

## DISTINGUISHED ORBITS OF REDUCTIVE GROUPS

M. JABLONSKI

**ABSTRACT.** We prove a generalization of a theorem of Borel-Harish-Chandra on closed orbits of linear actions of reductive groups. Consider a real reductive algebraic group  $G$  acting linearly and rationally on a real vector space  $V$ . The group  $G$  can be viewed as the real points of a complex reductive group  $G^{\mathbf{C}}$  which acts on  $V^{\mathbf{C}} := V \otimes \mathbf{C}$ . In [2] it was shown that  $G^{\mathbf{C}} \cdot v \cap V$  is a finite union of  $G$ -orbits; moreover,  $G^{\mathbf{C}} \cdot v$  is closed if and only if  $G \cdot v$  is closed, see [20]. We show that the same result holds not just for closed orbits but for the so-called distinguished orbits. An orbit is called *distinguished* if it contains a critical point of the norm squared of the moment map on a projective space. Our main result compares the complex and real settings to show that  $G \cdot v$  is distinguished if and only if  $G^{\mathbf{C}} \cdot v$  is distinguished.

In addition, we show that, if an orbit is distinguished, then under the negative gradient flow of the norm squared of the moment map, the entire  $G$ -orbit collapses to a single  $K$ -orbit. This result holds in both the complex and real settings.

We finish with applications to the study of left-invariant geometry of Lie groups; of particular interest are left-invariant Einstein and Ricci soliton metrics on solvable and nilpotent Lie groups. Using the above theorems, we obtain a procedure for recovering Ricci soliton metrics on nilpotent Lie groups.

**1. Introduction.** An analytic approach to finding closed orbits in the complex setting was developed by Kempf and Ness [19] and extended to the real setting by Richardson and Slodowy [20]. From their perspective, the closed orbits are those that contain zeros of the so-called moment map. However, one can consider more generally critical points of this moment map on projective space. Work on the moment map in the complex setting has been done by Ness [18] and Kirwan [10]. Following those works, the real moment map was explored by Marian [15] and Eberlein and Jablonski [3].

Consider a real linear reductive group  $G$  acting linearly and rationally on a real vector space  $V$ . There is a complex linear reductive group  $G^{\mathbf{C}}$  such that  $G$  is a finite index subgroup of the real points of  $G^{\mathbf{C}}$ ;

---

Received by the editors on December 23, 2009.

DOI:10.1216/RMJ-2012-42-5-1521 Copyright ©2012 Rocky Mountain Mathematics Consortium

moreover,  $G^{\mathbf{C}}$  acts on the complexification  $V^{\mathbf{C}}$  of  $V$ . The linear action of  $G$ , respectively  $G^{\mathbf{C}}$ , extends to an action on real projective space  $\mathbf{P}V$ , respectively complex projective space  $\mathbf{CP}(V^{\mathbf{C}})$ . For  $v \in V$ , we call an orbit  $G \cdot v$ , or  $G \cdot [v]$ , *distinguished* if the orbit  $G \cdot [v]$  in real projective space contains a critical point of  $\|m\|^2$ , the norm square of the real moment map. Similarly, for  $v \in V^{\mathbf{C}}$ , we call an orbit  $G^{\mathbf{C}} \cdot v$ , or  $G^{\mathbf{C}} \cdot \pi[v]$ , *distinguished* if the orbit  $G^{\mathbf{C}} \cdot \pi[v]$  in complex projective space contains a critical point of  $\|\mu^*\|^2$ , the norm square of the complex moment map. Here  $\pi : \mathbf{RP}V^{\mathbf{C}} \rightarrow \mathbf{CP}(V^{\mathbf{C}})$  is the natural projection. Our main theorems are:

**Theorem 4.7.** *Given  $G \curvearrowright V$ ,  $G^{\mathbf{C}} \curvearrowright V^{\mathbf{C}}$ , and  $[v] \in \mathbf{P}V$ , we have*

$$G \cdot [v] \text{ is a distinguished orbit in } \mathbf{P}V \text{ if and only if} \\ G^{\mathbf{C}} \cdot \pi[v] \text{ is a distinguished orbit in } \mathbf{CP}(V^{\mathbf{C}}).$$

Here  $\pi : \mathbf{P}V \subseteq \mathbf{RP}V^{\mathbf{C}} \rightarrow \mathbf{CP}(V^{\mathbf{C}})$  is the usual projection.

**Theorem 5.1.** *For  $x \in \mathbf{CP}(V^{\mathbf{C}})$ , suppose  $G^{\mathbf{C}} \cdot x \subseteq \mathbf{CP}(V^{\mathbf{C}})$  contains a critical point of  $\|\mu^*\|^2$ . If  $z \in \mathfrak{C} \subseteq \mathbf{CP}(V^{\mathbf{C}})$  is such a critical point, then  $\mathfrak{C} \cap G^{\mathbf{C}} \cdot x = U \cdot z$ . Moreover,  $U \cdot z = \cup_{g \in G^{\mathbf{C}}} \omega(gx)$ .*

**Theorem 5.2.** *For  $x \in \mathbf{P}V$ , suppose  $G \cdot x \subseteq \mathbf{P}V$  contains a critical point of  $\|m\|^2$ . If  $z \in \mathfrak{C}_{\mathbf{R}} \subseteq \mathbf{P}V$  is such a critical point, then  $\mathfrak{C}_{\mathbf{R}} \cap G \cdot x = K \cdot z$ . Moreover,  $K \cdot z = \cup_{g \in G} \omega(gx)$ .*

Here  $\mu^*$  is the moment map for the action of  $G^{\mathbf{C}}$  on  $\mathbf{CP}(V^{\mathbf{C}})$  and  $\mathfrak{C}$  is the set of critical points of  $\|\mu^*\|^2$  in  $\mathbf{CP}(V^{\mathbf{C}})$ , while  $m$  is the moment map for the action of  $G$  on  $\mathbf{P}V$  and  $\mathfrak{C}_{\mathbf{R}}$  is the set of critical points of  $\|m\|^2$  in  $\mathbf{P}V$ . In the theorems above,  $U$  is a specific maximal compact subgroup of  $G^{\mathbf{C}}$  and  $K$  is a specific maximal compact subgroup of  $G$ ; these subgroups are described in the sequel.

The fact that  $\mathfrak{C} \cap G^{\mathbf{C}} \cdot x = U \cdot z$  was proved in [18] in the complex setting; the fact that  $\mathfrak{C}_{\mathbf{R}} \cap G \cdot x = K \cdot z$  was proved in [15] in the real setting. The fact that the orbit collapses under the negative gradient flow of  $\|\mu^*\|^2$ , respectively  $\|m\|^2$ , to a single  $U$ -orbit, respectively  $K$ -orbit, is our new contribution (see Definition 4.5 for the definition of the  $\omega$ -limit set).

To motivate the reader, we present the following examples.

**Example 1.1.** A space of Lie algebra structures. Consider the vector space  $V = \wedge^2(\mathbf{R}^n)^* \otimes \mathbf{R}^n = \{\mu : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n \mid \mu \text{ is bilinear and skew-symmetric}\}$ . Observe that a Lie bracket  $\mu$  on the vector space  $\mathbf{R}^n$  is an element of  $V$ . The usual action of  $GL_n\mathbf{R}$  on  $\mathbf{R}^n$  extends to an action on  $V$  defined as follows. For  $\mu \in V$ ,  $g \in GL_n\mathbf{R}$ , and  $X, Y \in \mathbf{R}^n$  we have

$$g \cdot \mu(X, Y) = g\mu(g^{-1}X, g^{-1}Y).$$

This is a ‘change of basis’ action on  $V$ . Here the  $GL_n\mathbf{R}$  orbits are precisely the isomorphism classes of algebras.

Consider  $\mu \in V$  which is a Lie algebra on  $\mathbf{R}^n$ , and let  $SL_n\mathbf{R}$  act on  $V$  by restricting the  $GL_n\mathbf{R}$  action above; observe that the  $GL_n\mathbf{R}$  orbits are the same as the  $SL_n\mathbf{R}$  orbits in projective space  $\mathbf{P}V$ . The Lie algebra  $\mu$  is a semi-simple algebra if and only if the orbit  $SL_n\mathbf{R} \cdot \mu$  is closed in  $V$  (see [12] for details). This fact demonstrates that the geometry of  $GL_n\mathbf{R}$  orbits is related to the algebraic nature of the Lie algebra  $\mu$ .

As distinguished orbits are a natural generalization of closed orbits, one would like to know what it means for an orbit  $GL_n\mathbf{R} \cdot \mu$  to be distinguished. If  $\mu$  is a nilpotent Lie algebra, then  $GL_n\mathbf{R} \cdot \mu$  is distinguished if and only if the (simply connected) nilpotent Lie group associated to  $\mu$  admits a left-invariant Ricci soliton metric. This relationship between the left-invariant geometry of nilpotent Lie groups and the geometry of  $GL_n\mathbf{R}$ -orbits is our primary motivation for studying the property of being a distinguished orbit. See Section 6 and [12] for more details on distinguished orbits for this particular representation.

**Example 1.2.** The adjoint representation. Let  $G$  be a (real or complex) semi-simple Lie group acting on its Lie algebra  $\mathfrak{g}$  by the adjoint action. It is a classical result of Borel et al. that the adjoint orbit  $\text{Ad } G \cdot X$  is closed if and only if  $X$  is a semi-simple element of  $\mathfrak{g}$ ; that is, if and only if  $\text{ad } X$  is a semi-simple endomorphism of  $\mathfrak{g}$ . The remaining distinguished (non-closed) orbits will lie in the nullcone of  $\mathfrak{g}$ ; recall that the nullcone consists of  $X \in \mathfrak{g}$  for which  $\text{ad } X$  is a nilpotent transformation of  $\mathfrak{g}$ .

It is a fact that the collection of orbits in the nullcone is finite. Naturally, we want to know which of these are distinguished in the sense presented here. It turns out that every orbit in the nullcone is a distinguished orbit. To prove this result, one first proves it for the complex Lie group  $SL_n \mathbf{C}$ , then one deduces the result for all complex semi-simple Lie groups from this case. To obtain the result for real semi-simple Lie groups, one must compare the real and complex cases using Theorem 4.7. We do not present the details here; instead, we refer the interested reader to [7]. For more information on the moment map of the adjoint representation, see Example 2.5.

One application of Theorem 4.7 is as follows. Since  $G^{\mathbf{C}} \cdot v \cap V$  is a finite union of  $G$ -orbits, if we can show that one of these  $G$ -orbits is distinguished, then all of them are. This has been applied to the problem of finding generic 2-step nilpotent Lie groups which admit soliton metrics. (See [8] and Section 6 for more information on the soliton problem.) Another application of the aforementioned theorem is:

**Theorem 6.5.** *Let  $N_1$  and  $N_2$  be two real simply connected nilpotent Lie groups whose complexifications  $N_1^{\mathbf{C}}$ ,  $N_2^{\mathbf{C}}$  are isomorphic. Then  $N_1$  is an Einstein nilradical if and only if  $N_2$  is an Einstein nilradical.*

This theorem stands out in that it is special to the case of nilpotent Lie groups. For example,  $SU(2)$  and  $SL_2 \mathbf{R}$  are simple groups with isomorphic complexifications. However,  $SU(2)$  does admit a left-invariant Einstein metric while  $SL_2 \mathbf{R}$  cannot.

In addition to Section 6, the technical results of this work are applied to construct new examples of nilpotent Lie groups which cannot admit left-invariant Ricci soliton metrics. For details on the construction of such spaces, see [7].

**2. Notation and technical preliminaries.** Our goal is to study closed reductive subgroups  $G$  of  $GL(E)$  which are more or less algebraic. Here  $E$  is a real vector space, and we denote its complexification by  $E^{\mathbf{C}} = E \otimes \mathbf{C}$ . We call a subgroup  $H$  of  $GL(E)$  a *real algebraic group* if  $H$  is the zero set of polynomials on  $GL(E)$  with real coefficients; that is, polynomials in  $\mathbf{R}[GL(E)]$ .

Consider a closed subgroup  $H \subseteq GL(E)$  with finitely many connected components and its Lie algebra  $\mathfrak{h} \subseteq \mathfrak{gl}(E)$ . Let  $\mathfrak{z}$  denote the center of

$\mathfrak{h}$ . We say that  $H$ , or  $\mathfrak{h} = L(H)$ , is *reductive* if  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus \mathfrak{z}$ ,  $[\mathfrak{h}, \mathfrak{h}]$  is semi-simple, and  $\mathfrak{z} \subseteq \mathfrak{gl}(E)$  consists of semi-simple endomorphisms. Reductive groups in this sense are precisely the groups that are completely reducible, see [2, Section 1.2].

We say that a group  $G \subset GL(E)$  is a *real linear reductive group* if  $G$  is a finite index subgroup of a real algebraic reductive group  $H$ ; that is,  $G$  satisfies  $H_0 \subseteq G \subseteq H$ , where  $H_0$  is the Hausdorff identity component of  $H$ . For complex algebraic groups, the Hausdorff and Zariski identity components coincide. However, this need not be true for real algebraic groups. Given  $G$ , it is well known that a complex (algebraic) reductive group  $G^{\mathbf{C}}$  exists defined over  $\mathbf{R}$  such that  $G$  is Zariski dense in  $G^{\mathbf{C}}$  and is a finite index subgroup of the real points  $G^{\mathbf{C}}(\mathbf{R}) := G^{\mathbf{C}} \cap GL(E)$  of  $G^{\mathbf{C}}$ ; that is,  $G^{\mathbf{C}}(\mathbf{R})_0 \subseteq G \subseteq G^{\mathbf{C}}(\mathbf{R})$ . For completeness we construct this group.

Consider  $G, H$  as above. The ideal of polynomials that describes  $H \subseteq GL(E)$  also describes a variety  $\overline{H} \subseteq GL(E^{\mathbf{C}})$  which is defined over  $\mathbf{R}$ . This variety  $\overline{H}$  is the Zariski closure of  $H$  in  $GL(E^{\mathbf{C}})$ . As  $H$  is a subgroup of  $GL(E^{\mathbf{C}})$ , it follows that  $\overline{H}$  is actually a complex algebraic subgroup of  $GL(E^{\mathbf{C}})$ , see [1, I.2.1]. Moreover,  $\overline{H}$  is smooth and we have  $\dim_{\mathbf{C}} \overline{H} = \dim_{\mathbf{R}} H$ , see [22]. By comparing the dimensions of these groups and their tangent spaces at the identity, one sees  $L(\overline{H}) = L(H) \otimes \mathbf{C}$ . Thus,  $\overline{H}$  is reductive as  $H$  is reductive.

To construct  $G^{\mathbf{C}}$  we consider  $\overline{H}_0$ , the Hausdorff identity component of  $\overline{H}$ . Recall that  $\overline{H}_0$  is an algebraic group as the Hausdorff and Zariski identity components of  $\overline{H}$  coincide, see [1, I.1]. Define  $G^{\mathbf{C}} = \overline{H}_0 \cdot G$ . This is a subgroup of  $\overline{H}$  as  $\overline{H}_0$  is normal in  $\overline{H}$ ; moreover, as  $\overline{H}_0$  has finite index in  $\overline{H}$  and  $G^{\mathbf{C}}$  contains  $\overline{H}_0$ , it follows that  $G^{\mathbf{C}}$  has finite index in  $\overline{H}$ . Equivalently, we can write  $G^{\mathbf{C}} = \cup_n (g_n \cdot \overline{H}_0)$  where  $\{g_n\} \subset G$  is a finite collection. Thus,  $G^{\mathbf{C}}$  is an algebraic group as it can be described as a union of varieties. Additionally, we observe that each component  $g_n \cdot \overline{H}_0$  of  $G^{\mathbf{C}}$  intersects  $G$  and that  $G^{\mathbf{C}}$  is the Zariski closure of  $G$ . The importance of this observation will be made clear when extending certain inner products on real vector spaces to their complexifications; see Proposition 2.4.

We call group  $G^{\mathbf{C}}$  the *complexification* of  $G$ . We choose the complex subgroup  $G^{\mathbf{C}}$  instead of  $\overline{H}$  as  $\overline{H}$  might have topological components which do not intersect  $G$ . This is the complexification used by Mostow,

see [16, Section 2]. We point out that we are always working in the usual topology and will explicitly state when we are talking about Zariski closed sets.

Let  $V$  be a real vector space, and denote its complexification by  $V^{\mathbf{C}} = V \otimes \mathbf{C}$ . We will consider representations  $\rho : G \rightarrow GL(V)$  that are the restrictions of morphisms  $\rho^{\mathbf{C}} : G^{\mathbf{C}} \rightarrow GL(V^{\mathbf{C}})$  of algebraic groups. See [1] for more information on algebraic groups and morphisms between them. We will call such a representation a *rational representation* of  $G$ . Note: We will denote the induced Lie algebra representation by the same letter.

**2.1. Cartan involutions.** Let  $E$  be a finite-dimensional real vector space. A *Cartan involution* of  $GL(E)$  is an involution of the form  $\theta(g) = (g^t)^{-1}$ , where  $g^t$  denotes the metric adjoint with respect to some inner product on  $E$ . At the Lie algebra level this involution is  $\theta(X) = -X^t$ .

**Proposition 2.1** [16]. *A Cartan involution  $\theta$  of  $GL(E)$  exists such that  $G^{\mathbf{C}}(\mathbf{R})$  is  $\theta$ -stable.*

**Proposition 2.2** [2, Proposition 13.5]. *Let  $\rho : G^{\mathbf{C}}(\mathbf{R}) \rightarrow GL(V)$  be a rational representation. Let  $\theta$  be a Cartan involution of  $GL(E)$  such that  $G^{\mathbf{C}}(\mathbf{R})$  is  $\theta$ -stable. Then a Cartan involution  $\theta_1$  of  $GL(V)$  exists such that  $\rho \circ \theta = \theta_1 \circ \rho$ .*

This proposition is extended in the next proposition which follows from Sections 1 and 2 of [20].

**Proposition 2.3.** *Let  $G$  be defined as above and  $\rho : G \rightarrow GL(V)$  a rational representation. Then:*

a. *There exists a  $K$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $V$  such that  $G$  is self-adjoint. Hence, the Lie algebra  $L(G) = \mathfrak{g}$  is also self-adjoint. That is, Cartan involutions  $\theta$ ,  $\theta_1$  exist on  $G$ ,  $\rho(G)$ , respectively, such that  $\rho \circ \theta = \theta_1 \circ \rho$ .*

b. *There exist decompositions of  $G$  and  $\mathfrak{g}$ , called Cartan decompositions, so that  $G = KP$  as a product of manifolds and  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ .*

Here  $K = \{g \in G \mid \theta(g) = g\}$  is a maximal compact subgroup of  $G$ ,  $\mathfrak{k} = L(K) = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ ,  $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$ , and  $P = \exp(\mathfrak{p})$ . Moreover, an  $\text{Ad } K$ -invariant inner product  $\langle\langle \cdot, \cdot \rangle\rangle$  exists on  $\mathfrak{g}$  so that  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is orthogonal and, for  $X \in \mathfrak{p}$ ,  $\text{ad } X$  is a symmetric transformation relative to  $\langle\langle \cdot, \cdot \rangle\rangle$ .

c. Relative to the inner product  $\langle \cdot, \cdot \rangle$  on  $V$ ,  $\rho(X)$  are symmetric transformations for  $X \in \mathfrak{p}$  and  $\rho(X)$  are skew-symmetric transformations for  $X \in \mathfrak{k}$ .

The subspaces  $\mathfrak{k}$  and  $\mathfrak{p}$  that arise in the Cartan decomposition above have the following set of relations

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

This is easy to see since  $\mathfrak{k}$  and  $\mathfrak{p}$  are the  $+1, -1$  eigenspaces, respectively, of the Cartan involution  $\theta$ . We point out that our  $\text{Ad } K$ -invariant inner product on  $\mathfrak{g}$  restricts to such on  $\mathfrak{p}$  as the relations above show that  $\mathfrak{p}$  is  $\text{Ad } K$ -invariant. Additionally, if the group  $G$  were semi-simple, then up to scaling the only choice for  $\langle\langle \cdot, \cdot \rangle\rangle$  would be  $-B(\theta(\cdot), \cdot)$  on each simple factor of  $\mathfrak{g}$ , where  $B$  is the Killing form of  $G$ .

Our Cartan involution  $\theta$  on  $G$  is the restriction of a Cartan involution on  $G^{\mathbb{C}}$ , see [20, 2.8 and Section 8] and [16]. This gives Cartan decompositions  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u} \oplus \mathfrak{q}$  and  $G^{\mathbb{C}} = U \cdot Q$ , where  $U$  is a maximal compact subgroup of  $G^{\mathbb{C}}$ ,  $Q = \exp(\mathfrak{q})$ , and  $U \cap Q = \{1\}$ .

We observe that the maximal compact groups  $U$  and  $K$  are related by  $U = KU_0$ . To see this, it suffices to prove  $KU_0Q = UQ = G^{\mathbb{C}}$  since  $KU_0 \subseteq U$  and  $U \cap Q = \{1\}$ . Since  $U_0Q = \overline{H}_0$  and  $P \subseteq Q$ , we obtain  $KU_0Q = KP \cdot \overline{H}_0 = G \cdot \overline{H}_0 = \overline{H}_0 \cdot G = G^{\mathbb{C}}$ .

The subspaces  $\mathfrak{u}, \mathfrak{q} \subseteq \mathfrak{g}^{\mathbb{C}}$  are related to  $\mathfrak{k}, \mathfrak{p} \subseteq \mathfrak{g}$  as follows

$$\begin{aligned} \mathfrak{u} &= \mathfrak{k} \oplus i\mathfrak{p} \\ \mathfrak{q} &= i\mathfrak{k} \oplus \mathfrak{p}. \end{aligned}$$

These two subspaces of  $LG^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$  have a nice interpretation relative to a particular inner product on  $V^{\mathbb{C}}$ . Our construction of this inner product on  $V^{\mathbb{C}}$  is similar to that done in [20, Sections 2 and 8]. We will be consistent with their notation.

**Proposition 2.4.** *The  $K$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $V$ , described in Proposition 2.3, extends to a  $U$ -invariant inner product  $S$  on  $V^{\mathbf{C}}$  with a similar list of properties for  $G^{\mathbf{C}}$ . Additionally, the inner product  $\langle\langle \cdot, \cdot \rangle\rangle$  on  $\mathfrak{g}$  extends to an  $\text{Ad } U$ -invariant inner product  $\mathbf{S}$  on  $\mathfrak{g}^{\mathbf{C}}$ .*

*Proof.* The proof of this fact follows the construction of  $S$  in A2 (proof of 2.9) in [20]. Define the inner product on  $V^{\mathbf{C}}$  as

$$S(v_1 + i v_2, w_1 + i w_2) = \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle.$$

In this way,  $V$  and  $iV$  are orthogonal under  $S$  and  $i$  acts as a skew-symmetric transformation on  $V^{\mathbf{C}}$  relative to  $S$ .  $S$  is positive definite on  $V^{\mathbf{C}}$ .

Recall that  $U = KU_0$  (see the remark above), and observe that  $S$  is  $K$ -invariant as  $K$  preserves  $V$ ,  $iV$  and  $\langle \cdot, \cdot \rangle$  is  $K$ -invariant. Thus, to show  $U$ -invariance, one just needs to show  $U_0$ -invariance. This follows since  $\rho(\mathfrak{u})$  acts skew-symmetrically and  $U_0 = \exp(\mathfrak{u})$ .

We leave to the reader the details of showing that  $\rho(\mathfrak{u})$  acts skew-symmetrically and  $\rho(\mathfrak{q})$  acts symmetrically relative to  $S$ . Lastly, the extension of  $\langle\langle \cdot, \cdot \rangle\rangle$  on  $\mathfrak{g}$  to  $\mathbf{S}$  on  $\mathfrak{g}^{\mathbf{C}}$  is a special case of the above work.  $\square$

We say that the inner products on our complex spaces are *compatible* with the inner products on the underlying real spaces. The inner product  $S$  constructed here gives rise to a  $U$ -invariant Hermitian form  $H = S + iA$  on  $V^{\mathbf{C}}$  where we define  $A(x, y) = S(x, iy)$ . This Hermitian form is compatible with the real structure  $V$  in the sense of Richardson and Slodowy, that is,  $A = 0$  when restricted to  $V \times V$ ; see Sections 2 and 8 of [20].

**2.2. Moment maps.** Next we define our moment maps. The motivation for these definitions comes from symplectic geometry and the actions of compact groups on compact symplectic manifolds. In the complex setting, this moment map coincides with the one from the symplectic structure on  $\mathbf{CP}(V^{\mathbf{C}})$ . For more information, see [18] and [4].

**Real moment maps.** Given  $G \curvearrowright V$  we define  $\tilde{m} : V \rightarrow \mathfrak{p}$  implicitly by

$$\langle\langle \tilde{m}(v), X \rangle\rangle = \langle Xv, v \rangle$$



for all  $X \in \mathfrak{p}$ . Notice that  $\tilde{m}(v)$  is a real homogeneous polynomial of degree 2. Equivalently, we could define  $\tilde{m} : V \rightarrow \mathfrak{g}$ ; then using  $K$ -invariance and  $\mathfrak{k} \perp \mathfrak{p}$ , we obtain  $\tilde{m}(V) \subseteq \mathfrak{p}$ .

**Example 2.5** (The adjoint representation). Consider a real semi-simple Lie group  $G$  acting on its Lie algebra  $\mathfrak{g}$  by the adjoint action. Fix a Cartan involution  $\theta$  on  $G$ , and  $\mathfrak{g}$ . Let  $K = G^\theta$  be the fixed set of  $\theta$ ; then  $K$  is a maximal compact subgroup of  $G$ . Moreover, denoting the Killing form of  $\mathfrak{g}$  by  $B$ , we see that  $\langle \cdot, \cdot \rangle = -B(\cdot, \theta(\cdot))$  is a  $K$ -invariant inner product on  $\mathfrak{g}$  satisfying all of the hypotheses above.

Choosing the inner product on  $\mathfrak{g} = \text{Lie } G$  to be the same, that is,  $\langle \langle \cdot, \cdot \rangle \rangle = \langle \cdot, \cdot \rangle$ , we see that the moment map of this action is

$$m(X) = -[X, \theta(X)].$$

Embedding  $G \subset GL(V)$ , the Cartan involution can be written as  $\theta(g) = (g^t)^{-1}$ , where  $t$  denotes the metric adjoint with respect to some inner product on  $V$ . In doing so, our moment map becomes  $m(X) = XX^t - X^tX$ . See [3, 7, 20] for more information on the adjoint representation from this perspective.

We can just as well construct a moment map for the action  $G^{\mathbb{C}} \curvearrowright V^{\mathbb{C}}$  where we regard  $G^{\mathbb{C}}$  as a real Lie group. We use the inner product  $\mathbf{S}$  on  $V^{\mathbb{C}}$ . The (real) moment map for  $G^{\mathbb{C}} \curvearrowright V^{\mathbb{C}}$ , denoted by  $\tilde{n} : V^{\mathbb{C}} \rightarrow \mathfrak{q}$ , is defined by

$$\mathbf{S}(\tilde{n}(v), Y) = S(Yv, v)$$

for  $Y \in \mathfrak{q}$  and  $v \in V^{\mathbb{C}}$ .

Since these polynomials are homogeneous, they give rise to well-defined maps on (real) projective space. Define

$$\begin{aligned} m : \mathbf{P}V &\rightarrow \mathfrak{p} & n : \mathbf{R}P V^{\mathbb{C}} &\rightarrow \mathfrak{q} \\ m[v] &= \tilde{m}\left(\frac{v}{|v|}\right) = \frac{\tilde{m}(v)}{|v|^2} & n[w] &= \tilde{n}\left(\frac{w}{|w|}\right) = \frac{\tilde{n}(w)}{|w|^2}, \end{aligned}$$

where  $|w|^2 = S(w, w)$  and  $S = \langle \cdot, \cdot \rangle$  on  $V$ . Since  $V \subseteq V^{\mathbb{C}}$ , we have  $\mathbf{P}V \subseteq \mathbf{R}P V^{\mathbb{C}}$ ; this is our main reason for studying the real moment map on  $G^{\mathbb{C}}$ . The next lemma compares these two real moment maps.

**Lemma 2.6.**  *$n$  restricted to  $\mathbf{P}V$  equals  $m$ .*

*Proof.* Recall that  $n$  takes values in  $\mathfrak{q} = i\mathfrak{k} \oplus \mathfrak{p}$  and  $m$  takes values in  $\mathfrak{p} \subseteq \mathfrak{q}$ . Take  $v \in V$  and  $X \in \mathfrak{k}$ . Then

$$\mathbf{S}(\tilde{n}(v), iX) = S(iX \cdot v, v) = 0,$$

as  $V \perp iV$  (see Proposition 2.4), and we are using  $(iX) \cdot v = i(X \cdot v)$ , i.e.,  $\mathfrak{g}^{\mathbf{C}}$  acts  $\mathbf{C}$ -linearly on  $V^{\mathbf{C}}$ . Since  $\mathfrak{g} \perp i\mathfrak{g}$  under  $\mathbf{S}$ , we have  $i\mathfrak{k} \perp \mathfrak{p}$ . Thus,  $\tilde{n}(v) \in \mathfrak{p} \subseteq \mathfrak{q}$ . Now take  $X \in \mathfrak{p}$ .

$$\begin{aligned} \mathbf{S}(\tilde{n}(v), X) &= \langle \tilde{n}(v), X \rangle && \text{by compatibility of } \mathfrak{g} \subseteq \mathfrak{g}^{\mathbf{C}} \\ &\parallel \\ S(Xv, v) &= \langle Xv, v \rangle && \text{by compatibility of } V \subseteq V^{\mathbf{C}} \\ &= \langle \tilde{m}(v), X \rangle && \text{by definition/construction of } \tilde{m}. \end{aligned}$$

Therefore,  $\tilde{n}(v) = \tilde{m}(v)$  for  $v \in V \subseteq V^{\mathbf{C}}$ , which implies  $n[v] = m[v]$  for  $[v] \in \mathbf{P}V \subseteq \mathbf{R}P V^{\mathbf{C}}$ .  $\square$

**Complex moment maps.** We choose a notation that is similar to Ness [18] as we are following her definitions; the only difference is that we use  $\mu$  where she uses  $m$ . For  $v \in V^{\mathbf{C}}$ , consider  $\rho_v : G^{\mathbf{C}} \rightarrow \mathbf{R}$  defined by  $\rho_v(g) = |g \cdot v|^2$ , where  $|w|^2 = H(w, w) = S(w, w)$ . Define a map  $\mu : \mathbf{CP}(V^{\mathbf{C}}) \rightarrow \mathfrak{q}^* = \text{Hom}(\mathfrak{q}, \mathbf{R})$  by  $\mu(x) = (d\rho_v(e))/|v|^2$ , where  $v \in V^{\mathbf{C}}$  sits over  $x \in \mathbf{CP}(V^{\mathbf{C}})$ , cf. [18, Section 1]. We define the complex moment map  $\mu^* : \mathbf{CP}(V^{\mathbf{C}}) \rightarrow \mathfrak{q}$  by  $\mu = \mathbf{S}(\mu^*, \cdot)$ . Note that, taking the norm square of our complex moment map will give us the norm square of the moment map in Kirwan's setting; in Kirwan's language,  $i\mu$  would be the moment map [18, Section 1].

Let  $\pi$  denote the projection  $\pi : \mathbf{R}P V^{\mathbf{C}} \rightarrow \mathbf{CP}(V^{\mathbf{C}})$ .

**Lemma 2.7.** *The complex and real moment maps for  $G^{\mathbf{C}}$  are related by  $\mu^* \circ \pi = 2n$ .*

*Proof.* Many of our computations have the same flavor as those of Ness; we employ her ideas. Take an orthonormal basis  $\{\alpha_i\}$  of  $i\mathfrak{u} = \mathfrak{q}$

under  $\mathbf{S}$ . Also, let  $x = \pi[v] \in \mathbf{CP}(V^{\mathbf{C}})$  for  $v \in V^{\mathbf{C}}$ . Then

$$\begin{aligned}\mu^*(x) &= \sum_i \mathbf{S}(\mu^*(x), \alpha_i) \alpha_i \\ &= \sum_i [\mu(x) \alpha_i] \alpha_i \\ &= \sum_i \frac{1}{\|v\|^2} d\rho_v(e)(\alpha_i) \alpha_i \\ &= \sum_i \frac{1}{\|v\|^2} \frac{d}{dt} \bigg|_{t=0} \|\exp t\alpha_i \cdot v\|^2 \alpha_i.\end{aligned}$$

Here the norm on  $V^{\mathbf{C}}$  is from  $H = S + iA$ . But  $S$  is the inner product being used on  $V^{\mathbf{C}}$ , and so  $H(w, w) = S(w, w)$  tells us that  $\mu^*(x)$

$$\begin{aligned}&= \sum_i \frac{1}{\|v\|^2} 2S(\alpha_i v, v) \alpha_i \\ &= \sum_i 2\mathbf{S}(\tilde{n}[v], \alpha_i) \alpha_i \\ &= 2\tilde{n}[v]. \quad \square\end{aligned}$$

*Remark.* Since  $\mathbf{PV}$  is not a subspace of  $\mathbf{CP}(V^{\mathbf{C}})$ , we use  $\mathbf{RPV}^{\mathbf{C}}$  and the real moment map of  $G^{\mathbf{C}}$  to work between the known results of Kirwan and Ness to get information about our real group  $G \circ \mathbf{PV}$ .

**3. Comparison of real and complex cases.** Most of algebraic geometry and geometric invariant theory has been worked out exclusively for fields which are algebraically closed. We are interested in the real category and will exploit all the work that has already been done over  $\mathbf{C}$ . We use and refer the reader to [22] as our main reference for real algebraic varieties.

Recall that our representation  $\rho : G \rightarrow GL(V)$  is the restriction of a representation of  $G^{\mathbf{C}}$ . The following is Proposition 2.3 of [2] and Section 8 of [20]. Originally this was stated as a comparison between  $G^{\mathbf{C}}(\mathbf{R})_0$ -orbits and  $G^{\mathbf{C}}$ -orbits; however, it can be restated as a comparison between  $G$  and  $G^{\mathbf{C}}$  orbits, for any  $G$  satisfying

$G^{\mathbf{C}}(\mathbf{R})_0 \subseteq G \subseteq G^{\mathbf{C}}(\mathbf{R})$ . This is true as  $G^{\mathbf{C}}(\mathbf{R})_0$  has finite index in  $G$ .

**Theorem 3.1.** *Let  $v \in V$ . Then*

$$G^{\mathbf{C}} \cdot v \cap V = \bigcup_{i=1}^m X_i$$

where each  $X_i$  is a  $G$ -orbit. Moreover,  $G^{\mathbf{C}} \cdot v$  is closed in  $V^{\mathbf{C}}$  if and only if  $G \cdot v$  is closed in  $V$ .

Complex group orbits have some nice properties that we don't enjoy over the real numbers. For example, the Hausdorff and Zariski closures of a group orbit are the same for a complex linear algebraic group. One property that does translate to the reals is that the boundary of an orbit consists of orbits of strictly lower dimension. See [6, subsection 8.3] for the complex setting, and see below for the real setting. For some interesting examples of semi-simple real algebraic groups whose orbit closure is not the Zariski closure, see [3].

**Proposition 3.2.** *Let  $G$  and  $G^{\mathbf{C}}$  be defined as above. Take  $v \in V \subseteq V^{\mathbf{C}}$ . Then*

- a.  $\dim_{\mathbf{R}} G \cdot v = \dim_{\mathbf{C}} G^{\mathbf{C}} \cdot v$ .
- b.  $\partial(G \cdot v) = \overline{G \cdot v} - G \cdot v$  consists of  $G$ -orbits of strictly smaller dimension.
- c.  $G^{\mathbf{C}} \cdot v \cap \overline{G \cdot v} = G \cdot v$ .

*Proof of a.* Notice that, for a real Lie group  $H$ ,  $H \cdot v \simeq H/H_v$ . Let  $\mathfrak{h}$  be the Lie algebra of  $H$ . Then, at the Lie algebra level, it is easy to see that  $(\mathfrak{h}_v)^{\mathbf{C}} = (\mathfrak{h}^{\mathbf{C}})_{v^{\mathbf{C}}}$ , for  $v \in V \subseteq V^{\mathbf{C}}$ . As  $\dim_{\mathbf{R}} G = \dim_{\mathbf{C}} G^{\mathbf{C}}$ , we are done.

*Proof of b.* Recall two facts about complex group orbits. First, the boundary  $\overline{G^{\mathbf{C}} \cdot v} - G^{\mathbf{C}} \cdot v$  of the complex group orbit  $G^{\mathbf{C}} \cdot v$  consists of  $G^{\mathbf{C}}$ -orbits of strictly smaller dimension, see [6, subsection 8.3]. Second,

$$G^{\mathbf{C}} \cdot v \cap V = \bigcup_1^m X_i,$$

where each  $X_i$  is a  $G$ -orbit. Moreover, each  $X_i$  is closed in  $G^{\mathbf{C}} \cdot v \cap V$  as it is a finite union of connected components of  $G^{\mathbf{C}} \cdot v \cap V$ , see [2, Proposition 2.3]. If  $v \in X_i$  for  $1 \leq i \leq m$ , then  $G \cdot v = X_i$  and  $\overline{G \cdot v} \cap G^{\mathbf{C}} \cdot v = X_i$ . If  $w \in \overline{G \cdot v} - G \cdot v$ , then  $w \in \overline{G^{\mathbf{C}} \cdot v} - G^{\mathbf{C}} \cdot v$ , and it follows from a and the first remark above that  $G \cdot w$  has smaller dimension than  $G \cdot v$ .

*Proof of c.* This follows immediately from b and its proof.  $\square$

**3.3. Orbits in projective space.** Since our groups act linearly on vectors spaces, we can consider the induced actions on projective space  $G \curvearrowright \mathbf{P}V$  and  $G^{\mathbf{C}} \curvearrowright \mathbf{R}P V^{\mathbf{C}}$ .

**Lemma 3.3.** *For  $v \in V$ ,  $G^{\mathbf{C}} \cdot [v] \cap \overline{G \cdot [v]} = G \cdot [v]$  in  $\mathbf{R}P V^{\mathbf{C}}$ .*

This is the same result in projective space that we had for our vector spaces.

*Proof.* The actions of  $\mathbf{R}^* \times G$  and  $G$  on  $\mathbf{P}V$  are the same; moreover,  $(\mathbf{R}^* \times G)^{\mathbf{C}} = \mathbf{C}^* \times G^{\mathbf{C}}$ . Given  $v \in V$ , take  $g_n \in G$  and  $g \in G^{\mathbf{C}}$  such that  $[g_n v] \rightarrow [gv]$  in  $\mathbf{P}V$ . Then we want to show  $[gv] \in G \cdot [v]$ . Now take  $r_n, r \in \mathbf{R}$  such that  $r_n g_n v, r g v$  have unit length in  $V^{\mathbf{C}}$ . We can assume  $r_n g_n v \rightarrow r g v$  by passing to  $-r$  and a subsequence if necessary.

Then  $r_n g_n v \rightarrow r g v \in \mathbf{C}^* \times G^{\mathbf{C}} \cdot v \cap \overline{\mathbf{R}^* \times G \cdot v}$ . Therefore,  $r g v \in \mathbf{R}^* \times G \cdot v$  using Proposition 3.2 c, and our result follows.  $\square$

**4. Closed and distinguished orbits.** We begin with a theorem of Richardson and Slodowy (for real groups) which follows the work of Kempf and Ness (for complex groups). To find which orbits are closed, one looks for the infimum of  $|g \cdot v|^2$  along the orbit. Such a vector is called a *minimal vector*, and it occurs on the orbit precisely when our orbit is closed. Let  $\mathfrak{M}$  denote the set of minimal vectors in  $V$ . Although not stated using the moment map, the following was proven in [20, Theorem 4.4, 7.3].

**Theorem 4.1.**  *$G \cdot v$  is closed if and only if  $w \in G \cdot v$  exists such that  $\tilde{m}(w) = 0$ . Such a vector  $w$  is minimal. Moreover,  $\mathfrak{M} = \tilde{m}^{-1}(0)$  and  $\overline{G \cdot v} \cap \mathfrak{M}$  is a single  $K$ -orbit.*

Equivalently we could find the zeroes of  $\|\tilde{m}\|^2$  to find the minimal vectors. Minimal vectors are used to understand the *semi-stable points*, that is, all the vectors whose orbit closure does not contain zero. In contrast, the *null cone* is the set of vectors whose orbit closure does contain zero. To study the null cone, we move to projective space. Clearly, we cannot use minimal vectors to study the geometry of the null cone, so instead of looking for zeros of  $\|m\|^2$  on  $\mathbf{P}V$ , we look for critical points of  $\|m\|^2$ .

**Definition 4.2.** We say that  $v \in V$  or  $[v] \in \mathbf{P}V$  is distinguished if  $\|m\|^2 : \mathbf{P}V \rightarrow \mathbf{R}$  has a critical point at  $[v]$ . We say that an orbit  $G \cdot v$  or  $G \cdot [v]$  is distinguished if it contains a distinguished point. Analogously, we define distinguished points and  $G^{\mathbf{C}}$ -orbits in  $V^{\mathbf{C}}$  and  $\mathbf{CP}(V^{\mathbf{C}})$  using  $\|\mu^*\|^2$ .

Minimal vectors distinguished as zero is an absolute minimum of the function  $\|m\|^2$ . Our goal is to find an analogue of Theorem 3.1 for distinguished orbits. To understand critical points of  $\|m\|^2$ , we will relate this function to  $\|\mu^*\|^2$  by means of  $\|n\|^2$ . Recall that  $\|\mu^*\|^2$  has been studied extensively in [10, 18].

Our first observation is that the only closed orbits  $G \cdot [v] \subseteq \mathbf{P}V$  occur when  $G \cdot [v] = K \cdot [v]$ . This is well known, but an elegant and geometric proof is easily obtained using properties of the moment map, see e.g., [15, Theorem 1]. So our main interest is in the remaining distinguished orbits.

**Proposition 4.3.** *If  $[v] \in \mathbf{P}V$ , then  $\text{grad } \|n^*\|^2[v] = \text{grad } \|m^*\|^2[v] \in T_{[v]}G \cdot [v]$ . Hence,  $\|n^*\|^2$  has a critical point at  $[v] \in \mathbf{P}V \subseteq \mathbf{RP}V^{\mathbf{C}}$  if and only if  $\|m^*\|^2$  does so. Moreover, if  $[v] \in \mathbf{P}V$ , and  $\varphi_t[v]$  is the integral curve of  $-\text{grad } \|n^*\|^2$  starting at  $[v]$ , then  $\varphi_t[v] \in G \cdot [v] \subseteq \mathbf{P}V$  for all  $t$ .*

Before proving the proposition, we study the gradients of these functions. Let  $\phi : G^{\mathbf{C}} \times V^{\mathbf{C}} \rightarrow V^{\mathbf{C}}$  denote the action of  $G^{\mathbf{C}}$  on  $V^{\mathbf{C}}$ , and let  $\phi_v : G^{\mathbf{C}} \rightarrow V^{\mathbf{C}}$  denote the induced map for every  $v \in V^{\mathbf{C}}$ . We

define vector fields on  $V^{\mathbf{C}}$  and  $\mathbf{RPV}^{\mathbf{C}}$  as follows. On  $V^{\mathbf{C}}$  we define

$$\tilde{X}_{\alpha}(v) := d\phi_v(\alpha) = \left. \frac{d}{dt} \right|_{t=0} \exp t\alpha \cdot v$$

for  $\alpha \in \mathfrak{g}^{\mathbf{C}}$ . And, on  $\mathbf{RPV}^{\mathbf{C}}$ ,

$$X_{\alpha}[v] := \pi_* \tilde{X}_{\alpha}(v)$$

where  $\pi : V^{\mathbf{C}} \rightarrow \mathbf{RPV}^{\mathbf{C}}$  is projection. Note, this is well defined as our action  $G^{\mathbf{C}} \curvearrowright V^{\mathbf{C}}$  is linear.

**Lemma 4.4.** *For  $x \in \mathbf{PV}$ ,  $\text{grad} \|m^*\|^2(x) = 4X_{m(x)}(x)$ . For  $x \in \mathbf{RPV}^{\mathbf{C}}$ ,  $\text{grad} \|n^*\|^2(x) = 4X_{n(x)}(x)$ .*

Marian proves the first statement for  $\|m^*\|^2$  on  $\mathbf{PV}$ , see [15, Lemma 2]. Her proof carries over to obtain the statement for  $\|n^*\|^2$  on  $\mathbf{RPV}^{\mathbf{C}}$ .

*Proof of Proposition 4.3.* The first assertion follows from Lemma 4.4, Lemma 2.6 and the fact that  $m[v] \in \mathfrak{p} \subseteq \mathfrak{g}$  for  $[v] \in \mathbf{PV}$ . The second and third assertions follow immediately from the first.  $\square$

Next we relate the actions of our complex group  $G^{\mathbf{C}}$  on  $\mathbf{RPV}^{\mathbf{C}}$  and  $\mathbf{CP}(V^{\mathbf{C}})$ . By Lemma 2.7, we know that  $\|\mu^* \circ \pi[v]\|^2 = 4\|n[v]\|^2$  for  $v \in V^{\mathbf{C}}$  and  $\pi : \mathbf{RPV}^{\mathbf{C}} \rightarrow \mathbf{CP}(V^{\mathbf{C}})$ . This shows that  $\|n\|^2$  is not just  $U$ -invariant, it is also  $U \times \mathbf{C}^*$ -invariant. We wish to relate the actions of  $G^{\mathbf{C}}$  on  $\mathbf{RPV}^{\mathbf{C}}$  and  $\mathbf{CP}(V^{\mathbf{C}})$  by comparing their gradients from the natural Riemannian structures on these projective spaces.

**4.1. The Riemannian structures and gradients on projective space.** Recall that projective space can be endowed with a natural Riemannian metric so that projection from the vector space is a Riemannian submersion. This natural Riemannian metric is called the Fubini-Study metric and is defined as follows. Take  $\zeta_i \in T_{[w]}\mathbf{KP}(V^{\mathbf{C}})$ , where  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{C}$ . Let  $\Pi^{\mathbf{K}} : V^{\mathbf{C}} \rightarrow \mathbf{KP}(V^{\mathbf{C}})$  be the usual projection, and take  $\xi_i \in T_w V^{\mathbf{C}}$  such that  $\Pi_*^{\mathbf{K}}(\xi_i) = \zeta_i$ . The Fubini-Study metric on  $\mathbf{KP}(V^{\mathbf{C}})$  is defined by

$$(\zeta_1, \zeta_2) = \frac{(\xi_1, \xi_2)(w, w) - (\xi_1, w)(\xi_2, w)}{(w, w)}.$$

One can naturally identify the tangent space  $T_{\Pi\kappa(w)}\mathbf{KP}(V^{\mathbf{C}})$  with the orthogonal complement of  $\mathbf{K}$ -span  $\langle w \rangle$  in  $T_w V^{\mathbf{C}}$ . In our setting, we are using  $S$ , the extension of  $\langle \cdot, \cdot \rangle$  on  $V$ , as our inner product on  $V^{\mathbf{C}}$ . Using these natural choices of Riemannian structures on  $\mathbf{RPV}^{\mathbf{C}}$  and  $\mathbf{CP}(V^{\mathbf{C}})$ , we see that  $\pi : \mathbf{RPV}^{\mathbf{C}} \rightarrow \mathbf{CP}(V^{\mathbf{C}})$  is also a Riemannian submersion.

We are interested in the negative gradient flow of the moment map. Let  $\varphi_t$  denote the negative gradient flow of  $\|n\|^2$  on  $\mathbf{RPV}^{\mathbf{C}}$  and  $\|\mu^*\|^2$  on  $\mathbf{CP}(V^{\mathbf{C}})$ .

**Definition 4.5.** The  $\omega$ -limit set of  $\varphi_t(p) \subseteq \mathbf{RPV}^{\mathbf{C}}$  is the set  $\{q \in \mathbf{RPV}^{\mathbf{C}} \mid \varphi_{t_n}(p) \rightarrow q \text{ for some sequence } t_n \rightarrow \infty \text{ in } \mathbf{R}\}$ . We denote this set by  $\omega(p)$ .

Analogously, we can define the  $\omega$ -limit set of  $\varphi_t(p) \subseteq \mathbf{CP}(V^{\mathbf{C}})$ , and we denote this set by  $\omega(p)$  also. It is easy to see that  $\omega(p)$  is invariant under  $\varphi_t$  for all  $t$ .

*Remark.* We observe that points in the  $\omega$ -limit set of a negative gradient flow are fixed points of the flow, that is, critical points of the given function. In general, this is not true for  $\omega$ -limit points associated to non-gradient flows. We include a brief argument for the reader.

Consider  $F : M \rightarrow \mathbf{R}$ , and let  $\varphi_t(p)$  denote the integral curve of  $-\text{grad } F$  starting at  $p \in M$ . Observe that  $F$  is decreasing along  $\varphi_t(p)$ . Suppose that  $\omega(p)$  is non-empty. Then we can define  $c = \lim_{t \rightarrow \infty} F(\varphi_t(p))$  to obtain  $\omega(p) \subseteq F^{-1}(c)$ . Thus, for  $q \in \omega(p)$ , we see that  $\varphi_t(q) \subseteq F^{-1}(c)$ . Hence,  $\text{grad } F(q) = 0$ . That is, points in the  $\omega$ -limit set of  $-\text{grad } F$  are critical points for  $F$ .

**Proposition 4.6.** *Endow  $\mathbf{RPV}^{\mathbf{C}}$  and  $\mathbf{CP}(V^{\mathbf{C}})$  with the Riemannian metrics so that the projections from  $V^{\mathbf{C}}$  are Riemannian submersions. Then the following are true for  $[v] \in \mathbf{RPV}^{\mathbf{C}}$ :*

- a.  $4\pi_* \text{grad } \|n\|^2[v] = \text{grad } \|\mu^*\|^2(\pi[v])$ .
- b.  $[v] \in \mathbf{RPV}^{\mathbf{C}}$  is a critical point of  $\|n\|^2$  if and only if  $\pi[v] \in \mathbf{CP}(V^{\mathbf{C}})$  is a critical point of  $\|\mu^*\|^2$ .



c.  $\varphi_t \circ \pi = \pi \circ \varphi_{4t}$ , where  $\varphi_t$  denotes the negative gradient flow of  $\|n\|^2$  on  $\mathbf{RPV}^{\mathbf{C}}$  or  $\|\mu^*\|^2$  on  $\mathbf{CP}(V^{\mathbf{C}})$ .

d.  $\pi(\omega([v])) = \omega(\pi[v])$ , where  $\omega(p)$  denotes the  $\omega$ -limit set of the negative gradient flow starting from  $p$ .

*Proof.* Applying Lemma 2.7, we have

$$\begin{aligned} 4\langle \text{grad } \|n\|^2[v], w_{[v]} \rangle &= 4 \frac{d}{dt} \Big|_{t=0} \|n[v + tw]\|^2 \\ &= \frac{d}{dt} \Big|_{t=0} \|\mu^* \pi[v + tw]\|^2 \\ &= \langle \text{grad } \|\mu^*\|^2(\pi[v]), \pi_* w_{[v]} \rangle \end{aligned}$$

Since  $\pi_*$  is a submersion, we have that  $\pi_*$  maps the horizontal subspace of  $T_{[v]}\mathbf{RPV}^{\mathbf{C}}$  isometrically onto  $T_{\pi[v]}\mathbf{CP}(V^{\mathbf{C}})$  and part a is proven. Thus, if  $[v]$  is a critical point for  $\|n\|^2$ , then  $\pi[v]$  is one for  $\|\mu^*\|^2$ . To obtain the reverse direction use the  $\mathbf{C}^*$ -invariance of  $\|n\|^2$ . This proves part b.

Proof of part c. Let  $[v] \in \mathbf{RPV}^{\mathbf{C}}$ . Consider the curve  $\pi \circ \varphi_{4t}[v]$  in  $\mathbf{CP}(V^{\mathbf{C}})$ . This curve satisfies the following differential equation

$$\frac{d}{dt} \pi \circ \varphi_{4t}[v] = \pi_* 4(-\text{grad } \|n\|^2)(\varphi_{4t}[v]) = -\text{grad } \|\mu^*\|^2(\pi \circ \varphi_{4t}[v]).$$

That is, the curve  $\pi \circ \varphi_{4t}[v]$  is the integral curve of the negative gradient flow of  $\|\mu^*\|^2$  starting at  $\pi[v]$ . Thus,  $\pi \circ \varphi_{4t} = \varphi_t \circ \pi$ .

Proof of part d. We will show containment in both directions. Take  $p \in \omega[v]$ . Then a sequence of  $t_n \rightarrow \infty$  exists such that  $\varphi_{t_n}[v] \rightarrow p$  in  $\mathbf{RPV}^{\mathbf{C}}$ . Using part c, we have  $\varphi_{t_n/4}(\pi[v]) = \pi \circ \varphi_{t_n}[v] \rightarrow \pi(p)$ . That is,  $\pi(p) \in \omega(\pi[v])$ , or  $\pi(\omega[v]) \subseteq \omega(\pi[v])$ . To obtain the other direction, take  $q \in \omega(\pi[v])$  and  $t_n \rightarrow \infty$  so that  $\varphi_{t_n}(\pi[v]) \rightarrow q$  in  $\mathbf{CP}(V^{\mathbf{C}})$ . Consider the set  $\varphi_{4t_n}[v]$  in  $\mathbf{RPV}^{\mathbf{C}}$ . Since  $\mathbf{RPV}^{\mathbf{C}}$  is compact, we can find a limit point of this set and, passing to a subsequence, we may assume  $\varphi_{4t_n}[v] \rightarrow p$ . Then,  $p \in \omega[v]$ ,  $\pi(p) = q$  by (c), and we have shown  $q \in \omega(\pi[v])$ . That is,  $\omega(\pi[v]) \subseteq \omega(\omega[v])$ .  $\square$

We finish the section by stating our main theorem and some corollaries.

**Theorem 4.7.** *Given  $G \curvearrowright V$ ,  $G^{\mathbf{C}} \curvearrowright V^{\mathbf{C}}$ , and  $[v] \in \mathbf{P}V$ , we have*

$$G \cdot [v] \text{ is a distinguished orbit in } \mathbf{P}V \text{ if and only if} \\ G^{\mathbf{C}} \cdot \pi[v] \text{ is a distinguished orbit in } \mathbf{CP}(V^{\mathbf{C}}).$$

Here  $\pi : \mathbf{P}V \subseteq \mathbf{RP}V^{\mathbf{C}} \rightarrow \mathbf{CP}(V^{\mathbf{C}})$  is the usual projection.

*Remark.* Analysis of the proof of Theorem 4.7 shows the following. Given  $v \in V \subseteq V^{\mathbf{C}}$ , the orbits  $G \cdot [v] \subseteq \mathbf{P}V$  and  $G^{\mathbf{C}} \cdot \pi[v] \subseteq \mathbf{CP}(V^{\mathbf{C}})$  being distinguished is equivalent to  $G^{\mathbf{C}} \cdot [v] \subseteq \mathbf{RP}V^{\mathbf{C}}$  being distinguished using  $\|n\|^2$  on  $\mathbf{RP}V^{\mathbf{C}}$ .

**Corollary 4.8.** *Suppose we have  $v_1, v_2 \in V$  with distinct  $G$ -orbits but whose  $G^{\mathbf{C}}$ -orbits are the same. Then  $G \cdot [v_1]$  is distinguished if and only if  $G \cdot [v_2]$  is distinguished.*

*Remark.* The phenomenon of two vectors having different real orbits but the same complex orbit happens often (see the following examples). This corollary was a necessary ingredient in the solution to the problem of showing that generic 2-step nilmanifolds admit soliton metrics (see [8]). This corollary is also used to prove other interesting geometric results, see e.g., Theorem 6.5.

**Example 4.9** (Adjoint action of  $SL_2\mathbf{R}$ ). Consider the elements  $X_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $X_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$  of  $\mathfrak{sl}_2\mathbf{R}$ . It is a simple computation that these elements lie on distinct orbits of  $SL_2\mathbf{R}$ ; however, they must lie on the same  $SL_2\mathbf{C}$  orbit (i.e., they have the same Jordan normal form). Using Example 2.5, we compute that the moment map at  $X_1$  is  $m(X_1) = X_1X_1^t - X_1^tX_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ . As  $m(X_1) \cdot X_1 = m(X_1)X_1 - X_1m(X_1) = X_1$ , we see that  $X_1$  is a distinguished point of this action. Applying the corollary above, we see that the orbit  $SL_2\mathbf{R} \cdot X_2$  is also distinguished. The general case of adjoint actions of semi-simple Lie groups is studied in [7] where it is shown that every nilpotent orbit is distinguished.

**Example 4.10** (Quadratic forms on  $\mathbf{R}^n$  and  $\mathbf{C}^n$ ). Another classical example that demonstrates the propensity for multiple real orbits lying on the same complex orbit is the similarity action of  $GL_n\mathbf{R}$  on the

set of symmetric real matrices  $\text{Symm}(n, \mathbf{R})$ ; recall that the space  $\text{Symm}(n, \mathbf{R})$  is the space of symmetric (real) bilinear forms on  $\mathbf{R}^n$  which is the same as the space of quadratic forms on  $\mathbf{R}^n$ . Here the action is defined by  $g \cdot M = gMg^t$  for  $g \in GL_n \mathbf{R}$  and  $M \in \text{Symm}(n, \mathbf{R})$ . Likewise, we have the action of  $GL_n \mathbf{C}$  on  $\text{Symm}(n, \mathbf{C})$ , the space of symmetric complex matrices, which is the space of quadratic forms on  $\mathbf{C}^n$ .

Recall that Sylvester's theorem says a real similarity class  $[M]$  (or orbit  $GL_n \mathbf{R} \cdot M$ ) is completely determined by its signature  $(p, q, k)$  where  $p = \#$  of positive eigenvalues,  $q = \#$  of negative eigenvalues, and  $k = \text{nullity} = n - (p + q)$ . However, the complex similarity classes are completely determined by their nullity and so two real orbits  $GL_n \cdot M_1, GL_n \mathbf{R} \cdot M_2$  are on the same complex orbit if their nullities are equal; that is, if  $p_1 + q_1 = p_2 + q_2$ . Clearly, there are several distinct real orbits in a given complex orbit. Moreover, using the inner product  $\langle X, Y \rangle = \text{tr}(XY^t)$  on both  $\mathfrak{gl}_n \mathbf{R}$  and  $\text{Symm}(n, \mathbf{R})$ , one can show that every orbit in this representation is distinguished.

**5. Proofs of main theorems.** Here we prove Theorem 4.7 on distinguished orbits. To do this, we first prove a statement for complex moment maps in the complex setting. Then we will relate the complex moment map information to the real moment map for the  $G^{\mathbf{C}}$  action.

*Remark.* For  $x \in \mathbf{CP}(V^{\mathbf{C}})$ , the critical points of  $\|\mu^*\|^2$  restricted to  $G^{\mathbf{C}} \cdot x$  are precisely the critical points of  $\|\mu^*\|^2$  as a function on  $\mathbf{CP}(V^{\mathbf{C}})$ . This is because  $\text{grad } \|\mu^*\|^2(x)$  is always tangent to  $G^{\mathbf{C}} \cdot x$ . We denote the set of critical points of  $\|\mu^*\|^2$  in  $\mathbf{CP}(V^{\mathbf{C}})$  by  $\mathfrak{C}$ .

**Theorem 5.1.** *For  $x \in \mathbf{CP}(V^{\mathbf{C}})$ , suppose  $G^{\mathbf{C}} \cdot x \subseteq \mathbf{CP}(V^{\mathbf{C}})$  contains a critical point of  $\|m\|^2$ . If  $z \in \mathfrak{C} \subseteq \mathbf{CP}(V^{\mathbf{C}})$  is such a critical point, then  $\mathfrak{C} \cap G^{\mathbf{C}} \cdot x = U \cdot z$ . Moreover,*

$$U \cdot z = \bigcup_{g \in G^{\mathbf{C}}} \omega(gx).$$

Let  $\mathfrak{C}_{\mathbf{R}}$  denote the set of critical points of  $\|m\|^2$  on  $\mathbf{PV}$ . We have a real analogue of the theorem above.

**Theorem 5.2.** *For  $x \in \mathbf{PV}$ , suppose  $G \cdot x \subseteq \mathbf{PV}$  contains a critical point of  $\|m\|^2$ . If  $z \in \mathfrak{C}_{\mathbf{R}} \subseteq \mathbf{PV}$  is such a critical point, then  $\mathfrak{C}_{\mathbf{R}} \cap G \cdot x = K \cdot z$ . Moreover,*

$$K \cdot z = \bigcup_{g \in G} \omega(gx).$$

Before proving Theorems 5.1 and 5.2, we apply Theorem 5.1 to prove Theorem 4.7.

*Proof of Theorem 4.7.* Suppose first that  $G \cdot [v]$  is distinguished. Then  $G \cdot [v] = G \cdot [w]$  where  $[w]$  is a critical point of  $\|m\|^2$ . But now Proposition 4.3 implies that  $[w]$  is a critical point of  $\|n\|^2$  and Proposition 4.6 implies that  $\pi[w]$  is a critical point of  $\|\mu^*\|^2$ ; that is,  $G^{\mathbf{C}} \cdot \pi[v]$  is distinguished.

Now suppose  $G^{\mathbf{C}} \cdot \pi[v]$  is distinguished. Our goal is to show that the orbit  $G \cdot [v]$  in  $\mathbf{PV}$  contains a critical point of  $\|m\|^2$ . We will use the  $G^{\mathbf{C}}$  action on  $\mathbf{RPV}^{\mathbf{C}}$  and the real moment map of this action. As  $G^{\mathbf{C}} \cdot \pi[v]$  is distinguished, and  $\pi : G^{\mathbf{C}} \cdot [v] \rightarrow G^{\mathbf{C}} \cdot \pi[v]$  is surjective, there exists a  $w \in G^{\mathbf{C}} \cdot [v]$  such that  $\pi[w] \in G^{\mathbf{C}} \cdot \pi[v]$  is a critical point of  $\|\mu^*\|^2$ .

Apply the negative gradient flow of  $\|n\|^2$  in  $\mathbf{RPV}^{\mathbf{C}}$  starting at  $[v] \in \mathbf{PV}$ . By Proposition 4.3, this is the negative gradient flow of  $\|m\|^2$ , and the  $\omega$ -limit set  $\omega[v] \subseteq \overline{G \cdot [v]}$  consists of critical points of  $\|n\|^2$  and  $\|m\|^2$  (see the remark following Definition 4.5). By Proposition 4.6 d and Theorem 5.1, we have  $\pi(\omega[v]) = \omega(\pi[v]) \subseteq U \cdot \pi[w]$ ; hence,  $\omega[v] \subseteq \pi^{-1}(U \cdot \pi[w]) = \mathbf{C}^* \times U \cdot [w] \subseteq \mathbf{C}^* \times G^{\mathbf{C}} \cdot [v]$ . This implies

$$\begin{aligned} \omega[v] &\subseteq \mathbf{C}^* \times G^{\mathbf{C}} \cdot [v] \cap \overline{G \cdot [v]} \subseteq \mathbf{C}^* \times G^{\mathbf{C}} \cdot [v] \cap \overline{\mathbf{R}^* \times G \cdot [v]} \\ &= \mathbf{R}^* \times G \cdot [v] \\ &= G \cdot [v] \end{aligned}$$

by Lemma 3.3 and the fact that  $(\mathbf{R}^* \times G)^{\mathbf{C}} = \mathbf{C}^* \times G^{\mathbf{C}}$ . Hence,  $\omega[v]$  consists of critical points of  $\|m\|^2$  that lie in  $G \cdot [v]$ . This proves Theorem 4.7.  $\square$

Before proving Theorem 5.1, we prove Theorem 5.2. The proof of this theorem is actually embedded in the proof of Theorem 4.7. We present it here.

*Proof of 5.2.* The fact that  $\mathfrak{C}_{\mathbf{R}} \cap G \cdot x$  constitutes a single  $K$ -orbit is the content of [15, Theorem 1]. In [15],  $G$  is taken to be semi-simple; however, all the results hold for  $G$  real reductive with the same proofs, mutatis mutandis. Our original contribution is the second statement of the theorem. We prove it here.

Suppose  $G \cdot x \subseteq \mathbf{P}V$  contains a critical point  $z$  of  $\|m\|^2$ . Then the orbit  $G^{\mathbf{C}} \cdot \pi(x)$  is distinguished in  $\mathbf{CP}(V^{\mathbf{C}})$  by Theorem 4.7. The proof of Theorem 4.7 shows, for  $g \in G$ ,  $\omega(gx)$  consists of critical points of  $\|m\|^2$  in  $G \cdot x$ . By Theorem 1 of [15], we have  $\omega(gx) \subseteq K \cdot z$ . Hence,

$$\bigcup_{g \in G} \omega(gx) = K \cdot z,$$

since  $\omega(y) = \{y\}$  for all  $y \in K \cdot z$ .  $\square$

Lastly we have to prove Theorem 5.1. The first statement is proven in [18, Theorem 6.2]. That is, the critical points of  $\|\mu^*\|^2$  on a  $G^{\mathbf{C}}$ -orbit comprise a single  $U$ -orbit. As in Theorem 5.2, our original contribution is the second statement. While the proof of Theorem 5.1 follows from well-known results in the classic literature, surprisingly, this theorem does not appear in either [10, 18]. (As stated earlier, this theorem is a necessary tool to understanding special solutions of the Ricci flow on nilpotent Lie groups, see Section 6.)

*Remark.* It has been pointed out to me by Jorge Lauret that the set  $\omega(x)$  consists of a single point, see [21, subsection 2.5]. However, our proof does not require the use of this fact.

*Proof of Theorem 5.1.* To obtain this result, we couple the works of Kirwan and Ness, from which the theorem follows quickly. We begin with the following theorem as motivation, see [18, Theorem 7.1].

**Theorem (Ness).** *Let  $z \in \mathbf{CP}(V^{\mathbf{C}})$  be a critical point of  $\|\mu^*\|^2$ . For  $g \in G^{\mathbf{C}}$ ,*

$$\|\mu^*\|^2(z) \leq \|\mu^*\|^2(gz)$$

*with equality if and only if  $g \in U \subset G^{\mathbf{C}}$ .*

On the orbit  $G^{\mathbf{C}} \cdot z$ , one expects the values of  $\|\mu^*\|^2$  to be bounded away from the minimum value  $\|\mu^*\|^2(z)$  outside of a neighborhood of  $U \cdot z$ . However, a priori, it is not clear if the following can occur: Letting  $z$  be a point at which  $\|\mu^*\|^2$  is minimized along the orbit  $G^{\mathbf{C}} \cdot z$ , does there exist a sequence  $g_n \in G^{\mathbf{C}}$  such that  $\lim \|\mu^*\|^2(g_n z) = \|\mu^*\|^2(z)$  but  $\lim g_n z \in \overline{G^{\mathbf{C}} \cdot z} - G^{\mathbf{C}} \cdot z$ ? The technical work of [18, Section 7] is not enough to avoid this scenario; we demonstrate below that this does not happen.

We recall the following structural results from [10]. There is a smooth stratification of  $\mathbf{CP}(V^{\mathbf{C}})$  into strata  $S_\beta$  which are  $G^{\mathbf{C}}$ -invariant. The strata are determined by a certain decomposition of the critical set  $\mathfrak{C}$  of  $\|\mu^*\|^2$  in  $\mathbf{CP}(V^{\mathbf{C}})$ . This critical set is a finite union  $\mathfrak{C} = \cup_{\beta \in B} C_\beta$  where  $\|\mu^*\|^2$  takes a constant value on  $C_\beta$  and each  $C_\beta$  is  $U$ -invariant. We will denote this constant value of  $\|\mu^*\|^2$  on  $C_\beta$  by  $M_\beta = \|\beta\|^2$ . Here  $B$  is actually a finite set in  $\mathfrak{g}^{\mathbf{C}}$ , and the norm  $\|\cdot\|$  comes from the prescribed inner product on  $\mathfrak{g}^{\mathbf{C}}$ .

For  $\beta \in B$ , the stratum  $S_\beta$  is defined to be the set of points which flow via the negative gradient flow to the critical set  $C_\beta$ , that is,  $S_\beta = \{x \in \mathbf{CP}(V^{\mathbf{C}}) \mid \omega(x) \subseteq C_\beta\}$ . In particular,  $C_\beta \subseteq S_\beta$ . See [10, Section 2] for a detailed discussion of this Morse theoretic approach to geometric invariant theory. If  $G^{\mathbf{C}} \cdot y \cap C_\beta \neq \emptyset$ , then

$$G^{\mathbf{C}} \cdot y \cap C_\beta = U \cdot z,$$

for  $z \in C_\beta$ , that is, the critical points in a  $G^{\mathbf{C}}$ -orbit comprise a single  $U$ -orbit, see [18, Theorem 7.1].

Fix  $\beta$ . We will be interested in  $z \in C_\beta$  and the orbit  $G^{\mathbf{C}} \cdot z$ . We define  $\mathcal{O}_\varepsilon = \{x \in \mathbf{CP}(V^{\mathbf{C}}) \mid \|\mu^*\|^2(x) \in [M_\beta, M_\beta + \varepsilon)\} \cap S_\beta$ . This is an open subset of  $S_\beta$  that contains  $C_\beta = \{x \in S_\beta \mid \|\mu^*(x)\|^2 = M_\beta\}$ . We observe that  $\mathcal{O}_\varepsilon$  is invariant under the forward flow  $\varphi_t$  of  $-\text{grad } \|\mu^*\|^2$  as  $\|\mu^*\|^2$  decreases along the trajectories  $t \rightarrow \varphi_t(x)$ . Since  $G^{\mathbf{C}} \cdot z$  is a submanifold of  $\mathbf{CP}(V^{\mathbf{C}})$ , hence also of  $S_\beta$ ,  $\mathcal{O}_\varepsilon \cap G^{\mathbf{C}} \cdot z$  is open in  $G^{\mathbf{C}} \cdot z$  and contains  $U \cdot z$  as  $C_\beta$  is  $U$ -invariant.

**Definition 5.3.** We define  $\{V_{\varepsilon,i}\}$  to be the collection of connected components of  $\mathcal{O}_\varepsilon \cap G^{\mathbf{C}} \cdot z$  that intersect  $U \cdot z$ . We define

$$V_\varepsilon := \bigcup_i V_{\varepsilon,i}.$$

*Remark.*  $V_\varepsilon$  is an open set of  $G^{\mathbf{C}} \cdot z$  that contains  $U \cdot z$ . As  $U$  has finitely many components,  $U = \cup_{i=1}^m \phi_i U_0$ , and we can write

$$V_\varepsilon = \bigcup_{i=1}^m V_{\varepsilon,i}$$

where  $\phi_i U_0(z) \subseteq V_{\varepsilon,i}$ . The  $V_{\varepsilon,i}$  are connected and open in  $G^{\mathbf{C}} \cdot z$  as  $\mathcal{O}_\varepsilon \cap G^{\mathbf{C}} \cdot z$  is open in  $G^{\mathbf{C}} \cdot z$  and  $G^{\mathbf{C}} \cdot z$  is locally connected, see [17, Theorem 25.3]. Moreover, since  $\mathcal{O}_\varepsilon$  and  $G^{\mathbf{C}} \cdot z$  are invariant under  $\varphi_t$ ,  $t > 0$ , we see that the components  $V_{\varepsilon,i}$  are invariant under forward flow, as well.

**Proposition 5.4.** *There exists a  $\varepsilon > 0$  such that  $\overline{V_\varepsilon} \subseteq G^{\mathbf{C}} \cdot z$ . Moreover,  $\omega(V_\varepsilon) = U \cdot z$  for small  $\varepsilon > 0$ .*

*Proof.* Before proving this statement, we will show that some open set  $A$  exists containing  $U \cdot z$  in  $G^{\mathbf{C}} \cdot z$  such that  $\overline{A}$  is a compact subset of  $G^{\mathbf{C}} \cdot z$ . Then we will show that  $V_\varepsilon \subseteq A$  for small  $\varepsilon$ . This would then prove the first assertion of the proposition.

Recall that  $G^{\mathbf{C}} = U \exp(iLU)$ . If we let  $B$  be the open unit ball in  $iLU$ , then  $A = U \exp(B) \cdot z$  has the said property, that is,  $\overline{A}$  is a compact subset of  $G^{\mathbf{C}} \cdot z$ .

**Lemma 5.5.** *Either  $V_\varepsilon \subseteq A$  or  $V_\varepsilon \cap \partial A \neq \emptyset$ . For small  $\varepsilon > 0$ ,  $V_\varepsilon \subseteq A$ .*

This will follow from

**Lemma 5.6.** *Either  $V_{\varepsilon,i} \subseteq A$  or  $V_{\varepsilon,i} \cap \partial A \neq \emptyset$ .*

To prove this lemma, suppose that  $V_{\varepsilon,i} \not\subseteq A$  and  $V_{\varepsilon,i} \cap \partial A = \emptyset$ . Since  $V_{\varepsilon,i} \cap A$  intersects  $U \cdot z$ , we see that  $V_{\varepsilon,i} = (V_{\varepsilon,i} \cap A) \cup (V_{\varepsilon,i} \setminus \overline{A})$ ; that is,  $V_{\varepsilon,i}$  is separated by these disjoint open sets. This contradicts the connectedness of  $V_{\varepsilon,i}$ , and the lemma is proved.

We continue with the proof of the first lemma. Suppose  $V_\varepsilon \not\subseteq A$  for every  $\varepsilon > 0$ . Then for each  $\varepsilon$ , there exists some point  $p_\varepsilon \in V_\varepsilon \cap \partial A$ . By

definition,  $\|\mu^*\|^2(p_\varepsilon) \leq M_\beta + \varepsilon$ . Letting epsilon go to zero, we can find a limit point  $p_\infty \in \partial A$  as  $\partial A$  is compact. Hence,  $p_\infty \in G^{\mathbf{C}} \cdot z - A \subseteq G^{\mathbf{C}} \cdot z - U \cdot z$ . Moreover,  $\|\mu^*\|^2(p_\infty) = M_\beta$ , and we have found a point in  $G^{\mathbf{C}} \cdot z$  which is not on  $U \cdot z$  but minimizes  $\|\mu^*\|$  on  $G^{\mathbf{C}} \cdot z$ . This is a contradiction since  $G^{\mathbf{C}} \cdot z \cap C_\beta = U \cdot z$  by [18, Theorem 7.1]. Therefore,  $V_\varepsilon \subseteq A$  for small  $\varepsilon$ . This proves the first lemma and the first claim in the proposition.

To finish the proof of the proposition, we observe that  $U \cdot z = \omega(U \cdot z) \subseteq \omega(V_\varepsilon)$  since  $U \cdot z \subseteq C_\beta$  and  $\varphi_t$  fixes the points of  $C_\beta$  for all  $t$ . Thus, we just need to show containment in the other direction. Since the set  $V_\varepsilon$  is invariant under forward flow and  $V_\varepsilon \subseteq G^{\mathbf{C}} \cdot z \subseteq S_\beta$ , we see that  $\omega(V_\varepsilon) \subseteq \bar{V}_\varepsilon \cap C_\beta \subseteq G^{\mathbf{C}} \cdot z \cap C_\beta = U \cdot z$ .  $\square$

**Definition 5.7.** Let  $\mathcal{O} = \{x \in G^{\mathbf{C}} \cdot z \mid \omega(x) \subseteq U \cdot z\}$ .

**Lemma 5.8.** Consider the set  $\mathcal{O}$  defined above. Then  $\mathcal{O} = G^{\mathbf{C}} \cdot z$ .

To prove the lemma it suffices to show that  $\mathcal{O}$  is open and closed in  $G^{\mathbf{C}} \cdot z$  and intersects each component of  $G^{\mathbf{C}} \cdot z$ . To see that  $\mathcal{O}$  intersects each component of  $G^{\mathbf{C}} \cdot z$ , we observe that  $\mathcal{O}$  contains  $U \cdot z$  and that each component of  $G^{\mathbf{C}}$  intersects  $U$  since  $G^{\mathbf{C}} = UQ$  and  $Q = \exp(\mathfrak{q})$  is contractible, see the remarks before Proposition 2.4. Choose  $\varepsilon > 0$  as in Proposition 5.4.

**$\mathcal{O}$  is open.** We know for small  $\varepsilon > 0$ ,  $V_\varepsilon$  is open in  $G^{\mathbf{C}} \cdot z$ , contains  $U \cdot z$ , and  $V_\varepsilon$  is contained in  $\mathcal{O}$  by Proposition 5.4. It suffices to consider  $x \in \mathcal{O} \setminus U \cdot z$ . Then there exists a  $t_* > 0$  such that  $\varphi_{t_*}(x)$  belongs to  $V_\varepsilon$ , from the definition of  $\mathcal{O}$ . But  $\varphi_{-t_*} : V_\varepsilon \rightarrow \varphi_{-t_*}(V_\varepsilon)$  is a diffeomorphism of  $G^{\mathbf{C}} \cdot z$  (and also of  $S_\beta$ ). Thus,  $\varphi_{-t_*}(V_\varepsilon)$  is an open set in  $G^{\mathbf{C}} \cdot z$  containing  $x$ , which is contained in  $\mathcal{O}$ . Therefore,  $\mathcal{O}$  is open.

**$\mathcal{O}$  is closed.** We will show  $\partial\mathcal{O} = \emptyset$ ; here we mean the boundary of  $\mathcal{O}$  in the topological space  $G^{\mathbf{C}} \cdot z$ . Take  $y_n \in \mathcal{O}$  such that  $y_n \rightarrow y \in G^{\mathbf{C}} \cdot z$ . Since  $z \in C_\beta \subseteq S_\beta$  and  $S_\beta$  is  $G^{\mathbf{C}}$ -invariant, it follows that  $y \in G^{\mathbf{C}} \cdot z \subseteq S_\beta$ , and hence  $\omega(y) \subseteq C_\beta$ . Thus, there exists an  $M > 0$  such that  $\varphi_M(y) \in \mathcal{O}_\varepsilon$ . We will denote the component of  $\mathcal{O}_\varepsilon \cap G^{\mathbf{C}} \cdot z$  containing  $\varphi_M(y)$  by  $\mathcal{O}_\varepsilon^y$ ; again, this component is open



in  $G^{\mathbf{C}} \cdot z$  as  $G^{\mathbf{C}} \cdot z$  is locally connected. Observe that, for  $t \geq M$ ,  $\varphi_t(y) \in \mathcal{O}_\varepsilon^y$  and  $\varphi_s(\mathcal{O}_\varepsilon^y) \subseteq \mathcal{O}_\varepsilon^y$  for  $s \geq 0$  as  $\varphi_s$  leaves  $\mathcal{O}_\varepsilon \cap G^{\mathbf{C}} \cdot z$  invariant for  $s \geq 0$ . Since  $\varphi_t$  is a diffeomorphism on  $S_\beta$  which preserves  $G^{\mathbf{C}} \cdot z$ ,  $\varphi_M^{-1}(\mathcal{O}_\varepsilon^y)$  is an open set of  $G^{\mathbf{C}} \cdot z$  containing  $y$ .

We assert that  $\mathcal{O}_\varepsilon^y \cap V_\varepsilon \neq \emptyset$ . Since  $y_n \in \mathcal{O}$ , we know that there exists a  $T_n > 0$  such that  $\varphi_{T_n}(y_n) \in V_\varepsilon$ , by definition of  $\mathcal{O}$ . Additionally, for  $t \geq T_n$ ,  $\varphi_t(y_n) \in V_\varepsilon$  by the flow invariance of  $V_\varepsilon$ .

Pick  $N$  such that  $y_N \in \varphi_M^{-1}(\mathcal{O}_\varepsilon^y)$ , which we can do as  $\varphi_M^{-1}(\mathcal{O}_\varepsilon^y)$  is open and  $y_n \rightarrow y$ . Then we have  $\varphi_M(y_N) \in \mathcal{O}_\varepsilon^y$ , a single component of  $\mathcal{O}_\varepsilon \cap G^{\mathbf{C}} \cdot z$ , and  $\varphi_{T_N}(y_N) \in V_\varepsilon$ .

(i) If  $M \geq T_N$ , then  $\varphi_M(y_N) = \varphi_{M-T_N}(\varphi_{T_N}(y_N)) \in \varphi_{M-T_N}(V_\varepsilon) \subseteq V_\varepsilon$ . That is,  $\varphi_M(y_N) \in \mathcal{O}_\varepsilon^y \cap V_\varepsilon \neq \emptyset$ .

(ii) If  $T_N \geq M$ , then  $\varphi_{T_N}(y_N) = \varphi_{T_N-M}(\varphi_M(y_N)) \in \varphi_{T_N-M}(\mathcal{O}_\varepsilon^y) \subseteq \mathcal{O}_\varepsilon^y$ . That is,  $\varphi_{T_N}(y_N) \in \mathcal{O}_\varepsilon^y \cap V_\varepsilon \neq \emptyset$ .

Thus,  $\mathcal{O}_\varepsilon^y$  being a connected component of  $\mathcal{O}_\varepsilon \cap G^{\mathbf{C}} \cdot z$  which intersects  $V_\varepsilon$ , a union of connected components of  $\mathcal{O}_\varepsilon \cap G^{\mathbf{C}} \cdot z$ , we have  $\mathcal{O}_\varepsilon^y \subseteq V_\varepsilon$ . That is,  $y \in \mathcal{O}$  since  $\varphi_t(y) \in V_\varepsilon$  for  $t \geq M$  and  $\omega(V_\varepsilon) \subseteq U \cdot z$  by Proposition 5.4. This proves the lemma and Theorem 5.1.  $\square$

## 6. Applications to the left-invariant geometry of Lie groups.

We apply the previous results to the study of left-invariant metrics on nilpotent and solvable Lie groups. The relationship between left-invariant Einstein metrics on solvable Lie groups, left-invariant Ricci soliton metrics on nilpotent Lie groups, and geometric invariant theory was established and explored by Heber [5] and Lauret [11]. We present a sketch of this relationship below and refer the reader to [14] for more details.

To motivate the reader more familiar with Einstein metrics than Ricci solitons, we present the following.

**Theorem 6.1** [13]. *Let  $S$  be a solvable Lie group with nilradical  $N$ . If  $S$  admits a left-invariant Einstein metric, then  $N$  necessarily admits a left-invariant Ricci soliton metric. Moreover, a classification of solvable groups admitting left-invariant Einstein metrics reduces to classifying nilpotent groups admitting left-invariant Ricci soliton metrics.*

Left-invariant Ricci soliton metrics on nilpotent Lie groups are often referred to as *nilsoliton* metrics. Given the above theorem, a thorough understanding of nilsolitons is necessary for understanding Einstein metrics on solvmanifolds; however, they are very interesting in their own right.

The relationship between left-invariant metrics on nilpotent Lie groups and geometric invariant theory is as follows. Consider a nilpotent Lie group  $N$  with Lie algebra  $\mathfrak{N}$ . A left-invariant metric on  $N$  corresponds to a choice of inner product on  $\mathfrak{N}$ . Thus, the space of left-invariant metrics on  $N$  is the space of inner products on  $\mathfrak{N}$ . We can vary the inner products on  $\mathfrak{N}$  to search for nilsolitons, or we can fix our choice of inner product on  $\mathfrak{N}$  and instead vary the Lie algebra structure on  $\mathfrak{N}$ . This is the perspective taken by Lauret.

Consider the vector space  $\mathbf{R}^n$  with the usual inner product; that is, so that the standard basis is orthonormal. We consider the space of skew-symmetric, bilinear forms on  $\mathbf{R}^n$

$$V = \wedge^2(\mathbf{R}^n)^* \otimes \mathbf{R}^n$$

$$= \{\mu : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n \mid \mu \text{ is bilinear and skew-symmetric}\}.$$

The set of Lie algebra brackets is clearly a subset of the vector space above; in fact, the set of Lie algebra structures is a variety. Moreover, the set of nilpotent Lie algebra brackets is also a variety. It is described by the polynomials describing the Jacobi identity and nilpotency (via Cartan's criterion for nilpotency).

There is a natural  $GL_n\mathbf{R}$  action on  $V$  which preserves the varieties of Lie algebra structures. For  $\mu \in V$ ,  $g \in GL_n\mathbf{R}$  and  $X, Y \in \mathbf{R}^n$ , we have

$$g \cdot \mu(X, Y) = g\mu(g^{-1}X, g^{-1}Y).$$

In this setting, the  $GL_n\mathbf{R}$ -orbits are precisely the isomorphism classes of Lie algebra structures on  $\mathbf{R}^n$ .

The inner product on  $\mathbf{R}^n$  extends naturally to an inner product on  $V = \wedge^2(\mathbf{R}^n)^* \otimes \mathbf{R}^n$  as follows. Denote the inner product on  $\mathbf{R}^n$  by  $\langle \cdot, \cdot \rangle$ , and denote its extension to  $V$  by the same notation. Then for  $\mu, \lambda \in V$ , we define  $\langle \mu, \lambda \rangle = \sum_{ij} \langle \mu(e_i, e_j), \lambda(e_i, e_j) \rangle$  where  $\{e_i\}$  is an orthonormal basis of  $\mathbf{R}^n$ . On the Lie algebra  $\mathfrak{gl}_n\mathbf{R}$ , we use the usual inner product from the trace form, that is,  $\langle \langle X, Y \rangle \rangle = \text{tr}(XY^t)$  for  $X, Y \in \mathfrak{gl}_n\mathbf{R}$ .

Using these inner products, we can construct the moment map  $\widetilde{m}$  for the action of  $GL_n\mathbf{R}$  on  $V$  and similarly  $m$  for the action of  $GL_n\mathbf{R}$  on  $\mathbf{P}V$  (see Section 2).

**Theorem 6.2** (Lauret). *Let  $N_\mu$  denote the simply connected nilpotent Lie group with left-invariant metric whose Lie algebra  $\mathfrak{N}_\mu$  (with inner product) corresponds to the point  $\mu \in V$ . Then  $N_\mu$  is a nilsoliton if and only if  $\mu$  is a critical point of  $F(v) = \|m \circ \pi\|^2(v)$ . Equivalently,  $N_\mu$  is an Einstein nilradical if and only if the orbit  $GL_n\mathbf{R} \cdot \mu$  is distinguished.*

This theorem can be found in [14]. The last equivalence is not stated using the label of distinguished orbit but is stated using the idea. In Section 10 of [14], there are several open questions of interest which are presented. We state Question 5 from this list.

**Question 6.3.** Consider the function  $F : V \rightarrow \mathbf{R}$  defined by  $F(v) = \|m \circ \pi(v)\|^2$  where  $\pi : V \rightarrow \mathbf{P}V$  is the usual projection and  $m$  is the moment map on real projective space. Define  $\mu_t$  to be the integral curve of  $-\text{grad } F$  starting at  $\mu_0$  on the sphere of radius 2. Is  $\mu_\infty$  (the limit point along the integral curve) contained in the orbit  $GL_n\mathbf{R} \cdot \mu_0$  if  $N_{\mu_0}$  is an Einstein nilradical?

This is clearly a special case of our work (Theorem 5.2). The flow restricted to the sphere of radius 2 in  $V$  projects onto the flow in projective space; recall that the sphere is a two-to-one cover of  $\mathbf{P}V$ . Thus, convergence within the group orbit in the sphere is equivalent to convergence within the group orbit in projective space. Finally, as  $N_{\mu_0}$  being an Einstein nilradical is equivalent to the orbit  $GL_n\mathbf{R} \cdot \mu_0$  being distinguished, we have the following.

**Theorem 6.4.** *Let  $N_{\mu_0}$  be an Einstein nilradical. Let  $\mu_\infty$  denote the limit point of the negative gradient flow of the function  $F$  starting at  $\mu_0$ . Then  $\mu_\infty$  is contained in the orbit  $GL_n\mathbf{R} \cdot \mu_0$ ; that is,  $N_{\mu_0}$  and  $N_{\mu_\infty}$  are isomorphic Lie groups.*

Lastly, we apply Corollary 4.8 to the setting of real forms of complex Lie algebras to obtain another interesting geometric consequence of our work. Let  $N$  be a simply connected real nilpotent Lie group with Lie

algebra  $\mathfrak{N}$ . Let  $N^{\mathbf{C}}$  denote the simply connected complex Lie group with Lie algebra  $\mathfrak{N}^{\mathbf{C}} = \mathfrak{N} \otimes \mathbf{C}$ . We call  $N^{\mathbf{C}}$  the complexification of  $N$ .

**Theorem 6.5.** *Let  $N_1$  and  $N_2$  be two real simply connected nilpotent Lie groups whose complexifications  $N_1^{\mathbf{C}}$ ,  $N_2^{\mathbf{C}}$  are isomorphic. Then  $N_1$  is an Einstein nilradical if and only if  $N_2$  is an Einstein nilradical.*

*Remark.* This theorem has also been obtained by Nikolayevsky in [19, Theorem 6] where he studies closed orbits of a particular reductive group associated to each nilmanifold. There, the philosophy of comparing real and complex group orbits is also employed.

This theorem stands out in that it is special to the case of nilpotent Lie groups. For example,  $SU(2)$  and  $SL_2\mathbf{R}$  are simple groups with isomorphic complexifications. On semi-simple groups, if a left-invariant metric is an algebraic Ricci soliton, then it must be a left-invariant Einstein metric (see [11]). However,  $SU(2)$  does admit a left-invariant Einstein metric while  $SL_2\mathbf{R}$  cannot.

**Acknowledgments.** This note is a portion of my thesis work completed under the direction of Pat Eberlein at the University of North Carolina, Chapel Hill. I am grateful to P. Eberlein and J. Lauret for suggesting many improvements to the first version of this paper.

## REFERENCES

1. Armand Borel, *Linear algebraic groups*, Second ed., Grad. Texts Math. **126**, Springer-Verlag, New York, 1991.
2. Armand Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. Math. **75** (1962), 485–535.
3. Patrick Eberlein and Michael Jablonski, *Closed orbits of semisimple group actions and the real Hilbert-Mumford function*, New developments in Lie theory and geometry, Contemp. Math. **491**, American Mathematical Society, Providence, RI, 2009.
4. V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.
5. Jens Heber, *Noncompact homogeneous Einstein spaces*, Invent. Math. **133** (1998), 279–352.
6. James E. Humphreys, *Linear algebraic groups*, Grad. Texts Math. **21**, Springer-Verlag, New York, 1981.

7. Michael Jablonski, *Detecting orbits along subvarieties via the moment map*, Münster J. Math. **3** (2010), 67–88.
8. ———, *Real geometric invariant theory and Ricci soliton metrics on two-step nilmanifolds*, thesis, University of North Carolina, 2008.
9. G. Kempf and L. Ness, *The length of vectors in representation spaces*, Springer Lect. Notes **732** (Copenhagen), Algebraic Geometry, Proceedings, 1978.
10. Frances Clare Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Math. Notes **31**, Princeton University Press, Princeton, New Jersey, 1984.
11. Jorge Lauret, *Ricci soliton homogeneous nilmanifolds*, Math. Ann. **319** (2001), 715–733.
12. ———, *On the moment map for the variety of Lie algebras*, J. Funct. Anal. **202** (2003), 392–423.
13. ———, *Einstein solvmanifolds are standard*, Ann. Math. **172** (2010), 1859–1877.
14. ———, *Einstein solvmanifolds and nilsolitons*, New developments in Lie theory and geometry, Contemp. Math. **491**, American Mathematical Society, Providence, RI, 2009.
15. Alina Marian, *On the real moment map*, Math. Res. Lett. **8** (2001), 779–788.
16. G.D. Mostow, *Self-adjoint groups*, Ann. Math. **62** (1955), 44–55.
17. James R. Munkres, *Topology*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2000.
18. Linda Ness, *A stratification of the null cone via the moment map*, Amer. J. Math. **106** (1984), 1281–1329.
19. Y. Nikolayevsky, *Einstein solvmanifolds and the pre-Einstein derivation*, Trans. Amer. Math. Soc. **363** (2011), 3935–3958.
20. R.W. Richardson and P.J. Slodowy, *Minimum vectors for real reductive algebraic groups*, J. London Math. Soc. **42** (1990), 409–429.
21. Reyner Sjamaar, *Convexity properties of the moment mapping re-examined*, Adv. Math. **138** (1998), 46–91.
22. Hassler Whitney, *Elementary structure of real algebraic varieties*, The Annals Math. **66** (1957), 545–556.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OK 73019

Email address: michael.jablonski@gmail.com