonzero H $\in \mathfrak{P}$, and it follows that $I_G(V) \geq 2 > 1 = dim\ \rho(K)$. The result now follows from (4.2). $\square$

Corollary 4.8. *Let G* $= SL(2, \mathbb{R})$*, and let V be a G-module with dim V* ≥ 3*. If V has no trivial G-submodules, then G acts stably on V.*

Proof. If dim V ≥ 4, then the assertion follows from the previous result and (3.10). If dim V $= 3$, then the G-module is equivalent to the adjoint representation of G on $\mathfrak{G} = \mathfrak{sl}(2, \mathbb{R})$ since V has no trivial G-submodules. In this case the assertion follows from Example 1 in section 5. $\square$

$Remark$ The strict inequality $I_G(V) > dim\ K$ in the statement of (4.2) cannot be relaxed to the weak inequality $I_G(V) \geq dim\ K$. If $G = SL(2, \mathbb{R}), V = \mathfrak{G}$ and G acts on V by the adjoint representation,then the eigenvalues of a nonzero element X $\in \mathfrak{P}$ are $\lambda, 0$ and $-\lambda$ for some positive number λ. Hence $I_G(V) = dim\ K = 1$. However, M(v) ≥ 0 for all v in a nonempty subset of V that is Hausdorff open but not Zariski open. It is still true that G(v) is closed for v in a nonempty Zariski open subset of V. See Example 1 in section 5.

If V $= \mathbb{R}^2$ and G acts on V in the standard way, then G(v) $= \mathbb{R}^2 - \{0\}$ for all nonzero v $\in$ V, and hence M(v) > 0 for all nonzero v $\in \mathbb{R}^2$ by (3.8).

5. EXAMPLES

In this section we compute information about the M-function in several cases, and we give special attention to the case that M is negative somewhere on V.

Example 1(Adjoint representation of SL(2,$\mathbb{R}$)) Let G $= SL(2,\mathbb{R})$ and let V $= \mathfrak{G} = \{A \in M(2,\mathbb{R}) : trace A = 0\}$. We let G act on V by conjugation. Let $\langle , \rangle$ be the inner product on V given by $\langle A, B\rangle =$ trace AB^t, where B^t denotes the standard transpose operation in M(2,$\mathbb{R}$). For g$\in$ G let g* denote the metric transpose of g acting on V relative to the inner product $\langle , \rangle$. A routine computation shows that $g^* = g^t$, and we conclude that G is self adjoint relative to $\langle , \rangle$. Moreover, the Cartan involution on $\mathfrak{G}$ is the standard one, and the corresponding Cartan decomposition $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ is given by $\mathfrak{K} = \{X \in \mathfrak{G} : X^t = -X\}$ and $\mathfrak{P} = \{X \in \mathfrak{G} : X^t = X\}$

Proposition 5.1. *Let* $O_1 = \{A \in V : det\ A < 0\}, O_2 = \{A \in V : det\ A > 0\}$ *and* $\Sigma = \{A \in V : det\ A = 0\} = \{A \in V : A^2 = 0\}$. *Then*

a) The sets O_1, O_2 *and* Σ *are G-invariant, and V is their disjoint union. The sets* O_1 *and* O_2 *are nonempty open subsets of V in the standard topology of V.*

b) If $\mathfrak{M}$ *denotes the minimal vectors for the action of G on V, then* $\mathfrak{M} = \mathfrak{K} \cup \mathfrak{P}$.

c) G(A) is closed in V if $A \in O_1 \cup O_2$. *The zero matrix lies in the closure of G(A) if A*$\in \Sigma$.

d) M(A) $= 0$ *for all A* $\in O_1$ *; M(A)* $= -\sqrt{2}/2$ *for all A* $\in O_2$ *and M(A)* $= \sqrt{2}/2$ *for all A* $\in \Sigma$.

Remark Assertion d) shows that $\{v \in V : M(v) < 0\}$ is nonempty and open in the Hausdorff topology of V but is not open in the Zariski topology of V. Assertion d) also shows that $\{v \in V : M(v) = 0\}$ has nonempty interior.

Proof. Assertion a) is clearly true. We prove b). By Example 3 of (1.1) we know that A $\in \mathfrak{M} \Leftrightarrow AA^t = A^tA$. Since A $\in M(2,\mathbb{R})$ it is easy to show that A $\in \mathfrak{M} \Leftrightarrow A = A^t$ or $A = -A^t$, which proves b).

We prove c). Recall that G(A) is closed in V $\Leftrightarrow$ G(A) $\cap\ \mathfrak{M}$ is nonempty. Assertion c) now follows immediately from b) and the next result.

Lemma 1) If A $\in O_1$, then there exists g $\in$ G such that g(A) $= gAg^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \in \mathfrak{P}$, where $\lambda = |det\ A|^{1/2}$.

2) If A $\in O_2$, then there exists g $\in$ G such that g(A) $= gAg^{-1} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \in \mathfrak{K}$, where $\lambda = (det\ A)^{1/2}$.

3) If A $\in \Sigma$, then there exists a sequence $\{g_n\} \subset$ G such that $g_n(A) = \begin{pmatrix} 0 & \lambda_n \\ 0 & 0 \end{pmatrix}$, where $\lambda_n \to 0$ as n $\to \infty$.

Proof. For A $\in V = \mathfrak{G}$ we recall that the characteristic polynomial of A acting in standard fashion on $\mathbb{R}^2$ is given by $c_A(x) = x^2 + detA$.

1) If A $\in O_1$, then A has eigenvalues λ and $-\lambda$, where $\lambda = |det\ A|^{1/2}$. Let $\{v_1, v_2\}$ be a positively oriented basis of $\mathbb{R}^2$ such that $A(v_1) = \lambda\ v_1$ and $A(v_2) = -\lambda\ v_2$. Let g $\in GL(2,\mathbb{R})$ be an element with det g > 0 such that $g(v_1) = e_1$ and $g(v_2) = e_2$, where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. Write $g = ch$, where $c > 0$ and det h $= 1$. Then $h(A) = hAh^{-1} = gAg^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \in \mathfrak{P}$.

2) If A $\in O_2$, then A has eigenvalues λi and $-\lambda$i, where $\lambda = (det\ A)^{1/2}$. Let v_1, v_2 be vectors in V, not both zero, such that $A(v_1 + iv_2) =$ i $\lambda(v_1 + iv_2)$. It is routine to check

that v_1 and v_2 are linearly independent, $A(v_1) = -\lambda\, v_2$ and $A(v_2) = \lambda\, v_1$. Hence A has matrix $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ relative to the basis $\{v_1, v_2\}$ of $\mathbb{R}^2$. If the basis $\{v_1, v_2\}$ is positively oriented, then choose g $\in GL(2,\mathbb{R})$ with $det g > 0$, $g(v_1) = e_1$ and $g(v_2) = e_2$. If the basis $\{v_1, v_2\}$ is negatively oriented, then choose g $\in GL(2,\mathbb{R})$ with $det g > 0$, $g(v_1) = e_1$ and $g(v_2) = -e_2$. In either case choose c > 0 and h in G such that $g = ch$. It follows that $hAh^{-1} = gAg^{-1} = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ in the first case and $hAh^{-1} = gAg^{-1} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$ in the second case.

3) If $A \in \Sigma$, then $A^2 = 0$. It suffices to consider the case that A is nonzero. Choose a basis v_1, v_2 of $\mathbb{R}^2$ such that $A(v_1) = 0$ and $A(v_2) = v_1$. As in 2) we choose g $\in GL(2,\mathbb{R})$ with $det\ g > 0$ such that $g(v_1) = e_1$ and $g(v_2) = e_2$ or $g(v_1) = e_1$ and $g(v_2) = -e_2$, depending upon whether $\{v_1, v_2\}$ is a positively oriented basis or not. If we write $g = ch$, where $c > 0$ and h $\in$ G, then $hAh^{-1} = gAg^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. If $h_n = \begin{pmatrix} 1/n & 0 \\ 0 & n \end{pmatrix} \in G$, then $(h_n h)A(h_n h)^{-1} = \begin{pmatrix} 0 & n^{-2} \\ 0 & 0 \end{pmatrix} \to 0$ or $(h_n h)A(h_n h)^{-1} = \begin{pmatrix} 0 & -n^{-2} \\ 0 & 0 \end{pmatrix} \to 0$ as n $\to \infty$. This completes the proof of the lemma. □

We prove assertion d) of the proposition. Let $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $\{H_0, X, Y\}$ is a basis of $\mathfrak{G}$ such that $[H_0, X] = 2X, [H_0, Y] = -2Y$ and $[X, Y] = H_0$. The space $\mathfrak{P}$ is 2-dimensional and the 1-dimensional maximal compact subgroup K $\approx S^1$ acts transitively on the circle of vectors in $\mathfrak{P}$ with a fixed length c for every positive number c. If H $\in \mathfrak{P}$, then H has eigenvalues λ and $-\lambda$ for some real number λ, and $|H|^2 = trace(H^2) = 2\lambda^2$. It follows that H is a unit vector in $\mathfrak{P} \Leftrightarrow$ H has eigenvalues $\sqrt{2}/2$ and $-\sqrt{2}/2$. In particular, if H is any unit vector $\in \mathfrak{P}$, then there exists k $\in$ K such that $kHk^{-1} = H_0/2\sqrt{2}$.

We show that M(A) $= \sqrt{2}/2$ if A $\in \Sigma$. The argument in the proof of 3) of the lemma above shows that for any A $\in \Sigma$ there exist g $\in$ G and $\lambda \in \mathbb{R}$ such that $gAg^{-1} = \lambda X$. Hence $M(A) = M(\lambda X) = M(X)$ by the G-invariance of M and by (3.4) since λ Id commutes with G on V. It suffices to prove that $M(X) = \sqrt{2}/2$.

Note that $\mu(H_0, X) = 2$ since $[H_0, X] = 2\,X$. Hence $\mu(H_0/2\sqrt{2}, X) = \sqrt{2}/2$. Now let H be an arbitrary unit vector in $\mathfrak{P}$ and let k $\in$ K be an element such that $kHk^{-1} = H_0/2\sqrt{2}$. Choose a real number θ such that $k = \begin{pmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{pmatrix}$. Then $kXk^{-1} = -sin\,\theta\, cos\,\theta\, H_0 + cos^2\theta\, X - sin^2\theta\, Y$. If a) sin $\theta \neq 0$, then $\mu(H, X) = \mu(kHk^{-1}, kXk^{-1}) = \mu(H_0/2\sqrt{2}, kXk^{-1}) = -\sqrt{2}/2$. If b) sin $\theta = 0$, then k $= Id$ or k $= -Id$, which implies that $H_0/2\sqrt{2} = kHk^{-1} = H$ and X $= kXk^{-1}$. In this case $\mu(H, X) = \mu(H_0/2\sqrt{2}, X) = \sqrt{2}/2$. From a) and b) it follows that M(X) $= \sqrt{2}/2$.

We show that M(A) $= -\sqrt{2}/2$ for all A $\in O_2$. For A $\in O_2$ we write A $= \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = aH_0 + bX + cY$ for suitable real numbers a,b,c. By hypothesis $a^2 + bc = -det\ A < 0$, and hence b and c are always nonzero. It follows by inspection that $\mu(H_0, A) = -2$ and hence $\mu(H_0/2\sqrt{2}, A) = -\sqrt{2}/2$. If H is any unit vector in $\mathfrak{P}$, then choose k $\in$ K such that $kHk^{-1} = H_0/2\sqrt{2}$. By the argument above $\mu(H, A) = \mu(kHk^{-1}, kAk^{-1}) = \mu(H_0/2\sqrt{2}, kAk^{-1}) = -\sqrt{2}/2$. This proves that M(A) $= -\sqrt{2}/2$.

We prove that M(A) $= 0$ for all A $\in O_1$. Since A has eigenvalues λ and $-\lambda$ there exists g $\in$ G with $gAg^{-1} = \lambda H_0$ by 1) of the Lemma. Hence $M(A) = M(gAg^{-1}) = M(\lambda H_0) = M(H_0)$. It suffices to prove that $M(H_0) = 0$. Note that $H_0 \in Ker\ H_0$ since $H_0(H_0) = [H_0, H_0] = 0$, and hence $0 = \mu(H_0, H_0) = \mu(H_0/2\sqrt{2}, H_0)$. If H is any unit vector in $\mathfrak{P}$,then choose $k = \begin{pmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{pmatrix} \in$ K such that $kHk^{-1} = H_0/2\sqrt{2}$. Then $\mu(H, H_0) = \mu(kHk^{-1}, kH_0k^{-1}) = \mu(H_0/2\sqrt{2}, cos(2\theta)\ H_0 + sin(2\theta)\ X + sin(2\theta)\ Y)$. If $sin(2\theta) \neq 0$, then $\mu(H, H_0) = -\sqrt{2}/2$. If $sin(2\theta) = 0$, then $kH_0k^{-1} = \pm H_0$, and $\mu(H, H_0) = \pm\mu(H_0/2\sqrt{2}, H_0) = 0$. Hence $M(H_0) = max\{\mu(H, H_0) : H \in \mathfrak{P}, |H| = 1\} = 0$. □

Example 2 The adjoint representation of G on $\mathfrak{G}$

We generalize the first example. Before stating the main result (Proposition 5.5) we establish some terminology and recall some useful facts.

Let G be a connected, noncompact semisimple Lie group with Lie algebra $\mathfrak{G}$, and let G act on V $= \mathfrak{G}$ by the adjoint action. Let B : $\mathfrak{G}$ x $\mathfrak{G} \to \mathbb{R}$ denote the Killing form of $\mathfrak{G}$. By Proposition 7.4 of [H,p.184] there exists a decomposition $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ such that B is positive definite on $\mathfrak{P}$ and negative definite on $\mathfrak{K}$ and the linear map $\theta : \mathfrak{G} \to \mathfrak{G}$ given by $\theta(K + P) = K - P$ is an automorphism of $\mathfrak{G}$ of order two with $\mathfrak{K}$ and $\mathfrak{P}$ as the $+1$ and -1 eigenspaces. If $\langle , \rangle$ is the inner product on $\mathfrak{G}$ given by $\langle X, Y\rangle = -B(\theta(X), Y)$, then ad($\mathfrak{K}$) and ad($\mathfrak{P}$) consist of skew symmetric and symmetric linear maps of $\mathfrak{G}$ respectively. In particular, Ad(G) is a self adjoint subgroup of GL($\mathfrak{G}$). Fix $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}, \theta$ and $\langle , \rangle$ as above.

Semisimple elements, Cartan subalgebras, root space decomposition and rank

An element X of $\mathfrak{G}$ is said to be *semisimple* if the extension of ad X : $\mathfrak{G} \to \mathfrak{G}$ to $\mathfrak{G}^{\mathbb{C}}$ is diagonalizable. A subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ is a *Cartan subalgebra* of $\mathfrak{G}$ if $\mathfrak{A}$ is a maximal abelian subalgebra of $\mathfrak{G}$ and every element of $\mathfrak{A}$ is semisimple. Equivalently, a subalgebra $\mathfrak{A}$ is a Cartan subalgebra of $\mathfrak{G}$ if its complexification $\mathfrak{A}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{G}^{\mathbb{C}}$. Every semisimple element X of $\mathfrak{G}$ is contained in a Cartan subalgebra of $\mathfrak{G}$ (cf. Proposition 4.6, page 420 of [H]).

For a Cartan subalgebra $\mathfrak{B}$ of $\mathfrak{G}^{\mathbb{C}}$ one has the *root space decomposition* $\mathfrak{G}^{\mathbb{C}} = \mathfrak{B} \oplus \sum_{\lambda \in \Phi} \mathfrak{G}^{\mathbb{C}}_{\lambda}$, where ad B $= \lambda(B)\ Id$ on the 1-dimensional subspace $\mathfrak{G}^{\mathbb{C}}_{\lambda}$ for all $\lambda \in \Phi$ and all B $\in \mathfrak{B}$. The finite set $\Phi \subset Hom(\mathfrak{B}, \mathbb{C})$ is the set of *roots* determined by $\mathfrak{B}$.

Any two Cartan subalgebras of $\mathfrak{G}$ have the same dimension. The *rank* of a semisimple Lie algebra, real or complex, is the dimension of a Cartan subalgebra.

There are only finitely many orbits of Ad(G) acting on the set of Cartan subalgebras of $\mathfrak{G}$. For every Cartan subalgebra $\mathfrak{B}$ of $\mathfrak{G}$ there exists g $\in$ G such that Ad(g)($\mathfrak{B}$) is a θ - invariant Cartan subalgebra of $\mathfrak{G}$ (cf. Corollary 4.2, page 419 of [H]).

Regular elements

If X $\in \mathfrak{G}$, then let $Z(X) = \{Y \in \mathfrak{G} : [X, Y] = 0\}$ denote the centralizer of X in $\mathfrak{G}$. Note that $Z(X) = \mathfrak{G}_X$ since X = ad X on $\mathfrak{G}$ by the definition of the adjoint action. Let $\mathfrak{G}(X, 0) = \{Y \in \mathfrak{G} : (ad\ X)^k(Y) = 0 \text{ for some positive integer k}\} = \mathrm{Ker}\{(\mathrm{ad\,X})^{\dim\,\mathfrak{G}}\}$. An element X of $\mathfrak{G}$ is *regular* if dim $\mathfrak{G}(X, 0) \leq$ dim $\mathfrak{G}(Y, 0)$ for all Y $\in \mathfrak{G}$. Let $\mathfrak{R}$ denote the set of regular elements of $\mathfrak{G}$. In similar fashion we define $\mathfrak{G}^{\mathbb{C}}(X, 0)$ for X $\in \mathfrak{G}^{\mathbb{C}}$ and what it means for X to be regular in $\mathfrak{G}^{\mathbb{C}}$. We let $\mathfrak{R}^{\mathbb{C}}$ denote the regular elements of $\mathfrak{G}^{\mathbb{C}}$. We note that $\mathfrak{R}$ and $\mathfrak{R}^{\mathbb{C}}$ are nonempty Zariski open subsets of $\mathfrak{G}$ and $\mathfrak{G}^{\mathbb{C}}$ respectively.

Proposition 5.2. $\mathfrak{R} = \mathfrak{R}^{\mathbb{C}} \cap \mathfrak{G} = \{X \in \mathfrak{G} : dim\ \mathfrak{G}(X, 0) = rank\ \mathfrak{G}\}$. *If X* $\in \mathfrak{R}$*, then* $Z(X) = \mathfrak{G}(X, 0)$ *is a Cartan subalgebra of* $\mathfrak{G}$*.*

Proof. If X $\in \mathfrak{R}^{\mathbb{C}} \subset \mathfrak{G}^{\mathbb{C}}$, then it is well known that $\mathfrak{G}^{\mathbb{C}}(X, 0)$ is a Cartan subalgebra of $\mathfrak{G}^{\mathbb{C}}$; see for example Theorem 3.1 of [H, p. 163]. In particular $dim_{\mathbb{C}}\ \mathfrak{G}^{\mathbb{C}}(X, 0) = rank_{\mathbb{C}}\ \mathfrak{G}^{\mathbb{C}}$. By the definition of regularity in $\mathfrak{G}^{\mathbb{C}}$ it follows that $dim_{\mathbb{C}}\ \mathfrak{G}^{\mathbb{C}}(X, 0) \geq rank_{\mathbb{C}}\ \mathfrak{G}^{\mathbb{C}}$ for any X $\in \mathfrak{G}^{\mathbb{C}}$ with equality $\Leftrightarrow X \in \mathfrak{R}^{\mathbb{C}}$. If X $\in \mathfrak{G}$, then it is easy to see that $\mathfrak{G}(X, 0)^{\mathbb{C}} = \mathfrak{G}^{\mathbb{C}}(X, 0)$. Since $rank_{\mathbb{R}}\ \mathfrak{G} = rank_{\mathbb{C}}\ \mathfrak{G}^{\mathbb{C}}$ it follows that $dim_{\mathbb{R}}\ \mathfrak{G}(X, 0) \geq rank_{\mathbb{R}}\ \mathfrak{G}$ with equality $\Leftrightarrow X \in \mathfrak{R}^{\mathbb{C}} \cap \mathfrak{G}$. This proves the first assertion of the proposition. To prove the second assertion note that $Z(X) \subset \mathfrak{G}(X, 0)$ for all X $\in \mathfrak{G}$. If X $\in \mathfrak{R} \subset \mathfrak{R}^{\mathbb{C}}$, then $\mathfrak{G}(X, 0)^{\mathbb{C}} = \mathfrak{G}^{\mathbb{C}}(X, 0)$ is a Cartan subalgebra of $\mathfrak{G}^{\mathbb{C}}$. Hence $\mathfrak{G}(X, 0)$ is a Cartan subalgebra of $\mathfrak{G}$. Since $\mathfrak{G}(X, 0)$ is abelian and X$\in \mathfrak{G}(X, 0)$ it follows that $\mathfrak{G}(X, 0) \subset Z(X)$. Hence $\mathfrak{G}(X, 0) = Z(X) = \mathfrak{G}_X$ is a Cartan subalgebra of $\mathfrak{G}$. This completes the proof of the second assertion. □

$Remark$ We include some further information about regular elements of $\mathfrak{G}$, but we omit the details of the proofs since this information is not needed for the article. Note that the third assertion of the next statement together with the first assertion of (5.5) below shows that the set of regular elements in $\mathfrak{G}$ is the set of elements in $\mathfrak{G}$ whose orbits under Ad G are closed and of maximal dimension.

Proposition 5.3. *For a noncompact semisimple Lie algebra* $\mathfrak{G}$ *the following assertions are equivalent :*

1) X is a regular element of $\mathfrak{G}$*.*
2) X is semisimple and Z(X) $= \mathfrak{G}_X$ *is a Cartan subalgebra of* $\mathfrak{G}$*.*
3) X is semisimple and dim $\mathfrak{G}_X \leq$ *dim* $\mathfrak{G}_Y$ *for all Y* $\in \mathfrak{G}$*.*

$Minimal\ elements\ in$ $\mathfrak{G}$ By (5.3.1) of [RS] one knows that X $\in \mathfrak{G}$ is minimal for the action of Ad G on $\mathfrak{G} \Leftrightarrow 0 = [X, \theta(X)]$. By (2.9) $\mathfrak{M} = \{X \in \mathfrak{G} : \mathfrak{G}_X = Z(X) \text{ is invariant under } \theta\}$. We give a third description of $\mathfrak{M}$.

Proposition 5.4. *Let G be as above, and let* $\mathfrak{M}$ *denote the set of minimal vectors for the action of Ad G on* $\mathfrak{G}$*. Then* $\mathfrak{M}$ *is the union of all* θ*-invariant Cartan subalgebras of* $\mathfrak{G}$*.*

Proof. Let $\mathfrak{A}$ be a θ-invariant Cartan subalgebra of $\mathfrak{G}$. We show first that $\mathfrak{A} \subset \mathfrak{M}$. Let X be an element of $\mathfrak{A}$ and write $X = K + P$, where $K = (1/2)(X + \theta(X)) \in \mathfrak{A} \cap \mathfrak{K}$ and $P = (1/2)(X - \theta(X)) \in \mathfrak{A} \cap \mathfrak{P}$. Then $0 = [K, P] = (1/2)[\theta(X), X]$. Hence X $\in \mathfrak{M}$, which proves that $\mathfrak{A} \subset \mathfrak{M}$.

To complete the proof we first note that Ad K leaves invariant $\mathfrak{K}$ and $\mathfrak{P}$, and it follows immediately that θ commutes with the elements of Ad K. In particular, if $\mathfrak{A}$ is a θ-invariant Cartan subalgebra of $\mathfrak{G}$, then $Ad(\varphi)(\mathfrak{A})$ is also a θ-invariant Cartan subalgebra of $\mathfrak{G}$ for all $\varphi \in$ K.

It remains only to prove that if X is an element of $\mathfrak{M}$, then X lies in a θ-invariant Cartan subalgebra of $\mathfrak{G}$. Since X is minimal the orbit Ad G(X) is closed in $\mathfrak{G}$ by (1.2), and it follows from 1) of the next result that X is semisimple. By earlier remarks we may choose a Cartan subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ that contains X and an element g of G such that $\mathfrak{B} = Ad(g)(\mathfrak{A})$ is a θ-invariant Cartan subalgebra of $\mathfrak{G}$. The element Y = Ad(g)(X) lies in $\mathfrak{B} \subset \mathfrak{M}$ by the first paragraph of the proof, and hence X $\in Ad\ G(Y) \cap \mathfrak{M}$. By (1.2) it follows that $X = Ad(\varphi)(Y)$ for some $\varphi \in$ K. Hence $X \in Ad(\varphi)(\mathfrak{B})$, which is a θ-invariant Cartan subalgebra of $\mathfrak{G}$ by the discussion above. □

Proposition 5.5. *Let G act on V* $= \mathfrak{G}$ *by the adjoint action. Then*

1) Let $0 \neq X \in \mathfrak{G}$*. Then the orbit Ad G(X) is closed in* $\mathfrak{G} \Leftrightarrow$ *X is semisimple.*
2) Let $0 \neq X \in \mathfrak{G}$*. Then* $M(X) > 0 \Leftrightarrow$ *ad X :* $\mathfrak{G} \to \mathfrak{G}$ *is nilpotent.*

3) Let $0 \neq X \in \mathfrak{G}$. Then the following conditions are equivalent.

a)$M(X) < 0$.

b) The stability group G_X is compact.

c) $\mathfrak{G}_X = Z(X) \subset Ad(g)(\mathfrak{K})$ for some g ∈ G.

$Remark$ Assertion 1) of the result above is due to Borel-Harish-Chandra with a different proof. See Proposition 10.1 of [BH].

Proof. 1) Let $\theta : \mathfrak{G} \rightarrow \mathfrak{G}$ be the Cartan involution corresponding to the Cartan decomposition $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$. Let $X \in \mathfrak{G}$ be semisimple. By earlier discussion X ∈ $\mathfrak{B}$, where $\mathfrak{B}$ is a Cartan subalgebra of $\mathfrak{G}$. Choose g ∈ G such that $\mathfrak{A} = Ad(g)(\mathfrak{B})$ is a θ- invariant Cartan subalgebra of $\mathfrak{G}$. By the first paragraph of the proof of the previous result we see that $Y = Ad(g)(X)$ is a minimal element of $\mathfrak{G}$, and hence Ad(G)(Y) = Ad(G)(X) is closed in $\mathfrak{G}$ by (1.2).

Conversely, suppose that Ad(G)(X) is closed in $\mathfrak{G}$. By (1.2) there exists an element g ∈ G such that $Y = Ad(g)(X)$ is minimal. If we write $Y = K + P$, where $K \in \mathfrak{K}$ and $P \in \mathfrak{P}$, then by Lemma 5.3.1 of [RS] we obtain $0 = [\theta(Y), Y] = 2[K, P]$. Hence ad K and ad P commute. We observed earlier that ad K and adP are skew symmetric and symmetric respectively relative to the canonical inner product $\langle , \rangle$ on $\mathfrak{G}$. Hence both ad K and adP are semisimple on $\mathfrak{G}^{\mathbb{C}}$ and since they commute they have a common basis of eigenvectors in $\mathfrak{G}^{\mathbb{C}}$. Hence $Y = K + P$ is semisimple, and we conclude that $X = Ad(g^{-1})(Y)$ is semisimple since the set of semisimple elements of $\mathfrak{G}$ is invariant under all automorphisms of $\mathfrak{G}$.

2) Suppose first that ad X : $\mathfrak{G} \rightarrow \mathfrak{G}$ is nilpotent for some element X of $\mathfrak{G}$. Then $ad(\varphi(X)) = \varphi \circ ad\ X \circ \varphi^{-1}$ is nilpotent for all $\varphi \in$ Aut($\mathfrak{G}$). In particular adY : $\mathfrak{G} \rightarrow \mathfrak{G}$ is nilpotent for all Y ∈ $\overline{Ad\ G(X)}$, the closure in $\mathfrak{G}$ of the orbit Ad G(X). Note that Ad G(X) is not closed in $\mathfrak{G}$ by 1) ; ad X cannot be both semisimple and nilpotent unless ad X = 0, which implies that X = 0 since the center of a semisimple Lie algebra is trivial. By (1.6) there exists H ∈ $\mathfrak{P}$ and Y ∈ $\overline{Ad\ G(X)}$ such that Ad G(Y) is closed in $\mathfrak{G}$ and $Ad\ e^{tH}(X) = e^{t\ adH}(X) \rightarrow Y$ as t → ∞. Since Ad G(Y) is closed in $\mathfrak{G}$ it follows from 1) that ad Y is semisimple. Hence Y = 0 by the argument above since ad Y is also nilpotent. It follows from (3.8) that M(X) > 0.

Conversely, suppose that M(X) > 0 and choose a unit vector H ∈ $\mathfrak{P}$ such that $\mu(H, X) = M(X) > 0$. Let Λ denote the set of all eigenvalues of ad H, including zero, and let $\mathfrak{G}_\lambda \subset \mathfrak{G}$ denote the corresponding eigenspace for ad H.

Lemma Let Y ∈ $\mathfrak{G}$ be arbitrary. If ad X(Y) ≠ 0, then $\mu(H, ad\ X(Y)) \geq \mu(H, X) + \mu(H, Y)$.

Proof. Write $X = \sum_{\lambda \in \Lambda} X_\lambda$ and $Y = \sum_{\sigma \in \Lambda} Y_\sigma$. Then ad X(Y) = $\sum_{\lambda, \sigma \in \Lambda} [X_\lambda, Y_\sigma]$. Note that $[X_\lambda, Y_\sigma] \in \mathfrak{G}_{\lambda+\sigma}$ since ad H is a derivation of $\mathfrak{G}$. If $[X_\lambda, Y_\sigma] \neq 0$, then $X_\lambda \neq 0$, which implies that $\lambda \geq \mu(H, X)$, and $Y_\sigma \neq 0$, which implies that $\sigma \geq \mu(H, Y)$. Hence $\lambda + \sigma \geq \mu(H, X) + \mu(H, Y)$ if $[X_\lambda, Y_\sigma] \neq 0$. This proves the lemma. □

We now complete the proof of 2). Suppose that (ad X)^N(Y) is nonzero for some positive integer N and some element Y of $\mathfrak{G}$. From the lemma above it follows that $\mu(H, (ad\ X)^N(Y) \geq N\mu(H, X) + \mu(H, Y)$. If c_1 and c_2 are the smallest and the largest eigenvalues of ad H on $\mathfrak{G}$, then $c_2 \geq \mu(H, (ad\ X)^N(Y) \geq N\mu(H, X) + \mu(H, Y) \geq N\mu(H, X) + c_1$. We conclude that $N \leq (c_2 - c_1)/\mu(H, X) = (c_2 - c_1)/M(X)$. It follows that (ad X)$^N = 0$ on $\mathfrak{G}$ if $N > (c_2 - c_1)/M(X)$. Hence ad X : $\mathfrak{G} \rightarrow \mathfrak{G}$ is nilpotent if $M(X) > 0$.

We prove 3). The assertion a) ⇒ b) follows immediately from (3.9). We show b) ⇒ a). If G_X is compact, then the elements of the Lie algebra $\mathfrak{G}_X$ are skew symmetric hence semisimple relative to a G_X - invariant inner product on V = $\mathfrak{G}$. In particular ad X : $\mathfrak{G} \to \mathfrak{G}$ is semisimple, and by 1) it follows that Ad G(X) is closed in $\mathfrak{G}$. It follows that M(X) < 0 by (3.9).

We show a) ⇒ c). If M(X) < 0, then G_X is compact by (3.9). Let K* be a maximal compact subgroup of G that contains G_X, and let g ∈ G be an element such that $gKg^{-1} = K*$. Then $\mathfrak{G}_X = Z(X) \subset \mathfrak{K}^* = Ad(g)(\mathfrak{K})$.

We show c) ⇒ a). Choose g ∈ G such that Z(X) $\subset Ad(g)(\mathfrak{K})$ and let Y $= Ad(g^{-1})(X)$. Then $Z(Y) \subset \mathfrak{K}$ and M(Y) = M(X). It suffices to prove that M(Y) < 0. Since $Y \in \mathfrak{K}$ it follows that $\theta(Y) = Y$ and hence Y is minimal by (5.3.1) of [RS] since $[Y, \theta(Y)] = 0$. Since $\mathfrak{G}_Y \cap \mathfrak{P} = Z(Y) \cap \mathfrak{P} \subset \mathfrak{K} \cap \mathfrak{P} = \{0\}$ it follows that M(Y) < 0 by (3.11). □

We now reach the main result of this example, which generalizes the first example where $G = SL(2, \mathbb{R})$.

Proposition 5.6. *Let* $M^- = \{X \in \mathfrak{G} : M(X) < 0\}$. *Then*

1) M^- *is nonempty* ⇔ $rank\ \mathfrak{G} = rank\ \mathfrak{K}$, *where* $\mathfrak{K}$ *is the +1 eigenspace of the Cartan involution* $\theta : \mathfrak{G} \to \mathfrak{G}$.

2) Let $rank\ \mathfrak{G} = rank\ \mathfrak{K}$. *Then*

a) $M^- \subset \bigcup_{g \in G} Ad(g)(\mathfrak{K})$.

b) $\mathfrak{R} \cap \bigcup_{g \in G} Ad(g)(\mathfrak{K}) \subset M^-$.

Remark It is not difficult to show that $\bigcup_{g \in G} Ad(g)(\mathfrak{K}) = \{X \in \mathfrak{G} : \text{ad X is semisimple with eigenvalues in i}\mathbb{R}\}$. We omit the details of the proof.

Proof. We prove 1). If M^- is nonempty, then M(X) < 0 for some X∈ $\mathfrak{G}$. By 3) of (5.5) there exists g ∈ G such that $\mathfrak{G}_X = Z(X) \subset Ad(g)(\mathfrak{K})$. If Y $= Ad(g^{-1})(X))$, then Z(Y) $\subset \mathfrak{K}$. Since ad Y is skew symmetric on $\mathfrak{G}$ with respect to the canonical inner product it is semisimple on $\mathfrak{G}^{\mathbb{C}}$ and there exists a Cartan subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ with Y ∈ $\mathfrak{A}$. Hence Y $\in \mathfrak{A} \subset Z(Y) \subset \mathfrak{K}$ and it follows that $rank\ \mathfrak{K} = rank\ \mathfrak{G}$.

Conversely, suppose that $rank\ \mathfrak{K} = rank\ \mathfrak{G}$, and let $\mathfrak{A}$ be a Cartan subalgebra of $\mathfrak{G}$ with $\mathfrak{A} \subset \mathfrak{K}$. It suffices to show that there exists an element X of $\mathfrak{A}$ such that $Z(X) = \mathfrak{A}$, for then $X \in M^-$ by 3) of the previous result. Since $\mathfrak{A}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{G}^{\mathbb{C}}$ we have the root space decomposition $\mathfrak{G}^{\mathbb{C}} = \mathfrak{A}^{\mathbb{C}} \oplus \sum_{\lambda \in \Lambda} \mathfrak{G}^{\mathbb{C}}_{\lambda}$. If X is an element of $\mathfrak{A}$, then a routine argument shows that $Z(X)^{\mathbb{C}} = \{Z \in \mathfrak{G}^{\mathbb{C}} : [X, Z] = 0\} = \mathfrak{A}^{\mathbb{C}} \oplus \sum_{\lambda(X)=0} \mathfrak{G}_{\lambda}$. For every root λ we know that $\lambda : \mathfrak{A}^{\mathbb{C}} \to \mathbb{C}$ is nonzero, and hence Ker $\lambda \cap \mathfrak{A}$ must be a proper subspace of $\mathfrak{A}$. Since there are only finitely many roots λ we may choose a nonzero X ∈ $\mathfrak{A}$ such that $\lambda(X) \neq 0$ for all roots λ. It follows that $Z(X)^{\mathbb{C}} = \mathfrak{A}^{\mathbb{C}}$, which implies that $Z(X) = \mathfrak{A}$ and completes the proof of 1).

Let $rank\ \mathfrak{K} = rank\ \mathfrak{G}$, and let X ∈ M^-. By 3) of (5.5) X ∈ $Ad(g)(\mathfrak{K})$ for some g ∈ G, which proves 2a). We prove 2b). Let X ∈ $\mathfrak{R}$ be an element such that Y $= Ad(g)(X) \in \mathfrak{K}$ for some element g ∈ G. Note that M(Y) = M(X) by the G-invariance of M, and hence it suffices to prove that M(Y) < 0. Let $\mathfrak{A}$ be a maximal abelian subspace of $\mathfrak{K}$ that contains Y. It is known that Ad K acts transitively on the maximal abelian subspaces of $\mathfrak{K}$, and one of these subspaces is a Cartan subalgebra of $\mathfrak{G}$ since rank $\mathfrak{K}$ = rank $\mathfrak{G}$. Hence all maximal abelian subspaces of $\mathfrak{K}$, and in particular $\mathfrak{A}$, are Cartan subalgebras of $\mathfrak{G}$. Moreover, $\mathfrak{A} \subseteq$ Z(Y) and Z(Y) is a Cartan subalgebra of $\mathfrak{G}$ by (5.2) since X and $Y = Ad(g)X$ are regular. It follows that $\mathfrak{A} = Z(Y) \subset \mathfrak{K}$. The element Y is minimal for the action of Ad G since $[Y, \theta(Y)] = [Y, Y] = 0$, and $\mathfrak{G}_Y \cap \mathfrak{P} = Z(Y) \cap \mathfrak{P} = \{0\}$. It follows from (3.11) that M(Y) = M(X) < 0. This proves 2b), □

Admissible semisimple Lie algebras We say that a noncompact semisimple Lie algebra $\mathfrak{G}$ is *admissible* if $rank\ \mathfrak{G} = rank\ \mathfrak{K}$, where $\mathfrak{K}$ is a maximal compact subalgebra of $\mathfrak{G}$. We wish to determine the admissible noncompact semisimple Lie algebras. If $\mathfrak{G}$ is admissible and $\mathfrak{G}_c$ is compact and semisimple, then $\mathfrak{G} \oplus \mathfrak{G}_c$ is admissible. Hence, without loss of generality, we may assume that $\mathfrak{G}$ has no compact factors. Next we reduce to the case that $\mathfrak{G}$ is simple and noncompact.

Lemma 5.7. *Let $\mathfrak{G}$ be a semisimple Lie algebra with no compact factors, and write $\mathfrak{G} = \mathfrak{G}_1 \oplus\ \ldots\ \oplus \mathfrak{G}_N$, where $\{\mathfrak{G}_1,\ \ldots, \mathfrak{G}_N\}$ are simple noncompact Lie algebras. Then $\mathfrak{G}$ is admissible $\Leftrightarrow$ $\mathfrak{G}_k$ is admissible for $1 \leq k \leq N$.*

Proof. If $\mathfrak{K}_i$ is a maximal compact subalgebra of $\mathfrak{G}_i$ for $1 \leq i \leq N$, then $\mathfrak{K} = \mathfrak{K}_1 \oplus\ \ldots\ \oplus \mathfrak{K}_N$ is a maximal compact subalgebra of $\mathfrak{G} = \mathfrak{G}_1 \oplus\ \ldots\ \oplus \mathfrak{G}_N$. Hence $rank\ \mathfrak{K} = \sum_{i=1}^{N} rank\ \mathfrak{K}_i \leq \sum_{i=1}^{N} rank\ \mathfrak{G}_i = rank\ \mathfrak{G}$, with equality $\Leftrightarrow$ $rank\ \mathfrak{K}_i = rank\ \mathfrak{G}_i$ for $1 \leq i \leq N$. □

Admissible simple Lie algebras Before listing the admissible noncompact simple Lie algebras we recall the way that real noncompact simple Lie algebras are constructed, up to isomorphism. The results are due to Elie Cartan. For further discussion see for example [H, pp. 451-455].

Let $\mathfrak{U}$ be a real compact simple Lie algebra. Then $\mathfrak{U}^{\mathbb{C}}$ is a complex simple Lie algebra. Conversely, any complex simple Lie algebra is isomorphic to $\mathfrak{U}^{\mathbb{C}}$ for a real compact simple Lie algebra $\mathfrak{U}$, and the compact real form $\mathfrak{U}$ is uniquely determined up to isomorphism.

Let $\mathfrak{G}$ be a complex simple Lie algebra. A real simple Lie algebra $\mathfrak{G}_0$ is called a *real form* for $\mathfrak{G}$ if $\mathfrak{G}_0^{\mathbb{C}} = \mathfrak{G}$. The noncompact real forms of $\mathfrak{G}$ are determined as follows by the involutions of $\mathfrak{U}$, where $\mathfrak{U}$ is the compact real form of $\mathfrak{G}$. Let $\theta : \mathfrak{U} \to \mathfrak{U}$ be an automorphism of order two, and let $\mathfrak{U} = \mathfrak{K}_0 \oplus \mathfrak{P}_*$, where $\mathfrak{K}_0$ and $\mathfrak{P}_*$ are the $+1$ and -1 eigenspaces of θ in $\mathfrak{U}$. Let $\mathfrak{P}_0 = i\ \mathfrak{P}_* \subset \mathfrak{G}$, and let $\mathfrak{G}_0 = \mathfrak{K}_0 \oplus \mathfrak{P}_0$. Then $\mathfrak{G}_0$ is a real simple noncompact Lie algebra and a real form for $\mathfrak{G}$. Moreover, if $\theta_0 : \mathfrak{G}_0 \to \mathfrak{G}_0$ is the linear isomorphism whose $+1$ and -1 eigenspaces are $\mathfrak{K}_0$ and $\mathfrak{P}_0$ respectively, then θ_0 is an automorphism of $\mathfrak{G}_0$ of order two. The subalgebra $\mathfrak{K}_0$ is a maximal compact subalgebra of $\mathfrak{G}$. All noncompact real forms $\mathfrak{G}_0$ of $\mathfrak{G}$ and Cartan involutions θ_0 of $\mathfrak{G}_0$ arise in this fashion from an appropriate involutive automorphism θ of the compact real form $\mathfrak{U}$ of $\mathfrak{G}$.

Let $\mathfrak{G}_0$ be a real simple noncompact Lie algebra with Cartan involution θ_0, and let $\mathfrak{U}$ be the compact simple Lie algebra with involution θ that constructs $\{\mathfrak{G}_0, \theta_0\}$ as above. Since $\mathfrak{U}^{\mathbb{C}} = \mathfrak{G}_0^{\mathbb{C}}$ it follows that $rank\ \mathfrak{U} = rank\ \mathfrak{G} = rank\ \mathfrak{G}_0$. Hence we obtain the following criterion :

Lemma A real simple Lie algebra $\mathfrak{G}_0 = \mathfrak{K}_0 \oplus \mathfrak{P}_0$ is admissible $\Leftrightarrow$ $rank\ \mathfrak{U} = rank\ \mathfrak{K}_0$.

Using this criterion it is now easy to use the discussion on pages 451-455 and the Table on page 518 of [H] to reach the following conclusion, using the notation of Helgason :

Proposition 5.8. *1) The admissible real simple noncompact Lie algebras arise from involutions of type A III, D III, C I, C II, E II, E III, E V, E VI, E VII, E VIII, E IX, F I, F II, G.*

2) The nonadmissible real simple noncompact Lie algebras arise from involutions of type A I, A II, BD I, E I, E IV.

Example 3 The diagonal adjoint action of G on $\mathfrak{G}$ x ... X $\mathfrak{G}$ (p times)

The previous example lists necessary and sufficient conditions for M to take on negative values for the adjoint action of G on $\mathfrak{G}$. Even when M does take on negative values it does not do so on a Zariski open set as Examples 1 and 2 show. By contrast the situation is much simpler if G acts by the diagonal adjoint action on p ≥ 2 copies of $\mathfrak{G}$.

Proposition 5.9. *Let G act on V = $\mathfrak{G}$ x ... x $\mathfrak{G}$ (p times) by the diagonal adjoint action. If $p \geq 2$, then there exists a nonempty G - invariant Zariski open subset O of V such that G_v is finite and M(v) < 0 for all v in O.*

Proof. By (3.12) it suffices to show that $\mathfrak{G}_X = \{0\}$ for some $0 \neq X = (X_1, \, ... \, X_p) \in$ V. Hence it suffices to consider the case $p = 2$ since $\mathfrak{G}_X = \bigcap_{i=1}^{p} \mathfrak{G}_{X_i}$.

We use two preliminary results whose proofs we give in Appendix 1.

Lemma 1 Let $\mathfrak{G}$ be a finite dimensional real Lie algebra, and let p ≥ 2 be an integer. Let $\Sigma^p = \{(A_1, \, ... \, , A_p) \in \mathfrak{G}^p = \mathfrak{G} \, x \, ... \, \mathfrak{G}(\text{p times}) : \{\mathrm{A_1, \, ... \, , A_p}\} \text{ generate a proper subalgebra of } \mathfrak{G}\}$. Then Σ^p is a variety in $\mathfrak{G}^p$.

Lemma 2 Let $\mathfrak{G}$ be a finite dimensional real semisimple Lie algebra, and let $\Sigma^p \subset \mathfrak{G}^p$ be the variety of Lemma 1. Then Σ^p is a proper variety for every p ≥ 2.

We now complete the proof of the proposition. Let $O_1 = \{(X, Y) \in \mathfrak{G} \; x \; \mathfrak{G} : X \text{ and Y are regular elements of } \mathfrak{G}\}$. Then O_1 is a nonempty Zariski open subset of $\mathfrak{G} \; x \; \mathfrak{G}$ since the regular elements of $\mathfrak{G}$ form a Zariski open subset of $\mathfrak{G}$. Let $O_2 = \{(X, Y) \in \mathfrak{G} \; x \; \mathfrak{G} : \mathfrak{G} \text{ is the smallest subalgebra of } \mathfrak{G} \text{ containing X and Y}\}$. Then O_2 is a nonempty Zariski open subset of $\mathfrak{G} \; x \; \mathfrak{G}$ by Lemmas 1 and 2. We assert that if (X,Y) $\in O = O_1 \, \cap \, O_2$, which is nonempty and Zariski open in $\mathfrak{G} \; x \; \mathfrak{G}$, then $\mathfrak{G}_{(X,Y)} = \mathfrak{G}_X \, \cap \, \mathfrak{G}_Y = Z(X) \, \cap \, Z(Y) = \{0\}$.

Let $(X, Y) \in O$ and $\xi \in$ Z(X) $\cap$ Z(Y) be given. Then Z(ξ) is a subalgebra of $\mathfrak{G}$ that contains X and Y, and hence $Z(\xi) = \mathfrak{G}$ by the definition of O_2. It follows that $\xi = 0$ since $\mathfrak{G}$ is semisimple. □

Example 4 The action of H = SL(q,ℝ) x SL(p,ℝ) on V = $\mathfrak{so}(q, \mathbb{R}) \; x \; ... \; x \; \mathfrak{so}(q, \mathbb{R})$ (p times)

Let $G = SL(q, \mathbb{R})$ act on $\mathfrak{so}(q, \mathbb{R})$ by $g(C) = gCg^t$. Let $H = SL(q, \mathbb{R}) \; x \; SL(p, \mathbb{R})$ act on $V = \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ by $(g, h)(C \otimes v) = gCg^t \otimes h(v)$. Recall that $V = \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ is isomorphic to V = $\mathfrak{so}(q, \mathbb{R}) \; x \; ... \; x \; \mathfrak{so}(q, \mathbb{R})$ (p times). See the next example and the proof of (3.5) for further discussion.

We say that a pair (p,q) is $exceptional$ if H_C has positive dimension for all C in V. If (p,q) is a nonexceptional pair, then by Corollary 3.12 there exists a nonempty Zariski open subset O of $V = \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ such that if v $\in$ O, then H(v) is closed, H_v is finite and M(v) < 0.

If a pair (p,q) is exceptional, then so is the dual pair (D$-$p,q), where D $= (1/2)q(q - 1) = dim \; \mathfrak{so}(q, \mathbb{R})$. For a discussion of duality in this context see Corollary 5.8, Proposition 5.9 and Corollary 5.10 of [Eb3]. That discussion is a special case of a more general treatment of duality in Lemma 2 of [El].

The following is a complete list of exceptional pairs, up to the duality between (p,q) and (D$-$p,q),

TABLE OF EXCEPTIONAL PAIRS

(1,q) for q ≥ 2
(q(q$-$1)/2, q) for q ≥ 2
(2,2k+1)
(2,2k) for k ≥ 3
(2,4
(3,4)
(3,5)

(3,6)

The table above comes from Table 1 of the proposition in section 5.4 of [Eb2]. Table 1 is based on Table 6 of [El] and Tables 2a,2b of [KL].

Example 5 The action of G = SL(q,$\mathbb{R}$) on V = $\mathfrak{so}(q, \mathbb{R})$ x ... x $\mathfrak{so}(q, \mathbb{R})$ (p times)

Let G = $SL(q, \mathbb{R})$ act on $\mathfrak{so}(q, \mathbb{R})$ by $g(X) = gXg^t$ for g ∈ G and X ∈ $\mathfrak{so}(q, \mathbb{R})$. Let G act diagonally on $V = \mathfrak{so}(q, \mathbb{R})$ x ... x $\mathfrak{so}(q, \mathbb{R})$ (p times). Equivalently, if we identify V with $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ under the map $(C^1, \ldots, C^p) \rightarrow \sum_{i=1}^{p} C^i \otimes e_i$, then $g(C \otimes v) = gCg^t \otimes v$ for all C ∈ $\mathfrak{so}(q, \mathbb{R})$ and all v ∈ $\mathbb{R}^p$. Here $\{e_1, \ldots, e_p\}$ is the standard basis of $\mathbb{R}^p$.

For p ≥ 2 the action of G is stable on V in all cases except when (p,q) $= (2, 2k+1)$ and $(D-2, 2k+1)$, where $D = (1/2)(2k+1)(2k)$. However, it is not always the case that M < 0 on a nonempty Zariski open subset of V. We begin with a summary where the first entry is the value for (p,q). When an M value is designated as generic it means the value of M on a nonempty Zariski open subset of V. Otherwise, open for a subset means open in the Hausdorff topology of V.

In all cases $p \leq D = (1/2)q(q-1)$, and a statement valid for (p,q) is also valid for (D−p,q).

1) (2,2k). The generic stabilizer for G is isomorphic to $SL(2, \mathbb{R}) \oplus \ldots \oplus SL(2, \mathbb{R})$ (k times). M is zero generically.

2) (2,2k+1). A generic point of V has a G-orbit that is open in V. M is positive generically.

3) (3,4). The generic stabilizer of G is 3-dimensional. There exist nonempty disjoint open sets O_1, O_2 in V such that
a) M is negative on O_1, and the stabilizers of G on O_1 are isomorphic to SU(2).
b) M is zero on O_2, and the stabilizers of G on O_2 are isomorphic to $SL(2, \mathbb{R})$.

4) (3,6) The generic stabilizer of G is 1-dimensional. There exist nonempty disjoint open sets O_1, O_2 in V such that
a) M is negative on O_1, and the stabilizers of G on O_1 are isomorphic to SO(2) $= S^1$.
b) M is zero on O_2, and the stabilizers of G on O_2 are isomorphic to $\mathbb{R}$.

5) (p,2k+1), where p ≥ 3. The stabilizers of G are generically finite and M is negative generically.

6) (p,2k), where p ≥ 3 and k ≥ 4. The stabilizers of G are generically finite and M is negative generically.

We omit the details of 1) and 2). We give a brief outline of 3) and 4) in Appendix 2. We prove only 5) and 6), beginning with 5).

Proposition 5.10. *Let G $= SL(q, \mathbb{R})$ act on $\mathfrak{so}(q, \mathbb{R})$ by $g(X) = gXg^t$ for g ∈ G and X ∈ $\mathfrak{so}(q, \mathbb{R})$. Let G act diagonally on $V = \mathfrak{so}(q, \mathbb{R})$ x ... x $\mathfrak{so}(q, \mathbb{R})$ (p times), where $p \geq 3$. If q is odd, then there exists a Zariski open subset O of V such that M(D) < 0 and G_D is finite for all D ∈ O.*

Proof. Since q is odd there exists an irreducible representation of H = SU(2) on $\mathbb{R}^q$. If $\mathbb{R}^q$ is given an H-invariant inner product $\langle , \rangle$, then $\mathfrak{H} = \mathfrak{su}(2)$ may be identified with a 3-dimensional subalgebra of $\mathfrak{so}(q, \mathbb{R})$.

Next we prove a preliminary result that is valid for all positive integers q. An element $C = (C^1, \dots, C^p) \in V$ is said to be *irreducible* if the elements $\{C^1, \dots, C^p\}$ do not leave invariant any proper subspace of $\mathbb{R}^q$. It follows from the two lemmas of the previous example that the set of irreducible elements of V contains a nonempty Zariski open subset of V.

Lemma Let $\mathfrak{H}$ be a p-dimensional compact semisimple subalgebra of $\mathfrak{so}(q, \mathbb{R})$, and let $\{C^1, \dots, C^p\}$ be any basis of $\mathfrak{H}$. Let $C = (C^1, \dots, C^p) \in V$, and suppose that C is irreducible. Then M(C) < 0.

Proof. Let C be as above, and let $\mathfrak{H}$ = span $\{C^1, \dots, C^p\} \subset \mathfrak{so}(q, \mathbb{R})$. By hypothesis $\mathbb{R}^q$ is an irreducible $\mathfrak{H}$ - module. By the lemma in the proof of Proposition 3.21A of [EH] there exists a basis $\{D^1, \dots, D^p\}$ of $\mathfrak{H}$ such that $-$ trace $D^i D^j = \langle D^i, D^j \rangle = \delta_{ij}$ for $1 \leq i, j \leq p$ and $\sum_{i=1}^{p} (D^i)^2 = -\lambda \, Id$ for some positive number λ. If $D = (D^1, \dots, D^p)$, then D is minimal for the action of G on V by the first example of a moment map in section 1. By (3.5) it follows that M(C) = M(D) since span $\{C^1, \dots, C^p\}$ = span $\{D^1, \dots, D^p\} = \mathfrak{H}$. It suffices to prove that M(D) < 0.

Since D is minimal it follows from (1.1) that the Lie algebra $\mathfrak{G}_D$ is self adjoint. Equivalently, $\mathfrak{G}_D = \mathfrak{K}_D \oplus \mathfrak{P}_D$, where $\mathfrak{K}_D = \mathfrak{G}_D \cap \mathfrak{K}$ and $\mathfrak{P}_D = \mathfrak{G}_D \cap \mathfrak{P}$. To prove that M(D) < 0 we need to prove that $\mathfrak{P}_D = \{0\}$ by (3.11).

The elements of $\mathfrak{G}$ act on V by $X(C) = (XC^1 + C^1 X^t, \dots, XC^p + C^p X^t)$ for $C = (C^1, \dots, C^p) \in$ V and $X \in \mathfrak{G}$. If $X \in \mathfrak{P}_D$, then $0 = X(D)$, or equivalently, $XD^i + D^i X = 0$ for $1 \leq i \leq p$. It follows that X commutes with the elements $\{[D^i, D^j], 1 \leq i, j \leq p\}$, which generate the commutator ideal $[\mathfrak{H}, \mathfrak{H}]$. Note that $[\mathfrak{H}, \mathfrak{H}] = \mathfrak{H}$ since $\mathfrak{H}$ is semisimple, and hence X commutes with $\mathfrak{H}$. It follows that $\mathfrak{H}$ leaves invariant each eigenspace of the symmetric linear map X, and we conclude that X $= \lambda$ Id for some real number λ since $\mathbb{R}^q$ is an irreducible $\mathfrak{H}$ - module. Since $XD^i + D^i X = 0$ for $1 \leq i \leq p$ it follows that $\lambda = 0$. The proof of the lemma is complete. □

We complete the proof of the proposition. By (3.12) it suffices to prove that G_C is discrete for some C in V. If $C = (C^1, \dots, C^p) \in$ V, then $G_C = \bigcap_{i=1}^{p} G_{C^i}$. Hence it suffices to prove that G_C is discrete for some C $\in$ V in the case that p $= 3$.

As we observed above $\mathfrak{H} = \mathfrak{su}(2)$ is a 3-dimensional subalgebra of $\mathfrak{so}(q, \mathbb{R})$ such that $\mathbb{R}^q$ is irreducible under $\mathfrak{H}$. Let $D = (D^1, D^2, D^3) \in V = \mathfrak{so}(q, \mathbb{R})$ x $\mathfrak{so}(q, \mathbb{R})$ x $\mathfrak{so}(q, \mathbb{R})$ be the element constructed in the proof of the lemma above. We show that $\mathfrak{G}_D = \{0\}$).

In the proof of the lemma we showed that M(D) < 0 and $\mathfrak{G}_D = \mathfrak{K}_D \subset \mathfrak{K}$. Let X $\in \mathfrak{K}_D$ be given. Then $0 = X(D) = (XD^1 - D^1 X, \dots, XD^3 - D^3 X)$, which is equivalent to the statement that X commutes with the elements of span $\{D^1, \dots, D^3\} = \mathfrak{H}$. Hence the elements of $\mathfrak{H}$ commute with X^2, which is symmetric and negative semidefinite. Since $\mathbb{R}^q$ is an irreducible $\mathfrak{H}$ - module and $\mathfrak{H}$ leaves invariant every eigenspace of X^2 it follows that $X^2 = -\lambda \, Id$ for some $\lambda \geq 0$. If $\lambda = 0$, then $X = 0$. If $\lambda > 0$, then q must be even since Ker X $\neq \{0\}$ if $X \in \mathfrak{so}(q, \mathbb{R})$ and q is odd. In particular $\mathfrak{G}_D = \mathfrak{K}_D = \{0\}$ if q is odd, which completes the proof of the proposition. □

Remark 1 If $0 \neq \mathfrak{G}_D = \mathfrak{K}_D$, where $D = (D^1, \dots, D^p)$ is the minimal element of V discussed in the Lemma above, then the argument there shows that there exists a nonzero element X in $\mathfrak{K}_D$ such that $X^2 = -\, Id$ and X commutes with the elements of $\mathfrak{H}$. In particular $\mathbb{R}^q$ with q even becomes a complex vector space of dimension q/2, where the complex multiplication on $\mathbb{R}^q$ is given by (a + bi) v = a v + b Xv. Moreover, $\mathbb{R}^q$ becomes an irreducible complex $\mathfrak{H}$ module.

Conversely, suppose that $\mathfrak{H}$ is a compact, semisimple Lie algebra and V is an irreducible complex $\mathfrak{H}$ - module that is also irreducible as a real $\mathfrak{H}$ - module of dimension 2q. Let J $\in$ GL(V) denote multiplication by i. Then there exists an inner product $\langle , \rangle$ on V, regarded as a 2q-dimensional real vector space, such that J and the elements of $\mathfrak{H}$ are skew symmetric relative to $\langle , \rangle$ (see below). By the argument in the proof of 1) in the Lemma above there exists a basis $\{D^1, \dots D^p\}$ of $\mathfrak{H}$ such that $D = (D^1, \dots, D^p)$ is a minimal element of $\mathfrak{so}(2q, \mathbb{R})^p$ relative to the action of $G = SL(2q, \mathbb{R})$ on V and the inner product $\langle , \rangle$ on V. It follows that J is a nonzero element of $\mathfrak{K}_D$ since J commutes with $\mathfrak{H}$. The stability group G_D is compact by the proof of 1) above, but G_D is not discrete since J $\in \mathfrak{G}_D$.

We prove the existence of the inner product $\langle , \rangle$ on V with the properties stated above, regarding V as a real vector space of dimension 2q. First, consider the connected subgroup H of GL(V) with Lie algebra $\mathfrak{H}$. It is known that H is compact since the Killing form on $\mathfrak{H}$ is negative definite. See for example Chapter II, Proposition 6.6, Corollary 6.7 and Theorem 6.9 of [H]. If H′ is the subgroup of GL(V) generated by H and J, then H has index two in H′ since $J^2 = -Id$ and J commutes with the elements of H. It follows that H′ is compact. If $\langle , \rangle$ is any H′ - invariant inner product on V, then the elements of $\mathfrak{H}$ are skew symmetric and $JJ^t = Id$. Since $J^2 = -Id$ it follows that J is also skew symmetric.

Remark 2 Let $\mathfrak{H} = \mathfrak{su}(2)$ and let V be an irreducible complex $\mathfrak{H}$ - module of dimension q, where q is even. If V is regarded as a real vector space of dimension 2q, then it is known that V is also irreducible as a real $\mathfrak{H}$ - module. See sections 5 and 6 of [B-tD] for relevant discussion. The discussion of the remark above also applies to these $\mathfrak{H}$ - modules.

We conclude with the proof of 6) in the summary above.

Proposition 5.11. *Let $G = SL(q, \mathbb{R})$ and $V = \mathfrak{so}(q, \mathbb{R})^p$, where $p \geq 3$, $q \geq 3$ and (p,q) $\neq$ (3,4) or (3,6). Let G act on V as in (5.10). Then there exists a nonempty G - invariant Zariski open set O of V such that G_C is finite and M(C) < 0 for all $C = (C^1, \dots, C^p) \in$ O.*

Proof. By the argument used in the proof of (5.10) it suffices to prove that G_C is discrete for some C $\in$ V in the case that p = 3.

The assertion of the proposition for q odd was proved in the previous result. It remains only to consider the case that $p = 3$ and $q \geq 8$ is even. Let $H = SL(q, \mathbb{R})\ x\ SL(p, \mathbb{R})$ act on V $\approx \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ by $(g, h)(C) \otimes v) = (gCg^t \otimes h(v))$. Then V is an irreducible H-module since $SL(q, \mathbb{R})$ acts irreducibly on $\mathfrak{so}(q, \mathbb{R})$ and $SL(p, \mathbb{R})$ acts irreducibly on $\mathbb{R}^p$. From the table in Example 4 it is known, up to duality, that H_C is discrete on a nonempty Zariski open subset of V except in the following cases : a) $p = 1, q \geq 2$ b) $p = q(q-1)/2, q \geq 2$ c) $p = 2, q \geq 3$ d) $p = 3, q = 4, 5$ or 6. The proof is now complete since we are considering only the case that $p = 3$ and $q \geq 8$ is even. □

Appendix 1

In this appendix we give the proofs of two results that were used in the proof of (5.9).

Proof of Lemma 1 Let $\mathfrak{G}$ and $p \geq 2$ be given, and let $\{A_1, \dots, A_p\}$ be elements of $\mathfrak{G}$. We may assume without loss of generality that some A_k is nonzero. For A = $(A_1, \dots, A_p)$ set $P_1(A) = \{A_1, \dots, A_p\}$ and define inductively $P_{k+1}(A) = P_k(A) \cup adA_1(P_k(A)) \cup \dots adA_p(P_k(A))$. We regard the elements of $P_k(A)$ as formal Lie bracket expressions in the variables $A_1, \dots, A_p$. It follows that $|P_k(A)| = \sum_{i=1}^{k} p^i$.

Let $\mathfrak{G}_k(A) = \mathbb{R} - span(P_k(A))$ and let $\mathfrak{H}(A)$ be the Lie subalgebra of $\mathfrak{G}$ generated by $\{A_1, \dots, A_p\}$. Then

(1) $\mathfrak{G}_k(A) \subseteq \mathfrak{G}_{k+1}(A) \subseteq \mathfrak{H}(A)$ for all positive integers k.

Let N be the smallest positive integer such that $\mathfrak{G}_N(A) = \mathfrak{G}_{N+1}(A)$. If $\mathfrak{N}(A) = \{X \in \mathfrak{G} : adX(\mathfrak{G}_N(A)) \subset \mathfrak{G}_N(A)\}$, then $\mathfrak{N}(A)$ is a subalgebra of $\mathfrak{G}$ that contains

$\{A_1, \ldots, A_p\}$. Hence $\mathfrak{N}(A) \supset \mathfrak{H}(A) \supset \mathfrak{G}_N(A)$, which proves that $\mathfrak{G}_N(A)$ is a Lie algebra. We conclude that $\mathfrak{G}_N(A) = \mathfrak{H}(A)$. By (1) and the definition of N it follows that dim $\mathfrak{G}_k(A) <$ dim $\mathfrak{G}_{k+1}(A)$ for $1 \leq k \leq N-1$. This proves

(2) If $\mathfrak{G}_k(A) \neq \mathfrak{H}(A)$, then $dim\ \mathfrak{G}_k(A) \geq k$.

If $\mathfrak{H}(A)$ is a proper subalgebra of $\mathfrak{G}$, then $dim\ \mathfrak{G}_n(A) \leq dim\ \mathfrak{H}(A) \leq n-1$, where n = dim $\mathfrak{G}$. Conversely, if $dim\ \mathfrak{G}_n(A) \leq n-1$, then $\mathfrak{G}_n(A) = \mathfrak{H}(A)$ since otherwise $n \leq dim\ \mathfrak{G}_n(A)$ by (2). We have proved

(3) $\mathfrak{H}(A)$ is a proper subalgebra of $\mathfrak{G} \Leftrightarrow dim\ \mathfrak{G}_n(A) \leq n-1$, where $n = dim\ \mathfrak{G}$.

Let m $= |P_n(A)| = \sum_{i=1}^{n} p^i \geq n$, and let $\{\xi_1(A), \ldots, \xi_m(A)\}$ be an enumeration of the elements of $P_n(A)$. Let $\Phi(n) = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n : 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_n \leq m.\}$. For $\alpha \in \Phi(n)$ define a polynomial map $\Phi^\alpha : \mathfrak{G}^p \to \Lambda^n(\mathfrak{G}) \simeq \mathbb{R}$ by $\Phi^\alpha(A) = \Phi^\alpha(A_1, \ldots, A_p) = \xi_{\alpha_1}(A) \wedge \ldots \wedge \xi_{\alpha_n}(A)$. Then dim $\mathfrak{G}_n(A) \leq n-1 \Leftrightarrow$ any n elements of $P_n(A)$ are linearly dependent $\Leftrightarrow \Phi^\alpha(A) = 0$ for all $\alpha \in \Phi(n)$. This proves that $\Sigma = \{(A_1, \ldots, A_p) \in \mathfrak{G}^p : \{A_1, \ldots, A_p\}$ generate a proper subalgebra of $\mathfrak{G}\} = \{(\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{p}) \in \mathfrak{G}^\mathrm{p} : \Phi^\alpha(\mathrm{A}) = 0$ for all $\alpha \in \Phi(n)$. $\square$

Proof of Lemma 2 For p ≥ 2 let $\pi : \mathfrak{G}^p \to \mathfrak{G}^2$ be the projection given by $\pi(C^1, \ldots, C^p) = (C^1, C^2)$. Note that $\pi(\Sigma^p) \subset \Sigma^2$. If Σ^2 is proper, then Σ^p is proper for all p ≥ 2. Hence it suffices to consider the case p $= 2$.

Let $\mathfrak{G}^{\mathbb{C}}$ denote the complexification of $\mathfrak{G}$, and let $\mathfrak{B}$ denote a Cartan subalgebra of $\mathfrak{G}^{\mathbb{C}}$. Let $\Phi \subset Hom(\mathfrak{B}, \mathbb{C})$ denote the roots determined by $\mathfrak{B}$, and let $\mathfrak{G}^{\mathbb{C}} = \mathfrak{B} \oplus \sum_{\alpha \in \Phi} \mathfrak{G}^{\mathbb{C}}_\alpha$ denote the corresponding rootspace decomposition of $\mathfrak{G}^{\mathbb{C}}$.

Let $\Phi' = \Phi \cup \{\alpha - \beta : \alpha, \beta$ are distinct elements of $\Phi\} \subset \mathfrak{A}^*$. Choose $A \in \mathfrak{A} : \lambda(A) \neq 0$ for all $\lambda \in \Phi'$. Then $\{\alpha(A) : \alpha \in \Phi\}$ are distinct nonzero complex numbers. For B $\in \mathfrak{G}^{\mathbb{C}}$ we write $B = B_0 + \sum_{\alpha \in \Phi} B_\alpha$, where $B_0 \in \mathfrak{B}$ and $B_\alpha \in \mathfrak{G}^{\mathbb{C}}_\alpha$ for all $\alpha \in \Phi$. Let $U = \{B \in \mathfrak{G}^{\mathbb{C}} : B_\alpha \neq 0$ for all $\alpha \in \Phi\}$, and let $O = U \cap \mathfrak{G}$. Let B be any element in the nonempty Zariski open subset O of $\mathfrak{G}$.

We show that $\mathfrak{H}(A, B) = \mathfrak{G}$, where $\mathfrak{H}(A, B)$ denotes the subalgebra of $\mathfrak{G}$ generated by A and B. This will show that Σ^2 is a proper variety in $\mathfrak{G}^2$. It suffices to prove that $\mathfrak{H}(A, B)^{\mathbb{C}} = \mathfrak{G}^{\mathbb{C}}$.

For an element $\alpha \in \Phi$ we define a linear map $P_\alpha = (ad\ A) \circ \prod_{\beta \in \Phi, \beta \neq \alpha}(ad\ A - \beta(A)Id) : \mathfrak{G}^{\mathbb{C}} \to \mathfrak{G}^{\mathbb{C}}$. Note that P_α leaves invariant every subspace $\mathfrak{G}^{\mathbb{C}}_\beta, \beta \in \Phi, \mathfrak{B} \subset Ker\ A$ and $\mathfrak{G}^{\mathbb{C}}_\beta \subset Ker\ (ad\ A - \beta(A)\ Id)$ if $\beta \neq \alpha$. Hence $P_\alpha(B) = \lambda_\alpha B_\alpha$, where λ_α is a nonzero complex number, and $P_\alpha(B) \in \mathfrak{H}(A, B)^{\mathbb{C}}$. It follows that $\mathfrak{G}^{\mathbb{C}}_\alpha \subset \mathfrak{H}(A, B)^{\mathbb{C}}$ for all $\alpha \in \Phi$ since each $\mathfrak{G}^{\mathbb{C}}_\alpha$ is 1-dimensional. However, $[\mathfrak{G}^{\mathbb{C}}_\alpha, \mathfrak{G}^{\mathbb{C}}_{-\alpha}] = \mathbb{C}\ H_\alpha$, where $H_\alpha \in \mathfrak{A}$ is the root vector determined by α. Since $\mathfrak{B} = \mathbb{C} - span\{H_\alpha : \alpha \in \Phi\}$ it follows that $\mathfrak{B} \oplus \sum_{\alpha \in \Phi} \mathfrak{G}^{\mathbb{C}}_\alpha = \mathfrak{G}^{\mathbb{C}} \subset \mathfrak{H}(A, B)^{\mathbb{C}}$.$\square$

Appendix 2

1) We discuss the case (p,q) = (3,4), which is case 3) of the summary of the action of G $= SL(q, \mathbb{R})$ on $V = \mathfrak{so}(q, \mathbb{R})^p$, as stated just before (5.10).

Let $\mathbb{H}$ denote the quaternions, and let P denote the purely imaginary quaternions. In $\mathbb{H}$ we have the canonical inner product $\langle x, y \rangle = Re(x\overline{y})$. In P we have the Lie algebra structure [x,y] = xy − yx. For $\alpha, \beta \in$ P define $L_{\alpha,\beta} : \mathbb{H} \to \mathbb{H}$ by $L_{\alpha,\beta}(x) = \alpha x - x\beta$. If $\mathfrak{L} = \{L_{\alpha,\beta} : \alpha, \beta \in P\}$, then $\mathfrak{L}$ is a Lie algebra isomorphic to $\mathfrak{so}(4, \mathbb{R})$ when given the bracket structure $[L_{\alpha,\beta}, L_{\gamma,\delta}] = L_{\alpha,\beta}\ L_{\gamma,\delta} - L_{\gamma,\delta}\ L_{\alpha,\beta} = L_{[\alpha,\gamma],[\beta,\delta]}$. Note that $\mathfrak{L}$ has commuting ideals $\mathfrak{L}_1 = \{L_{\alpha,0} : \alpha \in P\}$ and $\mathfrak{L}_2 = \{L_{0,\beta} : \beta \in P\}$, both of which are isomorphic to $\mathfrak{so}(3, \mathbb{R})$.

In V $= \mathfrak{L}^3 = \mathfrak{so}(4, \mathbb{R})^3$ we define

a) $L_1 = (L_{\alpha_1,0}, L_{\alpha_2,0}, L_{\alpha_3,0})$, where $\alpha_1, \alpha_2, \alpha_3$ are linearly independent elements of P.

b) $L_2 = (L_{\lambda_1\alpha,\beta_1}, L_{\lambda_2\alpha,\beta_2}, L_{\lambda_3\alpha,\beta_3})$, where $\alpha, \beta_1, \beta_2, \beta_3$ are elements of P, $\alpha \neq 0$, W = span $\{\beta_1, \beta_2, \beta_3\}$ is a 2-dimensional subspace of P and $\lambda_1, \lambda_2, \lambda_3$ are real numbers, not all zero, such that $\sum_{k=1}^3 \lambda_k \beta_k = 0$. Then

L_1, L_2 are minimal elements for the action of $G = SL(4,\mathbb{R})$ on V since $\sum_{k=1}^3 (L_{\alpha_k,0})^2 = -\lambda\, Id$ and $\sum_{k=1}^3 (L_{\lambda_k\alpha,\beta_k})^2 = -\mu\, Id$, where $\lambda = \sum_{k=1}^3 |\alpha_k|^2$ and $\mu = \sum_{k=1}^3 |\beta_k|^2 + |\alpha|^2 (\sum_{k=1}^3 \lambda_k^2)$.

The generic stabilizer of G on $V = \mathfrak{so}(4,\mathbb{R})^3$ is 3-dimensional (cf. [KL]). One may show that there exist nonempty open subsets O_1, O_2 of V such that $L_1 \in O_1$ and M is negative on O_1 while $L_2 \in O_2$ and M is zero on O_2. The stabilizers of G in O_1, O_2 are isomorphic to SU(2) and $SL(2,\mathbb{R})$ respectively. Moreover, the sets O_1, O_2 are invariant under the involution of V induced by the involution $L_{\alpha,\beta} \to L_{\beta,\alpha}$ on $\mathfrak{L} \approx \mathfrak{so}(4,\mathbb{R})$. The action of G on V is stable by (3.10).

2) We discuss the case (p,q) = (3,6), which is case 4) of the summary of the action of G = $SL(q,\mathbb{R})$ on $V = \mathfrak{so}(q,\mathbb{R})^p$, as stated just before (5.10).

Let $\{C^1, C^2, C^3\}$ be an orthonormal basis of $\mathfrak{so}(3,\mathbb{R})$ with respect to the inner product on $\mathfrak{so}(3,\mathbb{R})$ given by $\langle X, Y\rangle = -traceXY$. Then $\sum_{k=1}^3 (C^k)^2 = -\, Id$ (cf. the lemma in Proposition 3.21A of [EH]). For $1 \leq i \leq 3$ let E^i, F^i be the elements of $\mathfrak{so}(6,\mathbb{R})$ given in 3 x 3 block matrix form as $E^i = \begin{pmatrix} C^i & 0 \\ 0 & C^i \end{pmatrix}$ and $F^i = \begin{pmatrix} C^i & 0 \\ 0 & -C^i \end{pmatrix}$. Then $E = (E^1, E^2, E^3)$ and $F = (F^1, F^2, F^3)$ are minimal elements in $V = \mathfrak{so}(6,\mathbb{R})^3$ for the action of $G = SL(6,\mathbb{R})$ since $\sum_{k=1}^3 (E^k)^2 = \sum_{k=1}^3 (F^k)^2 = -\, Id$. In particular $\mathfrak{G}_E$ and $\mathfrak{G}_F$ are self adjoint. If we write elements of $\mathfrak{G}$ in 3 x 3 block matrix form as $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then it is routine to compute :

1) $\mathfrak{P}_E = \{0\}, \mathfrak{K}_E = \{\begin{pmatrix} 0 & \lambda\, Id \\ -\lambda\, Id & 0 \end{pmatrix} : \lambda \in \mathbb{R}\}$

2) $\mathfrak{K}_F = \{0\}, \mathfrak{P}_F = \{\begin{pmatrix} 0 & \lambda\, Id \\ \lambda\, Id & 0 \end{pmatrix} : \lambda \in \mathbb{R}\}$

The generic stabilizer of G on $V = \mathfrak{so}(6,\mathbb{R})^3$ is 1-dimensional (cf. [KL]).One may show that there exist nonempty open subsets O_1, O_2 of V such that $E \in O_1$ and M is negative on O_1 while $F \in O_2$ and M is zero on O_2. The stabilizers of G in O_1, O_2 are isomorphic to S^1 and $\mathbb{R}$ respectively. The action of G on V is stable by (3.10).

References

[BH] A.Borel and Harish-Chandra, " Arithmetic subgroups of algebraic groups ", Annals of Math. 75 (1962), 485-535.

[Bor] A.Borel, *Linear Algebraic Groups*, Second Enlarged Edition, Springer, Berlin, 1991.

[Bou] N. Bourbaki, *Elements of Mathematics : Lie Groups andLie Algebras, Part 1, Chapters 1 − 3*, Hermann, Paris, 1975.

[B-tD] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer, Berlin, 1985.

[C] C. Chevalley, *Theorie des Groupes de Lie : Tome II, Groupes Algebriques*, L'Université de Nancago, Hermann, Paris, 1951.

[Eb1] P. Eberlein, *Manifolds of Nonpositive Curvature*, Univ. Chicago Press, Chicago, 1996.

[Eb2] ——-, " The moduli space of 2-step nilpotent Lie algebras of type (p,q) ", Contemp.Math. 332 (2003), 37-72.

[Eb3] ——-, " Riemannian 2-step nilmanifolds with prescribed Ricci tensor ", to appear in Contemp. Math.

[El] A.G. Elashvili, "Stationary subalgebras of points of the common state for irreducible linear Lie groups ", Func. Analys. App. 6 (1972), 139-148.

[EH] P. Eberlein and J. Heber, " Quarter pinched homogeneous spaces of negative curvature ", Int. Jour. Math. 4 (1996), 441-500.

[H] S. Helgason, *Differential Geometry , Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978

[J] N. Jacobson, *Lie Algebras*, Wiley and Sons, New York, 1962.

[KL] F. Knop and P. Littelmann, " Der Grad erzeugender Functionen von Invariantenringen ", Math. Zeit. 196 (1987), 211-229.

[KN] G. Kempf and L. Ness, " The length of vectors in representation spaces ", Lecture Notes in Mathematics 732, 233-243, Springer, New York, 1979.

[Ma] A. Marian, " On the real moment map ", Math. Res. Lett. 8 (2001), 779-788.

[Mo1] G.D. Mostow, " The extensibility of local Lie groups of transformations and groups on surfaces ", Annals of Math 52 (3), (1950), 606-636.

[Mo2] ——-, " Self adjoint groups ", Annals of Math. 62 (1955), 44-55.

[Mu] D. Mumford, *Geometric Invariant Theory*, Springer, New York, 1965.

[Nee] A. Neeman, " The topology of quotient varieties ", Annals of Math. 122 (1985),419-459.

[Nes] L. Ness (with an appendix by D. Mumford), "Stratification of the null cone via the moment map ", Amer. Jour. Math. 106 (1984), 1281-1329.

[New] P. E. Newstead, "*Introduction to Moduli Problems and Orbit Spaces*, 193 pages, Tata Institute of Fundamental Research, Bombay, India, Springer, Berlin, 1978.

[PV] V. L. Popov and E. B. Vinberg, *Invariant Theory* in *Algebraic Geometry IV*, pp. 123-284, Encyclopedia of Mathematical Sciences, vol. 55, Springer, Berlin, 1994.

[P] V. L. Popov, " Stability criteria for the action of a semisimple group on a factorial manifold ", Izv. Akad. Nauk SSSR, 34 (1970), 527-535.

[RS] R. Richardson and P. Slodowy, " Minimum vectors for real reductive algebraic groups ", Jour. London Math. Soc. 42 (1990), 409-429.

[S] G. Schwarz, " The topology of algebraic quotients ", *Topological Methods in Algebraic Transformation Groups*, (New Brunswick, NJ 1988), pp. 135-151, Prog. Math. 80, Birkhauser, Boston, 1989.

[Va] V. Varadarajan, *Lie Groups , Lie Algebras and their Representations*, Springer, New York, 1984.

[Vin] E. B. Vinberg, " On stability of actions of reductive algebraic groups ", pp. 188-202 in *Lie algebras, Rings and related topics*, papers of the Second Tainan-Moscow International Algebra Workshop, Tainan-Taiwan, January 11-17,1997, Springer, Hong Kong, 2000.

[W] H. Whitney, " Elementary structure of real algebraic varieties ", Annals of Math. 66 (1957), 545-556.