

Real Geometric Invariant Theory and Ricci Soliton Metrics on Two-step Nilmanifolds

Michael R. Jablonski

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Approved by

Advisor: Patrick B. Eberlein

Reader: James Damon

Reader: Shrawan Kumar

Reader: Michael Taylor

Reader: Jonathan Wahl

ABSTRACT

MICHAEL R. JABLONSKI: Real Geometric Invariant Theory and Ricci Soliton Metrics on Two-step Nilmanifolds

(Under the direction of Patrick B. Eberlein)

In this work we study Real Geometric Invariant Theory and its applications to left-invariant geometry of nilpotent Lie groups. We develop some new results in the real category that distinguish GIT over the reals from GIT over the complexes. Moreover, we explore some of the basic relationships between real and complex GIT over projective space to obtain analogues of the well-known relationships that previously existed in the affine setting.

This work is applied to the problem of finding left-invariant Ricci soliton metrics on two-step nilpotent Lie groups. Using our work on Real GIT, we show that most two-step nilpotent Lie groups admit left-invariant Ricci soliton metrics. Moreover, we build many new families of nilpotent Lie groups which cannot admit such metrics.

To the Lord my God.

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Introduction

The goal of this work is to explore real Geometric Invariant Theory and some of its practical applications.

Our main interests involve the orbit structure of algebraic groups acting on varieties. In the affine setting we are particularly interested in closed orbits. In the projective setting we are interested in the so-called distinguished orbits (cf. Definition 5.2). Specifically, we study actions of real and complex reductive groups on vector spaces, projective spaces, and homogeneous spaces. We apply our results to the geometry of two-step nilpotent Lie groups with left-invariant metrics.

In Chapter 2 we introduce the reader to the subject of Geometric Invariant Theory, over both \mathbb{R} and \mathbb{C} . Let G be a (real or complex) reductive group acting linearly on V . Some main problems of interest are the following. When is a particular orbit $G \cdot v$ closed in V ? Does there exist a purely local criterion to determine closedness of an orbit? When does there exist a Zariski open set of closed orbits?

In Chapter 3 we give a real version of Mumford's Numerical criterion. This is a local criterion to determine closedness of an orbit. Let G be a complex reductive group acting linearly on a complex vector space V . A point $v \in V$ is called *stable* if G_v is finite and $G \cdot v$ is closed in V ; more generally, we say that a point $v \in V$ is *semi-stable* if $0 \notin \overline{G \cdot v}$. In contrast, the *null-cone* consists of all points $v \in V$ such that $0 \in \overline{G \cdot v}$. The Hilbert-Mumford criterion compares the action of an algebraic group G with the actions of all algebraic 1-parameter subgroups. This criterion can be summarized neatly using Mumford's Numerical function $M : V \rightarrow \mathbb{R}$ as follows.

Theorem 3.15. *Let G act on V and take $v \in V$. Then*

- (a) $M(v) > 0$ if and only if v is in the null cone
- (b) $M(v) = 0$ if and only if v is semi-stable, but not stable
- (c) $M(v) < 0$ if and only if v is stable

Over \mathbb{C} this is Theorem 2.12 in the text. If instead we consider the action of a real reductive group G on a real vector space V , then we can define a point $v \in V$ to be *real stable* if G_v is compact and $G \cdot v$ is closed (cf. Definition 3.2). In the real setting one can define a numerical function M analogously. With these natural adjustments to the real setting, we have obtained the above theorem in the real category, see Theorem 3.15.

This real version of the Hilbert-Mumford criterion obtains many more semi-stable representations that the traditional criterion misses, see Sections 3.2 and 3.3.

One application of the theorem above is the following.

Corollary 3.10. *Let $v \in V$ be such that $M(v) < 0$. Then there is an open neighborhood \mathcal{O} such that $M(w) < 0$ for $w \in \mathcal{O}$.*

If $M < 0$ for some $v \neq 0$, it follows that there is a nonempty Zariski open set of points whose G -orbits are all closed (cf. Proposition 2.8). We point out that this open set where $M < 0$ is in general only Hausdorff open (cf. Example 3.20).

Chapter 4 is concerned with complex linear reductive groups G and the homogeneous spaces G/F which are affine. It is well-known that G/F is affine precisely when F is reductive (this is Matsushima's Criterion). Let H be a reductive subgroup of G . We ask the following question: When is the orbit $H \cdot (gF)$ closed in G/F ? We have obtained the following.

Theorem 4.1. *Consider the induced action of H on G/F . Then generic H -orbits are closed in G/F ; that is, there is a nonempty Zariski open set of G/F such that the H -orbit of any point in this open set is closed.*

Our proof uses Weyl's Unitary Trick and exploits the beautiful interplay between real and complex Geometric Invariant Theory. We do not know of this result in the literature and would be interested in a proof that holds more generally for reductive groups over algebraically closed fields.

From this theorem we obtain some interesting corollaries in regards to intersections of reductive algebras, linear actions of reductive subgroups, and stratifications of closed orbits by closed orbits of reductive subgroups:

Corollary 4.3. *Let G be a reductive algebraic group. If H, F are generic reductive subgroups, then $H \cap F$ is also reductive. More precisely, take any two reductive subgroups H, F of G . Then $H \cap gFg^{-1}$ is reductive for generic $g \in G$.*

Corollary 4.4. *Let G be a reductive group acting linearly on V . Let H be a reductive subgroup of G . If G has generically closed orbits then H does also. Moreover, each closed G -orbit is stratified by H -orbits which are generically closed.*

We say that a representation V of G is *good* if generic G -orbits are closed in V .

Corollary 4.5. *Let G be a reductive group, and let V and W be good G -representations, that is, generic G -orbits are closed. Then $V \oplus W$ is also a good G -representation.*

In Chapter 5 we move to the projective setting and study orbits à la Kirwan and Ness. Let G be a (real or complex) reductive group acting linearly on a (real or complex) vector space V . This gives rise to a well-defined action of G on $\mathbb{P}V$.

A G -orbit in V is closed if and only if it contains a zero of the so-called moment map $\tilde{m} : V \rightarrow \mathfrak{g}$. Let G be a real reductive group acting on a real space V , then the Zariski closure $G^{\mathbb{C}}$ of G acts on $V^{\mathbb{C}} = V \otimes \mathbb{C}$. Consider $v \in V \subset V^{\mathbb{C}}$. Borel-Harish Chandra/Richardson-Slodowy have shown that $G \cdot v$ is closed if and only if $G^{\mathbb{C}} \cdot v$ is closed (see Chapter 2). We produce an analogue of this result for projective space; the analogue of closed orbits for projective space are the so-called distinguished orbits.

To study projective space and the G -orbits therein, Kirwan and Ness study the norm squared of the moment map $\|m\|^2 : \mathbb{P}V \rightarrow \mathbb{R}$. A point $[v] \in \mathbb{P}V$ is called *distinguished* if it is a critical point of $\|m\|^2$; an orbit $G \cdot [v]$ is called *distinguished* if it contains a distinguished point. We prove an analogue of the theorem by Borel-Harish Chandra/Richardson-Slodowy for distinguished orbits (this theorem is a necessary tool for the results in Chapter 7). Additionally, we prove a theorem on the behavior of the negative gradient flow of $\|m\|^2$ (these theorems provide tools that are used in Chapter 8). We state these results here. In the following theorems $m : \mathbb{P}V \rightarrow \mathfrak{g}$ denotes the real moment map and $\mathfrak{C}_{\mathbb{R}}$ denotes the critical points of $\|m\|^2$ while $\mu^* : \mathbb{CP}(V^{\mathbb{C}}) \rightarrow \mathfrak{g}^{\mathbb{C}}$ denotes the complex moment map and \mathfrak{C} denotes the critical points of $\|\mu^*\|^2$.

Theorem 5.7 *Given $G \curvearrowright V$, $G^{\mathbb{C}} \curvearrowright V^{\mathbb{C}}$, and $[v] \in \mathbb{P}V$ we have*

$G \cdot [v]$ is a distinguished orbit in $\mathbb{P}V$ if and only if $G^{\mathbb{C}} \cdot \pi[v]$ is a distinguished orbit in $\mathbb{CP}(V^{\mathbb{C}})$.

Here $\pi : \mathbb{P}V \subseteq \mathbb{RP}V^{\mathbb{C}} \rightarrow \mathbb{CP}(V^{\mathbb{C}})$ is the usual projection.

Theorem 5.9 *For $x \in \mathbb{CP}(V^{\mathbb{C}})$, suppose $G^{\mathbb{C}} \cdot x \subseteq \mathbb{CP}(V^{\mathbb{C}})$ contains a critical point of $\|\mu^*\|^2$. If $z \in \mathfrak{C} \subseteq \mathbb{CP}(V^{\mathbb{C}})$ is such a critical point, then $\mathfrak{C} \cap G^{\mathbb{C}} \cdot x = U \cdot z$. Moreover, $U \cdot z = \bigcup_{g \in G^{\mathbb{C}}} \omega(gx)$.*

Theorem 5.10 *For $x \in \mathbb{P}V$, suppose $G \cdot x \subseteq \mathbb{P}V$ contains a critical point of $\|m\|^2$. If $z \in \mathfrak{C}_{\mathbb{R}} \subseteq \mathbb{P}V$ is such a critical point, then $\mathfrak{C}_{\mathbb{R}} \cap G \cdot x = K \cdot z$. Moreover, $K \cdot z = \bigcup_{g \in G} \omega(gx)$.*

In Chapter 6 we introduce the reader to the basic results pertaining to left-invariant Ricci soliton metrics on nilpotent Lie groups N . Consider the *normalized Ricci flow* $\frac{\partial}{\partial t}g = -2ric + \frac{2sc(g)}{n}g$, where ric is the (2,0) Ricci tensor of g and $sc(g)$ is the scalar curvature of g . Let g_0 be a metric and consider a solution to the normalized Ricci flow which is of the form $g(t) = \sigma(t)\psi_t^*g_0$, where $\sigma(t)$ is a scalar function of time, ψ_t are diffeomorphisms. When such a solution exists, we call the metric g_0 a (*homothetic*) *Ricci soliton*.

Let g be a left-invariant metric on N . J. Lauret has given the following algebraic characterization of left-invariant Ricci soliton metrics on nilmanifolds. A nilpotent Lie group N with a left-invariant Ricci soliton metric is called a *nilsoliton*.

Proposition 6.5. *Let (N, g) be a nilpotent group N with left invariant metric g . Then g is a soliton metric if and only if*

$$ric_g = cI + D$$

for some $c \in \mathbb{R}$ and some symmetric $D \in Der(\mathfrak{N})$.

In Chapter 7 we develop the basic theory to study two-step nilpotent Lie groups in the search for left-invariant Ricci soliton metrics. Let N denote a nilpotent Lie group, \mathfrak{N} its Lie algebra, and $[\cdot, \cdot]$ the Lie algebra structure on \mathfrak{N} . The group N and the algebra \mathfrak{N} are said to be two-step nilpotent if $[\mathfrak{N}, [\mathfrak{N}, \mathfrak{N}]] = \{0\}$. A two-step nilpotent Lie algebra \mathfrak{N} is said to be of type (p, q) if $\dim[\mathfrak{N}, \mathfrak{N}] = p$ and $\text{codim}[\mathfrak{N}, \mathfrak{N}] = q$.

Here we study the action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ à la Lauret. This action is defined as follows. The group $GL(q, \mathbb{R})$ acts on $\mathfrak{so}(q, \mathbb{R})$ via $g \cdot M = gMg^t$ for $g \in GL(q, \mathbb{R})$ and $M \in \mathfrak{so}(q, \mathbb{R})$; the group $GL(p, \mathbb{R})$ acts on \mathbb{R}^p in the usual way. Hence $(g, h) \cdot (M \otimes v) = (gMg^t) \otimes h(v)$ for $(g, h) \in GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ and $M \otimes v \in \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$. In this setting an isomorphism class of algebras corresponds to a $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ -orbit in $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$. Our main result is the following. We point out that optimal metrics are metrics which are nilsolitons with additional strong geometric properties (cf. Definition 7.7).

Theorem 7.25. *A generic two-step nilmanifold admits a nilsoliton metric. Moreover, the types (p, q) other than $(1, 2k + 1), (2, 2k + 1), (D - 1, 2k + 1), (D - 2, 2k + 1)$ generically admit optimal metrics.*

In Chapter 8 we produce two procedures for building new two-step nilsolitons from ‘smaller’ ones. The first is called *concatenation* (see Section 8.2) and the second is *direct sum* (see Section 8.4).

Theorem 8.5. *Consider $q_1 \leq q_2$, $D = \frac{1}{2}q_2(q_2 - 1)$, and $1 \leq p \leq D$ with $p \neq D - 1, D - 2$. Let \mathfrak{N}_1 and \mathfrak{N}_2 be generic nilsolitons of types (q_1, p) and (q_2, p) , respectively. Then the concatenation $\mathfrak{N} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathfrak{Z}$ is also a nilsoliton.*

Moreover, we produce a construction that generates many new examples of two-step nilalgebras which cannot admit left-invariant Ricci soliton metrics. Our construction produces such algebras in most types (p, q) . This is the content of Proposition 8.10 and Section 8.4.

Part 1

Representations and Orbit Structures

CHAPTER 1

Preliminaries

1. Algebraic Geometry

We present some of the tools from algebraic geometry that will be useful in our study of semi-simple group actions and particularly the structure of their orbits. Our main references for algebraic geometry will be [Bor91, Sha88, PV94] and [Whi57] for real algebraic geometry.

Consider affine space \mathbb{C}^n . A set X in \mathbb{C}^n is called an *affine variety* if X is the vanishing set of a collection of polynomials $\{f_\alpha : \mathbb{C}^n \rightarrow \mathbb{C}\}$. Varieties can be defined more generally; however, we will not need the more abstract notion of variety and we restrict our attention to the affine setting. Associated to X we have the *ring of regular functions* $\mathbb{C}[X]$. These are all the functions from X to \mathbb{C} that can be described by polynomials (given a coordinate system on X).

Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two (affine) varieties. A morphism $f : X \rightarrow Y$ is called *regular* if $f = (f_1, \dots, f_m)$ and each f_i is a regular function on X . A regular function $f : X \rightarrow Y$ is equivalent to having a comorphism $f^* : \mathbb{C}[Y] \rightarrow \mathbb{C}[X]$ between their rings of regular functions. This comorphism is defined via precomposition. It can be shown that two affine varieties are isomorphic if and only if their rings of regular functions are isomorphic.

Our variety X is said to be *defined over \mathbb{R}* if X can be described as the zero set of a collection of polynomials with real coefficients. Here we have a fixed coordinate system for \mathbb{C}^n ; that is, we have a basis $\{e_1, \dots, e_n\}$ and the space $\mathbb{R}^n \subset \mathbb{C}^n$ is well-defined. The ring of regular functions of X being defined over \mathbb{R} means precisely that $\mathbb{C}[X] = \mathbb{R}[X] \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathbb{R}[X]$ is the ring of polynomials with real coefficients. Likewise, we say that a morphism $f : X \rightarrow Y$ is *defined over \mathbb{R}* if $f^* : \mathbb{R}[Y] \rightarrow \mathbb{R}[X]$.

Let X be defined over \mathbb{R} . We define the set of *real points* of $X \subset \mathbb{C}^n$ as $X(\mathbb{R}) := X \cap \mathbb{R}^n$. This set is a real algebraic variety in the following sense.

1.1. Real Algebraic Geometry. We say that a set $X \subset \mathbb{R}^n$ is a *real algebraic variety* if it is the zero set of a collection of polynomials with real coefficients. Note that the ideal of polynomials that vanish on X has many different sets of generators. We present some of the well-known and very useful results relating the real and complex settings, see [Whi57] for proofs and more detail.

As $X \subset \mathbb{R}^n \subset \mathbb{C}^n$, we can consider the smallest (complex) algebraic variety containing X . This is the Zariski closure of X in \mathbb{C}^n which we denote by \overline{X} . This variety \overline{X} is defined over \mathbb{R} by construction.

THEOREM 1.1. *Let X be a real algebraic variety. We can write X as a finite union of irreducible components. As in the complex setting this union is unique if the irreducible components are maximal. Moreover, X has finitely many topological components. At the manifold or smooth points we have $\dim_{\mathbb{R}} X = \dim_{\mathbb{C}} \overline{X}$.*

This is quite distinct from the complex setting. If X is a smooth, irreducible complex variety then viewed as a manifold X is connected, see [Sha88, II.2.1 Theorem 6, VII.2 Theorem 1]

We will call a set a *(real) semi-algebraic set* if it is a union of some of the topological components of a real algebraic variety. Some of these components might be real varieties themselves. Consider the following example [Whi57, section 12]

EXAMPLE 1.2. *Consider the real variety cut out by $f = x^2 + y^2 - y^3$ in \mathbb{R}^2 . This zero set consists of a curve and a point (the origin). Here the real variety has two topological components, one of which is a variety (the point) and the other is not as f is irreducible over \mathbb{R} .*

We will demonstrate many more examples exhibiting this kind of behavior. In fact, we will be interested in trying to detect/classify all the topological components that arise when our variety is the orbit of a semi-simple group.

2. Lie Groups, Algebraic Groups, and Representations

Our main references for Lie groups will be [Hoc65, BtD95, Hel01]; for algebraic groups we use [Hum81, PV94, Ser87, Bor91, Che55].

Let G be a group which is also an analytic manifold. Let $\mu : G \times G \rightarrow G$ and $\text{inv} : G \rightarrow G$ be the multiplication and inverse maps, respectively. We call G a *Lie group* if these are analytic maps between manifolds. The space of left-invariant vector fields on G is called the Lie algebra of G and is denoted by $L(G)$ or the gothic letter \mathfrak{g} . This vector space is isomorphic to $T_e G$. The bracket structure (the algebra structure) on \mathfrak{g} is the usual one when $\mathfrak{g} \subset \mathfrak{gl}_n$, that is, for $X, Y \in \mathfrak{g}$ the bracket is $[X, Y] = XY - YX$.

DEFINITION 1.3. *Let G be a group which is also a variety over \mathbb{C} . We say that G is a complex algebraic group if μ and inv are regular maps between varieties. If G is a (Zariski) closed subgroup of some $GL_n(\mathbb{C})$ then G is said to be a linear algebraic group.*

DEFINITION 1.4. *Let G be a subgroup of $GL_n(\mathbb{R})$ which is also a real algebraic variety. We call G a real algebraic group. Similarly, if G is a subgroup of $GL_n(\mathbb{R})$ and is just a semi-algebraic set, then we call G a semi-algebraic group.*

Our primary interest will be in linear algebraic groups over \mathbb{R} and their real points (which are then real algebraic groups). For a detailed exposition of algebraic groups over algebraically closed fields see [Bor91]. We will exploit much of the work that has been done over \mathbb{C} to obtain information about algebraic groups

over \mathbb{R} . The real algebraic groups of interest to us will all be linear groups, so we restrict our attention to these.

Sometimes we will abuse terminology and refer to semi-algebraic groups as algebraic groups. It should be clear from context which we mean. However, from the view point of our results there is no need to distinguish between them.

PROPOSITION 1.5. *Let G be a real semi-algebraic group in $GL_n(\mathbb{R})$. Let $G^{\mathbb{C}}$ denote the Zariski closure over \mathbb{C} of the set G in $GL_n(\mathbb{C})$. Then $G^{\mathbb{C}}$ is a complex (linear) algebraic group defined over \mathbb{R} such that the set of real points $G^{\mathbb{C}}(\mathbb{R})$ satisfies $G^{\mathbb{C}}(\mathbb{R})_0 \subset G \subset G^{\mathbb{C}}(\mathbb{R})$, where $G^{\mathbb{C}}(\mathbb{R})_0$ denotes the Hausdorff identity component. Moreover, the Lie algebras of these groups satisfy $L(G^{\mathbb{C}}) = L(G) \otimes \mathbb{C}$ and $\dim_{\mathbb{R}} G = \dim_{\mathbb{C}} G^{\mathbb{C}}$.*

PROOF. As G is a semi-algebraic group, there is some real algebraic group H such that $H_0 \subset G \subset H$, where H_0 is the Hausdorff identity component of H . The fact that the set $G^{\mathbb{C}}$ is an algebraic group is the content of [Bor91, Proposition 1.3]. Now the results stated above follow directly from Theorem 1.1 and the observations that $L(H) = L(G)$ and $\dim_{\mathbb{R}} H = \dim_{\mathbb{R}} G$. \square

For such a group G as above, we call $G^{\mathbb{C}}$ the *(algebraic) complexification*. This depends on the embedding of G as an algebraic subgroup of $GL_n(\mathbb{R})$ and is not necessarily the universal complexification of G as described by Hochschild [Hoc65, XVII.5]. For an example of different algebraic complexifications see Example 1.13 and the remark thereafter.

DEFINITION 1.6. *Let $G^{\mathbb{C}}$ denote a complex algebraic group. An algebraic one-parameter subgroup, or 1-PS, is a morphism of algebraic groups $\chi : \mathbb{C}^* \rightarrow G^{\mathbb{C}}$. Let G denote a real algebraic group. A real algebraic 1-PS is a (real) morphism of (real) algebraic groups $\chi : \mathbb{R}^* \rightarrow G$. (By a morphism of algebraic groups we mean a homomorphism of groups which is a morphism of varieties.)*

2.1. Special kinds of Lie algebras. Let \mathfrak{g} be a Lie algebra. We define the following basic notions; see [Ser87] for more information. The *lower central series* of \mathfrak{g} is a descending series of ideals defined by

$$\begin{aligned} C^1 \mathfrak{g} &= \mathfrak{g} \\ C^m \mathfrak{g} &= [\mathfrak{g}, C^{m-1} \mathfrak{g}] \end{aligned}$$

for $n \geq 2$. A Lie algebra \mathfrak{g} is called *nilpotent* if there exists k such that $C^k \mathfrak{g} = 0$; moreover, we call \mathfrak{g} *k-step nilpotent* if k is the smallest integer such that $C^k \mathfrak{g} = 0$. The *derived series* of \mathfrak{g} is defined as the following descending series

$$\begin{aligned} D^1 \mathfrak{g} &= \mathfrak{g} \\ D^n \mathfrak{g} &= [D^{n-1} \mathfrak{g}, D^{n-1} \mathfrak{g}] \end{aligned}$$

for $n \geq 2$. The algebra \mathfrak{g} is said to be *solvable* if there is some k such that $D^k \mathfrak{g} = 0$. Similarly, we say that \mathfrak{g} is k -step solvable if k is the smallest integer such that $D^k \mathfrak{g} = 0$. We say that G is nilpotent or solvable if its Lie algebra is so.

Given a Lie algebra \mathfrak{g} we can consider the bilinear form $B(X, Y) = \text{tr}(ad X \circ ad Y)$ called the *Killing form*. The algebra \mathfrak{g} is said to be *semi-simple* if B is non-degenerate. We say G is semi-simple if \mathfrak{g} is so. Next we define the notion of a reductive algebra/group. We give the definition in terms of being a subgroup, resp. subalgebra, of GL_n , resp. \mathfrak{gl}_n , as this is the setting of primary interest to us. For a more intrinsic definition of reductive see [Bor91].

Consider a closed subgroup $G \subseteq GL(E)$ with finitely many connected components and its Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}_n$. Let \mathfrak{z} denote the center of \mathfrak{g} . We say that G , or $\mathfrak{g} = L(G)$, is *reductive* if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}$, $[\mathfrak{g}, \mathfrak{g}]$ is semi-simple, and $\mathfrak{z} \subseteq \mathfrak{gl}_n$ consists of semi-simple endomorphisms. Reductive groups in this sense are precisely the groups that are completely reducible, see [BHC62, section 1.2].

2.2. Representations of real groups. Let G be a Lie group. A (*real linear*) *representation* of G is a continuous homomorphism of Lie groups $\varphi : G \rightarrow GL_n(\mathbb{R})$. Similarly one can define a complex linear representation. Our primary interest is in the representations of real semi-simple and reductive groups. For complex semi-simple groups we have the following fundamental result.

THEOREM 1.7. *Let G be a connected complex semi-simple Lie group. Then*

- (a) *There is a complex algebraic group structure on G , and one only, which is compatible with its analytic group structure.*
- (b) *If H is a complex algebraic group, every analytic homomorphism from G to H is algebraic.*

See [Ser87] for more details.

This is in contrast to the real setting. Consider $\widetilde{SL_2\mathbb{R}}$ the simply connected cover of $SL_2\mathbb{R}$. Since this group has infinite center it cannot be the real points of a complex linear algebraic group. However we have the following result.

PROPOSITION 1.8. *Let G be a connected real semi-simple subgroup of $GL_n\mathbb{R}$ for some n . Then G is a real semi-algebraic group.*

See [Che55, corollary of §§14]. An immediate consequence of this is the following.

COROLLARY 1.9. *Let $\phi : G \rightarrow GL_n\mathbb{R}$ be a real representation of a connected semi-simple group G . Then $\phi(G)$ is a real semi-algebraic group.*

PROOF. Since G is semi-simple, so is its image $\phi(G)$. Now the result follows from the proposition. \square

DEFINITION 1.10. *Consider a complex algebraic group $G^{\mathbb{C}}$ and a complex linear representation $\phi : G^{\mathbb{C}} \rightarrow GL_n(\mathbb{C})$. The representation is called a *rational representation* if it is a morphism of algebraic groups; that is, a homomorphism of groups which is also a variety morphism.*

DEFINITION 1.11. Consider G a real (semi) algebraic group and $G^{\mathbb{C}}$ its complexification. We say that a representation $\phi : G \rightarrow GL_n(\mathbb{R})$ is a rational representation if it is the restriction of a rational representation $\phi : G^{\mathbb{C}} \rightarrow GL_n(\mathbb{C})$ which is defined over \mathbb{R} .

In light of these propositions and corollaries it makes sense to have, at a minimum, a basic understanding of algebraic groups, their representations, and algebraic geometry in general so that we can use all the tools from this very rich geometric setting. We give some examples of Lie groups which are not algebraic groups and how they can be very poorly behaved.

EXAMPLE 1.12. Let $T^2 = S^1 \times S^1$ be the compact 2-torus. Let G be the Lie subgroup of T^2 which is a dense winding line. This subgroup is clearly not closed, but abstractly can be viewed as the algebraic group \mathbb{R} . We will present a (non-algebraic) representation of G whose orbits are not submanifolds.

Consider the representation $\phi : T^2 \rightarrow GL_4(\mathbb{R})$ where $T^2 = S^1 \times S^1$ acts on $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ by the usual S^1 action on \mathbb{R}^2 in each slot. This representation is linear and the orbits here will be 2-tori. Now restrict this representation to one on $G =$ winding line. The orbits here will be winding lines contained in compact tori, thus they do not inherit the subspace topology from \mathbb{R}^4 .

We will see that this is in sharp contrast to the algebraic setting. Moreover, for later reference we point out that this group G is also self-adjoint with respect to a certain inner product on \mathbb{R}^4 . The next example demonstrates that not all representations of real algebraic groups are forced to be algebraic. Again, this is in contrast to the complex setting.

EXAMPLE 1.13. Consider $SL_3(\mathbb{R}) \subset SL_3(\mathbb{C})$ and the adjoint representations $Ad : SL_3(k) \rightarrow Ad(SL_3)(k) \subset GL(\mathfrak{sl}(3, k))$, where $k = \mathbb{R}$ or \mathbb{C} . Observe that Ad restricted to $SL_3(\mathbb{R})$ is one-to-one and so defines an analytic isomorphism of Lie groups; that is, the center of $SL_3(\mathbb{R})$ is trivial. In contrast, $SL_3(\mathbb{C})$ does have non-trivial center (of order 3) and so $Ad : SL_3(\mathbb{C}) \rightarrow PSL_3(\mathbb{C})$ has nontrivial kernel. Note $Im(Ad(G)) \simeq G/Z(G)$, where $Z(G)$ is the center of G .

Thus $Ad^{-1} : Ad(SL_3(\mathbb{R})) \rightarrow SL_3(\mathbb{R})$ is a well-defined homomorphism of Lie groups, but it cannot be the restriction of a homomorphism between $PSL_3(\mathbb{C})$ and $SL_3(\mathbb{C})$.

Remark. This example also produces multiple ‘complexifications’ of $SL_3(\mathbb{R})$; namely, $SL_3(\mathbb{C})$ and $PSL_3(\mathbb{C})$ both arise as the Zariski closure of $SL_3(\mathbb{R})$ depending on the algebraic structure, or imbedding, placed on $SL_3(\mathbb{R})$. One can show that $PSL_3(\mathbb{R}) = PSL_3(\mathbb{C})(\mathbb{R})$, and it is obvious that $SL_3(\mathbb{R}) = SL_3(\mathbb{C})(\mathbb{R})$.

3. Real vs. Complex Algebraic Groups and Their Actions

Since every semi-simple subgroup of $GL_n \mathbb{R}$ can be realized as a semi-algebraic group, we will state the known results for complex algebraic groups over \mathbb{C} and show how to go between the real and complex categories. Unless otherwise said, we will assume that G is a semi-algebraic group and we denote its complexification by $G^{\mathbb{C}}$.

Complex group orbits on a variety or vector space have some nice properties that we don't enjoy over the reals. For example, the Hausdorff and Zariski closures of a group orbit are the same for a complex linear algebraic group. One property that does translate to the reals is that the boundary of an orbit consists of orbits of strictly lower dimension. See section 8.3 of [Hum81] for the complex setting and see below for the real setting. For some interesting examples of semi-simple real algebraic groups whose orbit closure is not the Zariski closure see [EJ].

PROPOSITION 1.14. *Let $G^{\mathbb{C}}$ be a complex algebraic group acting on a complex vector space $V^{\mathbb{C}}$; that is, we have a linear representation $\phi : G^{\mathbb{C}} \rightarrow GL(V^{\mathbb{C}})$. Let $v \in V^{\mathbb{C}}$. The following are true.*

- (a) *The Hausdorff and Zariski closures of $G^{\mathbb{C}} \cdot v$ coincide.*
- (b) *The boundary $\partial(G^{\mathbb{C}} \cdot v) = \overline{G^{\mathbb{C}} \cdot v} - G^{\mathbb{C}} \cdot v$ consists of $G^{\mathbb{C}}$ orbits of strictly smaller dimension.*
- (c) *The orbit $G^{\mathbb{C}} \cdot v$ is a locally closed, embedded submanifold. That is, $G^{\mathbb{C}} \cdot v$ is an open set of a closed set. Moreover, this closed set is actually the variety $\overline{G^{\mathbb{C}} \cdot v}$.*

Given a real vector space V we denote the complexification by $V^{\mathbb{C}} = V \otimes \mathbb{C}$.

PROPOSITION 1.15. *Let G be a semi-algebraic group and $\phi : G \rightarrow GL(V)$ a rational representation. Let ϕ also denote the representation $G^{\mathbb{C}} \rightarrow GL(V^{\mathbb{C}})$ which restricts to G . Then for $v \in V \subset V^{\mathbb{C}}$ the following are true.*

- (a) *The stabilizer subalgebras satisfy $(\mathfrak{g}^{\mathbb{C}})_v = (\mathfrak{g}_v)^{\mathbb{C}}$*
- (b) $\dim_{\mathbb{R}} G \cdot v = \dim_{\mathbb{C}} G^{\mathbb{C}} \cdot v$
- (c) $G^{\mathbb{C}} \cdot v \cap V = \bigcup_{i=1}^m X_i$ where each X_i is a G -orbit.
- (d) $G^{\mathbb{C}} \cdot v$ is closed in $V^{\mathbb{C}}$ if and only if $G \cdot v$ is closed in V
- (e) $\partial(G \cdot v)$ consists of G -orbits of strictly smaller dimension.
- (f) $G^{\mathbb{C}} \cdot v \cap \overline{G \cdot v} = G \cdot v$.
- (g) *The orbit $G \cdot v$ is a locally closed, embedded submanifold.*

Proof of a. The first claim is clear as $\mathfrak{g}^{\mathbb{C}} = LG^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ acts \mathbb{C} -linearly on $V^{\mathbb{C}}$.

Proof of b. Notice that for a real Lie group H , $H \cdot v \simeq H/H_v$. Let \mathfrak{h} be the Lie algebra of H . At the Lie algebra level part (a) shows that $(\mathfrak{h}_v)^{\mathbb{C}} = (\mathfrak{h}^{\mathbb{C}})_v$, for $v \in V \subseteq V^{\mathbb{C}}$. As $\dim_{\mathbb{R}} G = \dim_{\mathbb{C}} G^{\mathbb{C}}$, we are done.

Proof of c & d. This can be found in [BHC62, Proposition 2.3] and [RS90].

Proof of e. The previous proposition states that the boundary $\overline{G^{\mathbb{C}} \cdot v} - G^{\mathbb{C}} \cdot v$ of the complex group orbit $G^{\mathbb{C}} \cdot v$ consists of $G^{\mathbb{C}}$ -orbits of strictly smaller dimension, see [Hum81, section 8.3]. Additionally, $G^{\mathbb{C}} \cdot v \cap V = \bigcup_1^m X_i$, where each X_i is a G -orbit. Each X_i is closed in $G^{\mathbb{C}} \cdot v \cap V$ as it is a finite union of connected components of $G^{\mathbb{C}} \cdot v \cap V$, see [BHC62, Proposition 2.3]. If $v \in X_i$ for $1 \leq i \leq m$, then $G \cdot v = X_i$ and $\overline{G \cdot v} \cap G^{\mathbb{C}} \cdot v = X_i$. If $w \in \overline{G \cdot v} - G \cdot v$, then $w \in \overline{G^{\mathbb{C}} \cdot v} - G^{\mathbb{C}} \cdot v$, and it follows from (b) and the previous proposition above that $G \cdot w$ has smaller dimension than $G \cdot v$.

Proof of f. This follows immediately from (e) and its proof.

Proof of g. This follows from part (c) of the previous proposition and part (c) above. \square

DEFINITION 1.16. *We say that two distinct orbits $G \cdot v_1$ and $G \cdot v_2$ are $G^{\mathbb{C}}$ -conjugate or complex conjugate if $G^{\mathbb{C}} \cdot v_1 = G^{\mathbb{C}} \cdot v_2$.*

3.1. Stabilizer in General Position. Consider a complex algebraic group G acting linearly on V . A subgroup G' is called a *stabilizer in general position*, or s.g.p., if there exists an open set \mathcal{O} of V with the following property. Given $v \in \mathcal{O}$ there exists $g \in G$ such that $G_v = gG'g^{-1}$. The following theorem is due to Richardson and Luna, cf. [PV94].

THEOREM 1.17. *Let G be a complex reductive (algebraic) group acting linearly and rationally on V . Then the s.g.p. exists.*

Over \mathbb{R} there may not exist such a subgroup G' . In general, a real Zariski open set has many Hausdorff components. In this way, there is usually not a single ‘generic’ item. However, we can make the following simple observation.

PROPOSITION 1.18. *Let G be a real reductive group acting linearly and rationally on a real vector space V . There exist a finite collection of Lie algebras $\{\mathfrak{g}_1, \dots, \mathfrak{g}_k\}$ and a Zariski open set \mathcal{O} of V such that for $v \in \mathcal{O}$ the stabilizer \mathfrak{g}_v is isomorphic to one of the algebras in $\{\mathfrak{g}_i\}$. Here all the algebras $\{\mathfrak{g}_i\}$ have isomorphic complexifications.*

Remark. At the moment there is not a real analogue of Theorem 1.17 that we know of in the literature. It would be interesting to prove the existence of a finite collection of real s.g.p.’s. We intend to work on this problem in the future.

4. Riemannian Geometry

We use as our main references for Riemannian geometry Helgason [Hel01] and do Carmo [dC92].

A manifold M is called a *Riemannian manifold* if there is a smooth metric $g : TM \rightarrow \mathbb{R}$ on M , where TM is the tangent bundle of M . We usually denote this pair by (M, g) ; sometimes we will interchangeably use g or \langle, \rangle . Every Riemannian manifold comes equipped with a compatible connection ∇ called the Levi-Civita connection. Denote the set of smooth vector fields on M by $V(M)$.

Given such a manifold, there exist basic geometric invariants. Of particular interest are the different notions of curvature. The *curvature tensor* R of a Riemannian manifold is a correspondence that associates to every pair $X, Y \in V(M)$ a mapping $R(X, Y) : V(M) \rightarrow V(M)$ given by $R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z$ for $Z \in V(M)$. This is a tensor of type $(3, 1)$. Equivalently we can consider the tensor of type $(4, 0)$ defined as $(X, Y, Z, T) = \langle R(X, Y)Z, T \rangle$.

Given a 2-plane $\sigma \subset T_p M$, and a pair of orthonormal vectors $x, y \in \sigma$, we can define the *sectional curvature* of σ as $K(\sigma) = (x, y, x, y)$. A manifold is called a manifold of constant curvature if $K(\sigma)$ is constant for all choices of $\sigma \subset T_p M$ and $p \in M$. More generally, we can consider the *Ricci curvature* defined as follows. Let $x = z_n \in T_p M$ be a unit vector. We extend this to an orthonormal basis $\{z_1, \dots, z_n\}$ of $T_p M$. The Ricci curvature at p is $Ric_p(x) = \frac{1}{n-1} \sum (x, z_i, x, z_i)$. This is an average of scalar curvatures of all 2-planes containing x . A Riemannian manifold M is called an *Einstein manifold* if $Ric_p(x)$ is constant for all choices of unit $x \in T_p M$ and $p \in M$.

EXAMPLE 1.19. Let G be a Lie group. A metric \langle, \rangle is called *left-invariant* if $\langle X, Y \rangle_g = \langle L_{g^{-1}*}X, L_{g^{-1}*}Y \rangle_e$ for $g \in G$ and $X, Y \in T_g G$, where $L_{g^{-1}}$ denotes left translation by g^{-1} .

The set of left-invariant metrics is equivalent to the set of inner products on the Lie algebra $L(G)$. A problem of great interest is to find all solvable Lie groups with left-invariant Einstein metrics. See Chapter 6 for more information.

Closed Orbits of Semi-Simple and Reductive Groups

Let G denote a real linear semi-simple group with finitely many connected components; that is, $G \subset GL_n(\mathbb{R})$ for some n . Recall that G is semi-algebraic and we can consider the complexification of G , as defined in Section 1.2, which we denote $G^{\mathbb{C}}$. Recall that $G^{\mathbb{C}} \subset GL_n(\mathbb{C})$ is a linear algebraic group and is the Zariski closure of G in $GL_n(\mathbb{C})$.

More generally we can consider real linear reductive groups. We always assume that our linear groups are closed subgroups and semi-algebraic or algebraic. The real reductive groups are products of semi-simple groups having finitely many components and (algebraic) tori. Again, if G is a real reductive group then G is a finite index subgroup of the real points $G^{\mathbb{C}}(\mathbb{R})$ of a complex algebraic reductive group $G^{\mathbb{C}}$.

The following problems are of great interest to us. Let G be a real reductive group which acts on a real vector space V , linearly and rationally, and let $G^{\mathbb{C}}$ act on the complexification $V^{\mathbb{C}} = V \otimes \mathbb{C}$. Given $v \in V \subset V^{\mathbb{C}}$ we know that $G^{\mathbb{C}} \cdot v$ is a finite union of G -orbits, see Section 1.3.

QUESTION 2.1. *Consider $G^{\mathbb{C}} \cdot v \cap V$ for $v \in V$. This is a finite union of G -orbits which are said to be conjugate to each other. What are the G -orbits that appear in this intersection? Can they be classified using semi-algebraic invariants?*

A simpler question would be

QUESTION 2.2. *How many different G -orbits appear in $G^{\mathbb{C}} \cdot v \cap V$?*

QUESTION 2.3. *Which different diffeomorphism classes of orbits appear in $G^{\mathbb{C}} \cdot v \cap V$?*

QUESTION 2.4. *Which real stabilizers in general position appear? Or what are the different ‘generic’ diffeomorphism classes of orbits?*

Recall that real stabilizers ‘in general position’ are only general in the Hausdorff sense and not in the Zariski sense like the s.g.p of a complex reductive group, see Section 1.3.1.

QUESTION 2.5. *When is a G -orbit closed? Are there good criteria to determine this? Are there any local criteria for determining closedness of an orbit?*

Recall that any orbit of an algebraic group is a locally closed submanifold. Thus the problem of an orbit being closed is a global problem; that is, we are asking about the embedding of the orbit as a submanifold. In

this light, a local criteria to determine closedness of an orbit would be a welcome achievement. See Theorem 3.11 for a partial result which is the real analogue of Mumford's Numerical Criteria.

1. Geometric Invariant Theory over \mathbb{C}

Our main reference for Geometric Invariant Theory (GIT) will be Newstead [New78]. We aim to give a brief introduction to some of the results of GIT and how they can be useful towards our study of closed orbits. In this section, G will refer to a linear algebraic group over \mathbb{C} and X will be an affine variety on which G acts.

DEFINITION 2.6. *A categorical quotient of X by G is a pair (Y, ϕ) where Y is a variety and $\phi : X \rightarrow Y$ is a morphism such that*

- (a) *ϕ is constant on the orbits of the action*
- (b) *for any variety Z and any morphism $\psi : X \rightarrow Z$ which is constant on orbits, there is a unique morphism $\chi : Y \rightarrow Z$ such that $\chi \circ \phi = \psi$*

If in addition $\phi^{-1}(y)$ consists of a single orbit for all $y \in Y$, then (Y, ϕ) is called an *orbit space*. We point out that categorical quotients are uniquely determined up to isomorphism. This follows from the universal property in the definition.

Given a variety X , recall that the ring of regular functions on X is denoted by $\mathbb{C}[X]$. We will denote the G -invariant functions by $\mathbb{C}[X]^G$. The following is [New78, Theorem 3.5].

THEOREM 2.7. *There exists an affine variety Y and a morphism $\phi : X \rightarrow Y$ such that*

- (a) *ϕ is G -invariant*
- (b) *ϕ is surjective*
- (c) *if U is open in Y , then $\phi^* : \mathbb{C}[U] \rightarrow \mathbb{C}[\phi^{-1}(U)]$ is an isomorphism of $\mathbb{C}[U]$ onto $\mathbb{C}[\phi^{-1}(U)]^G$*
- (d) *if W is a closed invariant subset of X , then $\phi(W)$ is closed*
- (e) *if W_1, W_2 are disjoint closed invariant subsets of X , then $\phi(W_1) \cap \phi(W_2) = \emptyset$.*

The variety Y above is often denoted by $X//G$. Moreover, the ring of regular functions is $\mathbb{C}[X//G] = \mathbb{C}[X]^G$. Conversely, $X//G$ can be defined to be the isomorphism class of varieties whose ring of regular functions is isomorphic to $\mathbb{C}[X]^G$. This quotient $X//G$ from GIT is a categorical quotient, see [New78] for details. The following is [New78, Proposition 3.8].

PROPOSITION 2.8. *Suppose $G \cdot v$ is a closed orbit of maximal dimension in a variety X . Then there exists a Zariski open set $\mathcal{O} \subset X$ such that $G \cdot w$ is closed for $w \in \mathcal{O}$.*

Remark. Really, this result could have been stated in both directions. That is, if there exists such an open set, then we can pick out a closed orbit of maximal dimension. The proposition gives a crude criterion

for determining whether or not we generically have closed orbits. We say crude because in practice it is not always easy to pick out a closed orbit of maximal dimension. Additionally, we show that this result holds for real algebraic groups acting on real varieties.

PROOF OF 2.8 FOR REAL ALGEBRAIC GROUPS. Let $X \subset V$ be a real variety contained in a real vector space V ; that is, X is the zero set of a collection of real polynomials on V . Let G be a real algebraic group acting on X ; that is, $G \subset GL(V)$ is an algebraic subgroup of $GL(V)$ that acts on the space X .

Consider the complex vector space $V^{\mathbb{C}} = V \otimes \mathbb{C}$ and the Zariski closure $X^{\mathbb{C}}$ of X in $V^{\mathbb{C}}$ (cf. Section 1.1). Let $G^{\mathbb{C}}$ denote the Zariski closure of G in $GL(V^{\mathbb{C}})$; recall that $G^{\mathbb{C}}$ is an algebraic group (cf. Section 1.2).

We claim that $G^{\mathbb{C}}$ acts on $X^{\mathbb{C}}$. To see this, we denote the action of the groups $G, G^{\mathbb{C}}$ by μ ; that is, $\mu : G^{\mathbb{C}} \times X^{\mathbb{C}} \rightarrow V^{\mathbb{C}}$. This map is a regular map; that is, the map is continuous in the Zariski topology. By hypothesis $\mu : G \times X \rightarrow X \subset V \subset V^{\mathbb{C}}$. The following is an easy exercise from point-set topology. Let $F : M \rightarrow N$ be a continuous map and $U \subset M$, then $F(\overline{U}) \subset \overline{F(U)}$, where \overline{U} denotes the closure of U . Applying this to μ we see that $\mu(G^{\mathbb{C}} \times X^{\mathbb{C}}) = \mu(\overline{G \times X}) \subset \overline{\mu(G \times X)} \subset \overline{X} = X^{\mathbb{C}}$. This shows that $G^{\mathbb{C}}$ acts on $X^{\mathbb{C}}$.

By hypothesis there exists some $x \in X$ such that $G \cdot x$ is closed in X and has maximal real dimension. Recall from Proposition 1.15 that $\dim_{\mathbb{R}} G \cdot y = \dim_{\mathbb{C}} G^{\mathbb{C}} \cdot y$ for all $y \in X \subset X^{\mathbb{C}}$; moreover, $G \cdot x$ is closed in X if and only if $G^{\mathbb{C}} \cdot x$ is closed in $X^{\mathbb{C}}$. Consider the Zariski open set $\mathcal{O} \subset X^{\mathbb{C}}$ which consists of points whose $G^{\mathbb{C}}$ -orbit has maximal dimension. Since X is Zariski dense in $X^{\mathbb{C}}$, the set $X \cap \mathcal{O}$ is nonempty and Zariski dense in X . Moreover, the point $x \in X \cap \mathcal{O}$ by the arguments stated in this paragraph.

We now have a point $x \in X \subset X^{\mathbb{C}}$ whose $G^{\mathbb{C}}$ -orbit is closed in $X^{\mathbb{C}}$ and has maximal complex dimension. The proposition being true over \mathbb{C} implies there exists a Zariski open set \mathcal{O}' of $X^{\mathbb{C}}$ consisting of points whose $G^{\mathbb{C}}$ -orbit is closed and of maximal complex dimension. Again using the arguments of the previous paragraph $X \cap \mathcal{O}'$ is a nonempty Zariski open set consisting of points whose G -orbit is closed and of maximal real dimension. \square

DEFINITION 2.9. *Let G be a reductive algebraic group which acts on V . We say that $v \in V$ is a G -stable point, or just stable point, if $G \cdot v$ is closed and G_v is discrete. We say that $v \in V$ is a semi-stable point if $0 \notin \overline{G \cdot v}$. We say that $v \in V$ is good semi-stable if $G \cdot v$ is closed. We say that $v \in V$ is unstable if $0 \in \overline{G \cdot v}$.*

The set of unstable points is called the null cone.

There exists a general criterion for finding stable points. It is called the Hilbert-Mumford Criterion. Let λ be an algebraic one parameter subgroup of G , or 1-PS for short. We know that λ is diagonalizable, see [Bor91, 4.6]. Consider the eigenspace decomposition $V = \oplus_i V_i$. On each V_i , λ acts by $\lambda(c) = c^{r_i}$ for $c \in \mathbb{C}^*$, where $r_i \in \mathbb{Z}$. Now we define $\mu(v, \lambda) = \min\{r_i | v_i \neq 0\}$, where $v = \sum_i v_i$, $v_i \in V_i$, cf. [New78, pg. 104].

DEFINITION 2.10. *We call a point $v \in V$ λ -stable if it is stable under the action of the group $\lambda(\mathbb{C}^*) \subset G$.*

One can show that if $\mu(v, \lambda) \geq 0$ then $\lambda(\mathbb{C}^*) \cdot v$ is not closed. However, if $\mu(v, \lambda) < 0$ and $\mu(v, \lambda^{-1}) < 0$ then $\lambda(\mathbb{C}^*) \cdot v$ will be closed. These two inequalities are equivalent to the point v being λ -stable. This result can be seen easily from the geometric approach to GIT, see below. Next we state the criterion for stability.

THEOREM 2.11 (Hilbert-Mumford Criterion). *Let G be a reductive algebraic group acting linearly on a complex vector space V . Then a point $v \in V$ is G -stable if and only if it is λ -stable for all 1-PS λ of G . The theorem remains true if we replace stable with semi-stable and relax our strict inequalities to just inequalities.*

We can rephrase this theorem using a numerical criterion that encodes the information from all the 1-PS simultaneously. Define $M(v) = \max_{\lambda \in 1-PS} \{\mu(v, \lambda)\}$.

THEOREM 2.12 (Hilbert-Mumford Numerical Criterion). *Let G be a reductive algebraic group acting linearly on V . Then*

- (a) $M(v) < 0$ if and only if v is stable
- (b) $M(v) = 0$ if and only if v is semi-stable
- (c) $M(v) > 0$ if and only if v is in the nullcone

This theorem gives a local criterion for determining when an orbit might be closed. That is, if $M < 0$ then the orbit is closed, if $M = 0$ then maybe, and if $M > 0$ then the orbit is not closed. In practice this numerical criterion is very powerful and useful for finding stable points. However, we are interested in the more general setting of finding generic closed orbits, or good semi-stable points.

In contrast to the stable situation, we have the following result from GIT.

THEOREM 2.13. *Let G be a complex reductive linear algebraic group acting linearly on V and take $v \in V$. If $G \cdot v$ is not a closed orbit, then there exists an algebraic 1-PS λ such that $\lambda \cdot v$ is not closed. Moreover, there exists $v_0 \in \overline{\lambda \cdot v}$ such that $G \cdot v_0$ is closed.*

This theorem is also true for real algebraic reductive groups with λ being a real algebraic 1-PS. The result over both \mathbb{R} and \mathbb{C} was proven by Birkes [Bir71].

2. GIT over \mathbb{R} and the Geometric Approach

The geometric (metric) approach to GIT was first done in the complex setting by Kempf and Ness. Here Hermitian inner products are put on a vector space in such a way that the group remains closed under the metric adjoint operation. If an orbit is closed, then one can move along the orbit and come to the point closest to the origin, such a point is called a *minimal vector*. The same ideas were introduced by Richardson-Slodowy in the real setting to talk about closed orbits of real reductive groups. This seems to be the more natural setting and we present their ideas below.

Let $G \subset GL_n(\mathbb{R})$ be a real semi-simple semi-algebraic group. Much of the geometry of G and its orbits can be studied via the complexification of our real objects. Let $G^{\mathbb{C}} \subset GL_n(\mathbb{C})$ denote the complexification

of G . Consider a rational representation $\rho : G \rightarrow GL(V)$. By definition we know that ρ is the restriction to G of some rational representation $\rho^{\mathbb{C}} : G^{\mathbb{C}} \rightarrow GL(V^{\mathbb{C}})$, where $V^{\mathbb{C}} = V \otimes \mathbb{C}$ is the complexification of V . Note: We will denote the induced Lie algebra representation by the same letter.

Cartan Involutions. Let E be a finite dimensional real vector space. A *Cartan involution* of $GL(E)$ is an involution of the form $\theta(g) = (g^t)^{-1}$, where g^t denotes the metric adjoint with respect to some inner product on E . At the Lie algebra level this involution is $\theta(X) = -X^t$.

PROPOSITION 2.14 (Mostow [Mos55]). *There exists a Cartan involution θ of $GL(E)$ such that $G^{\mathbb{C}}(\mathbb{R})$ is θ -stable.*

PROPOSITION 2.15 (Borel, Proposition 13.5 [BHC62]). *Let $\rho : G^{\mathbb{C}}(\mathbb{R}) \rightarrow GL(V)$ be a rational representation. Let θ be a Cartan involution of $GL(E)$ such that $G^{\mathbb{C}}(\mathbb{R})$ is θ -stable. Then there exists a Cartan involution θ_1 of $GL(V)$ such that $\rho \circ \theta = \theta_1 \circ \rho$.*

This proposition is extended in the next proposition which follows from sections 1 and 2 of [RS90].

PROPOSITION 2.16. *Let G be defined as above and $\rho : G \rightarrow GL(V)$ a rational representation, then*

- (a) *There exists a K -invariant inner product on V such that G is self-adjoint; hence, the Lie algebra $L(G) = \mathfrak{g}$ is also self-adjoint. Moreover, there exist Cartan involutions θ, θ_1 on $G, \rho(G)$, respectively, such that $\rho \circ \theta = \theta_1 \circ \rho$.*
- (b) *There exist decompositions of G and \mathfrak{g} , called Cartan decompositions, so that $G = KP$ as a product of manifolds and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here $K = \{g \in G \mid \theta(g) = g\}$ is a maximal compact subgroup of G , $\mathfrak{k} = L(K) = \{X \in \mathfrak{g} \mid \theta(X) = X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$, and $P = \exp(\mathfrak{p})$. Moreover, there exists an AdK -invariant inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on \mathfrak{g} so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is orthogonal.*
- (c) *Relative to the K -invariant inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on V , $\rho(X)$ is a symmetric transformation on V for $X \in \mathfrak{p}$, and $\rho(X)$ is a skew-symmetric transformation on V for $X \in \mathfrak{k}$.*

The subspaces \mathfrak{k} and \mathfrak{p} that arise in the Cartan decomposition above have the following set of relations

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$$

This is easy to see since \mathfrak{k} and \mathfrak{p} are the $+1, -1$ eigenspaces, respectively, of the Cartan involution θ . We point out that our $Ad K$ -invariant inner product on \mathfrak{g} restricts to such on \mathfrak{p} as the relations above show that \mathfrak{p} is $Ad K$ -invariant. Additionally, if the group G were semi-simple, then up to scaling the only choice for $\langle\langle \cdot, \cdot \rangle\rangle$ would be $-B(\theta(\cdot), \cdot)$ on each simple factor of \mathfrak{g} , where B is the Killing form of G .

Our Cartan involution θ on G is the restriction of a Cartan involution on $G^{\mathbb{C}}$, see [RS90, 2.8 and section 8] and [Mos55]. This gives Cartan decompositions $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u} \oplus \mathfrak{q}$ and $G^{\mathbb{C}} = U \cdot Q$, where U is a maximal compact subgroup of $G^{\mathbb{C}}$, $Q = \exp(\mathfrak{q})$, and $U \cap Q = \{1\}$.

We observe that the maximal compact groups U and K are related by $U = KU_0$. To see this, it suffices to prove $KU_0Q = UQ = G^{\mathbb{C}}$ since $KU_0 \subseteq U$ and $U \cap Q = \{1\}$. Since $U_0Q = \overline{H}_0$ and $P \subseteq Q$, we obtain $KU_0Q = KP \cdot \overline{H}_0 = G \cdot \overline{H}_0 = \overline{H}_0 \cdot G = G^{\mathbb{C}}$.

The subspaces $\mathfrak{u}, \mathfrak{q} \subseteq \mathfrak{g}^{\mathbb{C}}$ are related to $\mathfrak{k}, \mathfrak{p} \subseteq \mathfrak{g}$ as follows

$$\begin{aligned}\mathfrak{u} &= \mathfrak{k} \oplus i\mathfrak{p} \\ \mathfrak{q} &= i\mathfrak{k} \oplus \mathfrak{p}\end{aligned}$$

These two subspaces of $LG^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$ have a nice interpretation relative to a particular inner product on $V^{\mathbb{C}}$. Our construction of this inner product on $V^{\mathbb{C}}$ is similar to that done in sections 2 and 8 of [RS90]. We will be consistent with their notation.

PROPOSITION 2.17. *The K -invariant inner product \langle, \rangle on V , described in Proposition 2.16, extends to a U -invariant inner product S on $V^{\mathbb{C}}$ with a similar list of properties for $G^{\mathbb{C}}$. Additionally, the inner product \ll, \gg on \mathfrak{g} extends to an Ad U -invariant inner product \mathbb{S} on $\mathfrak{g}^{\mathbb{C}}$.*

PROOF. The proof of this fact follows the construction of S in the appendix A2 (proof of 2.9) in [RS90]. Define the inner product S on $V^{\mathbb{C}}$ as

$$S(v_1 + i v_2, w_1 + i w_2) = \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle$$

In this way, V and iV are orthogonal under S and multiplication by i acts as a skew-symmetric transformation on $V^{\mathbb{C}}$ relative to S . S is positive definite on $V^{\mathbb{C}}$.

Recall that $U = KU_0$ (see the remark above), and observe that S is K -invariant as K preserves V , iV and \langle, \rangle is K -invariant. Thus to show U -invariance, one just needs to show U_0 -invariance. This follows since $\rho(\mathfrak{u})$ acts skew-symmetrically and $U_0 = \exp(\mathfrak{u})$.

From the definitions of $\mathfrak{u}, \mathfrak{q}$ and S it follows that $\rho(\mathfrak{u})$ acts skew-symmetrically and $\rho(\mathfrak{q})$ acts symmetrically relative to S . Lastly, the extension of \ll, \gg on \mathfrak{g} to \mathbb{S} on $\mathfrak{g}^{\mathbb{C}}$ is a special case of the above work.

□

We say that the inner products S, \mathbb{S} on our complex spaces $V^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}$ are *compatible* with the inner products $\langle, \rangle, \ll, \gg$ on the underlying real spaces V, \mathfrak{g} , respectively. The inner product S constructed here gives rise to a U -invariant Hermitian form $H = S + iA$ on $V^{\mathbb{C}}$ where we define $A(x, y) = S(x, iy)$. This Hermitian form is compatible with the real structure V in the sense of Richardson and Slodowy, that is, $A = 0$ when restricted to $V \times V$; see sections 2 and 8 of [RS90].

Given the above decomposition of our real group G , one would like to understand how orbits tend not to be closed, in a more refined way. Let $G = KP$ be our Cartan decomposition. Clearly the K orbit of a point will always be closed, as it is compact. This suggests then that the way in which an orbit tends to not

be closed is very much related to P . The following is a refinement of Birkes's Theorem, cf. Theorem 2.13; see [RS90, Lemma 3.3] for the establishment of the following.

LEMMA 2.18. *Let $v \in V$ and assume that $G \cdot v$ is not closed. Then there exists $X \in \mathfrak{p}$ such that $\lim_{t \rightarrow \infty} \exp(tX) \cdot v = v_0$ exists and the orbit $G \cdot v_0$ is closed. Moreover, X is the tangent vector of an algebraic one-parameter multiplicative \mathbb{R} -subgroup of G .*

DEFINITION 2.19. *A vector $v \in V$ is a minimal vector for G if $\|v\| \leq \|g \cdot v\|$ for all $g \in G$. Let \mathfrak{M} denote the set of minimal vectors in V .*

If $G \cdot v$ is a closed orbit, then clearly it contains a minimal vector. However, the converse is also true. The following are Theorems 4.3 and 4.4 from [RS90].

THEOREM 2.20. *Let $v \in V$. The the following are equivalent:*

- (a) $v \in \mathfrak{M}$
- (b) *the function $F_v : G \rightarrow \mathbb{R}$, defined by $F_v(g) = \|g \cdot v\|^2$, has a critical point at $e \in G$*
- (c) $\langle X \cdot v, v \rangle = 0$ for all $X \in \mathfrak{p}$

If v satisfies any of the conditions above, then G_v is self-adjoint (i.e., θ -stable).

THEOREM 2.21. *Let $v \in V$. Then the following are equivalent:*

- (a) *the orbit $G \cdot v$ is closed*
- (b) *$G \cdot v$ intersects \mathfrak{M}*

If v satisfies any of the conditions above, then $G \cdot v \cap \mathfrak{M}$ is a single K -orbit.

These theorems demonstrate the value of an inner product on V under which G is closed under the metric adjoint. Moreover, it gives a way of determining whether or not a particular G -orbit is closed, i.e., we can try to check to see if $G \cdot v$ contains a minimal vector. In light of the theorem above, we are looking for vectors that satisfy $\langle X \cdot v, v \rangle = 0$, for $X \in \mathfrak{p}$. Equivalently, we could define the following function and look for its zeros.

DEFINITION 2.22. *The moment map $\tilde{m} : V \rightarrow \mathfrak{p}$ is defined by $\langle \tilde{m}(v), X \rangle = \langle X \cdot v, v \rangle$.*

Rephrasing the above results in terms of this function we have:

COROLLARY 2.23. *The set of minimal vectors is $\mathfrak{M} = \tilde{m}^{-1}(0)$.*

Determining whether or not an orbit is closed is a global property of the orbit. Trying to determine closedness at a point on an orbit is hard if we are not at a minimal vector. In Section 3.1 we obtain a criteria which is local in nature, that is, uses only information about the point we are at, to determine closedness of the orbit.

3. Moment Maps

We recall the definition of the real moment map for the action of G on V . The motivation for these definitions comes from symplectic geometry and the actions of compact groups on compact symplectic manifolds. In the complex setting, this moment map coincides with the one from the symplectic structure on $\mathbb{CP}(V^\mathbb{C})$. For more information see [NM84] and [GS82].

Real moment maps. Given $G \curvearrowright V$ we define $\tilde{m} : V \rightarrow \mathfrak{p}$ implicitly by

$$\ll \tilde{m}(v), X \gg = \langle Xv, v \rangle$$

for all $X \in \mathfrak{p}$. Notice that $\tilde{m}(v)$ is a real homogeneous polynomial of degree 2. Equivalently, we really could define $\tilde{m} : V \rightarrow \mathfrak{g}$; then using K -invariance and $\mathfrak{k} \perp \mathfrak{p}$ we obtain $\tilde{m}(V) \subseteq \mathfrak{p}$.

We can just as well do this for $G^\mathbb{C} \curvearrowright V^\mathbb{C}$ where we regard $G^\mathbb{C}$ as a real Lie group. We use the inner products S on $V^\mathbb{C}$ and \mathbb{S} on $\mathfrak{g}^\mathbb{C}$. The (real) moment map for $G^\mathbb{C} \curvearrowright V^\mathbb{C}$, denoted by $\tilde{n} : V^\mathbb{C} \rightarrow \mathfrak{q}$, is defined by

$$\mathbb{S}(\tilde{n}(v), Y) = S(Yv, v)$$

for $Y \in \mathfrak{q}$ and $v \in V^\mathbb{C}$.

Since the polynomials \tilde{m}, \tilde{n} are homogeneous of degree 2, they give rise to well defined maps on (real) projective space. Define

$$\begin{aligned} m : \mathbb{P}V &\rightarrow \mathfrak{p} & n : \mathbb{RP}V^\mathbb{C} &\rightarrow \mathfrak{q} \\ m[v] &= \tilde{m}\left(\frac{v}{|v|}\right) = \frac{\tilde{m}(v)}{|v|^2} & n[w] &= \tilde{n}\left(\frac{w}{|w|}\right) = \frac{\tilde{n}(w)}{|w|^2} \end{aligned}$$

where $|w|^2 = S(w, w)$ and $S = \langle, \rangle$ on V . Since $V \subseteq V^\mathbb{C}$ we have $\mathbb{P}V \subseteq \mathbb{RP}V^\mathbb{C}$; this is our main reason for studying the real moment map on $G^\mathbb{C}$. The next lemma compares these two real moment maps.

LEMMA 2.24. *n restricted to $\mathbb{P}V$ equals m .*

PROOF. Recall that n takes values in $\mathfrak{q} = i\mathfrak{k} \oplus \mathfrak{p}$ and m takes values in $\mathfrak{p} \subseteq \mathfrak{q}$. Take $v \in V$ and $X \in \mathfrak{k}$ then

$$\mathbb{S}(\tilde{n}(v), iX) = S(iX \cdot v, v) = 0$$

as $V \perp iV$ (see Proposition 2.17), and we are using $(iX) \cdot v = i(X \cdot v)$, i.e., $\mathfrak{g}^\mathbb{C}$ acts \mathbb{C} -linearly on $V^\mathbb{C}$. Since $\mathfrak{g} \perp i\mathfrak{g}$ under \mathbb{S} , we have $i\mathfrak{k} \perp \mathfrak{p}$. Thus $\tilde{n}(v) \in \mathfrak{p} \subseteq \mathfrak{q}$. Now take $X \in \mathfrak{p}$.

$$\begin{aligned} \mathbb{S}(\tilde{n}(v), X) &= \ll \tilde{n}(v), X \gg && \text{by compatibility of } \mathfrak{g} \subseteq \mathfrak{g}^\mathbb{C} \\ &\parallel \\ S(Xv, v) &= \langle Xv, v \rangle && \text{by compatibility of } V \subseteq V^\mathbb{C} \\ &= \ll \tilde{m}(v), X \gg && \text{by definition/construction of } \tilde{m} \end{aligned}$$

Therefore, $\tilde{n}(v) = \tilde{m}(v)$ for $v \in V \subseteq V^\mathbb{C}$, which implies $n[v] = m[v]$ for $[v] \in \mathbb{P}V \subseteq \mathbb{RP}V^\mathbb{C}$. □

Complex moment maps. We choose a notation that is similar to Ness [NM84] as we are following her definitions; the only difference is that we use μ where she uses m . For $v \in V^\mathbb{C}$, consider $\rho_v : G^\mathbb{C} \rightarrow \mathbb{R}$ defined by $\rho_v(g) = |g \cdot v|^2$, where $|w|^2 = H(w, w) = S(w, w)$. Define a map $\mu : \mathbb{CP}(V^\mathbb{C}) \rightarrow \mathfrak{q}^* = \text{Hom}(\mathfrak{q}, \mathbb{R})$ by $\mu(x) = \frac{d\rho_v(e)}{|v|^2}$, where $v \in V^\mathbb{C}$ sits over $x \in \mathbb{CP}(V^\mathbb{C})$, cf. [NM84, section 1]. We define the complex moment map $\mu^* : \mathbb{CP}(V^\mathbb{C}) \rightarrow \mathfrak{q}$ by $\mu = \mathbb{S}(\mu^*, \cdot)$. Note, taking the norm square of our complex moment map will give us the norm square of the moment map in Kirwan's setting; in Kirwan's language $i\mu$ would be the moment map [NM84, section 1].

Let π denote the projection $\pi : \mathbb{RP}V^\mathbb{C} \rightarrow \mathbb{CP}(V^\mathbb{C})$.

LEMMA 2.25. *The complex and real moment maps for $G^\mathbb{C}$ are related by $\mu^* \circ \pi = 2n$*

PROOF. Many of our computations have the same flavor as those of Ness. We employ her ideas for the reals. Take an orthonormal basis $\{\alpha_i\}$ of $i\mathfrak{u} = \mathfrak{q}$ under \mathbb{S} . Also let $x = \pi[v] \in \mathbb{CP}(V^\mathbb{C})$ for $v \in V^\mathbb{C}$. Then

$$\begin{aligned} \mu^*(x) &= \sum_i \mathbb{S}(\mu^*(x), \alpha_i) \alpha_i \\ &= \sum_i [\mu(x) \alpha_i] \alpha_i \\ &= \sum_i \frac{1}{\|v\|^2} d\rho_v(e)(\alpha_i) \alpha_i \\ &= \sum_i \frac{1}{\|v\|^2} \left. \frac{d}{dt} \right|_{t=0} \|\exp t\alpha_i \cdot v\|^2 \alpha_i \end{aligned}$$

Here the norm on $V^\mathbb{C}$ is from $H = S + iA$. But S is the inner product being used on $V^\mathbb{C}$, and so $H(w, w) = S(w, w)$ tells us that $\mu^*(x)$

$$\begin{aligned} &= \sum_i \frac{1}{\|v\|^2} 2S(\alpha_i v, v) \alpha_i \\ &= \sum_i 2 \mathbb{S}(\tilde{n}[v], \alpha_i) \alpha_i \\ &= 2\tilde{n}[v] \end{aligned}$$

□

Remark. Since $\mathbb{P}V$ is not a subspace of $\mathbb{CP}(V^\mathbb{C})$, we use $\mathbb{RP}V^\mathbb{C}$ and the real moment map of $G^\mathbb{C}$ to work between the known results of Kirwan and Ness to get information about our real group $G \curvearrowright \mathbb{P}V$.

Examples of Moment Maps. Let G_1, G_2 be real reductive groups with the Cartan decompositions $G_i = K_i P_i$ and $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$, for $i = 1, 2$. Then the group $G = G_1 \times G_2$ has Cartan decompositions $G = KP$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $K = K_1 \times K_2$ and $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

PROPOSITION 2.26. *Let $G = G_1 \times G_2$ act on V , and let m, m_1, m_2 be the moment maps for G, G_1, G_2 , respectively. Then $m = m_1 + m_2$.*

This follows from the definitions and the fact that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

Let $V = \mathfrak{so}(q, \mathbb{R})^p := \mathfrak{so}(q, \mathbb{R}) \oplus \cdots \oplus \mathfrak{so}(q, \mathbb{R})$, p times. Let $G = GL(q, \mathbb{R})$ act on V diagonally by $g \cdot (C^1, \dots, C^p) = (gC^1g^t, \dots, gC^pg^t)$. The Lie algebra \mathfrak{g} acts diagonally by $X \cdot C^i = XC^i + C^iX^t$.

On V we define the inner product $\langle (C^1, \dots, C^p), (D^1, \dots, D^p) \rangle = -\sum \text{trace} C^i D^i$. This is the canonical extension of the canonical inner product on $\mathfrak{so}(q, \mathbb{R})$. Under this inner product, G is self-adjoint and the metric adjoint corresponds to the usual transpose. Hence $\mathfrak{k} = \mathfrak{so}(q, \mathbb{R})$ and $\mathfrak{p} = \{X \in \mathfrak{g} : X = X^t\}$.

EXAMPLE 2.27. Consider the action of $G = GL(q, \mathbb{R})$ on V . For $C = (C^1, \dots, C^p) \in V$, $m_G(C) = -2 \sum_{i=1}^p (C^i)^2$.

For the action of $H = SL(q, \mathbb{R})$ on V and $C \in V$, we have $m_H(C) = m_G(C) - \lambda(C)I_q = -2 \sum_{i=1}^p (C^i)^2 - \lambda(C)I_q$, where $\lambda(C) = \frac{2|C|^2}{q}$.

We show this for the action of $GL(q, \mathbb{R})$ first. Let $X \in \mathfrak{p}$ and $C \in V$ be given. For $\xi, \eta \in \mathfrak{g}$, we use the inner product $\langle \xi, \eta \rangle = \text{trace}(\xi\eta^t)$ on $\mathfrak{g} \subset M(q, \mathbb{R})$ and hence on \mathfrak{p} . Then for $X \in \mathfrak{p}$ we have $\langle m(C), X \rangle = \langle X(C), C \rangle = -\sum \text{trace}(XC^i + C^iX)(C^i) = -2 \sum \text{trace} X(C^i)^2 = \langle X, -2 \sum (C^i)^2 \rangle$.

To obtain the result for $SL(q, \mathbb{R})$, observe that the set \mathfrak{p} consists of traceless symmetric $q \times q$ matrices. As $-2 \sum_{i=1}^p (C^i)^2 - \lambda(C)I_q$ is traceless, the result follows from the work above.

Now consider $V = \mathfrak{so}(q, \mathbb{R})^p$ and observe that V is isomorphic to $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ via the map $C = (C^1, \dots, C^p) \mapsto \sum C^i \otimes e_i$, where $\{e_i\}$ is the standard basis of \mathbb{R}^p . Let $G = G_1 \times G_2$ where $G_1 = GL(q, \mathbb{R})$ and $G_2 = GL_p(\mathbb{R})$, then G acts on V in the usual way; that is, for $g = (g_1, g_2) \in G$ and $C = \sum_1^p C^i \otimes e_i$ we have

$$g \cdot C = (g_1, g_2) \cdot \sum_1^p C^i \otimes e_i = \sum_1^p (g_1 C^i g_1^t) \otimes g_2(e_i)$$

Here G_2 acts on \mathbb{R}^p in the standard fashion. Note, this action gives an action of $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ on V . The previously used inner product on V now becomes the unique inner product on $V = \mathfrak{so}(q) \otimes \mathbb{R}^p$ such that $\langle C \otimes v, D \otimes w \rangle = \langle C, D \rangle \langle v, w \rangle$ for $C, D \in \mathfrak{so}(q, \mathbb{R})$ and $v, w \in \mathbb{R}^p$, where $\langle C, D \rangle = -\text{trace}(CD)$ and \langle, \rangle is the standard inner product on \mathbb{R}^p for which the standard basis $\{e_i\}$ is orthonormal.

Observe that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ and the moment map $m : V \rightarrow \mathfrak{p}$ becomes $m(C) = (m_1(C), m_2(C))$, where $m_i : V \rightarrow \mathfrak{p}_i$ is the moment map for the action of G_i .

EXAMPLE 2.28. Consider the action of $G = GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ acting on $V = \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ defined above. Then the moment map is given by $m_G(C) = (m_1(C), m_2(C))$ where $m_1(C) = -2 \sum (C^i)^2$ as above and $m_2(C)$ is defined component wise as $m_2(C)_{ij} = \langle C^i, C^j \rangle$. We show that m_2 is the moment map for the action of $GL(p, \mathbb{R})$ on $\mathfrak{so} \otimes \mathbb{R}^p$.

For the action of $H = SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ we have $m_H(C) = (m_1(C) - \lambda(C)I_q, m_2(C) - \mu(C)I_p)$ where m_1, m_2 are defined as above, $\lambda(C) = \frac{2|C|^2}{q}$ and $\mu(C) = \frac{|C|^2}{p}$.

To see that the moment map is as described, by Proposition 2.26 we just need to check the action of $G_2 = GL(p, \mathbb{R})$. If $Y \in \mathfrak{p}_2$ and $C = \sum C^i \otimes e_i \in V$ are given, then $\langle Y(C), C \rangle = \langle \sum C^i \otimes Y(e_i), \sum C^j \otimes e_j \rangle = \sum \langle C^i, C^j \rangle \langle Y(e_i), e_j \rangle = \text{trace } m_2(C)Y = \langle m_2(C), Y \rangle$. Hence m_2 is the moment map for the action of $GL(p, \mathbb{R})$.

The result for H holds for the same reasons as in the previous example; that is, $m_1(C) - \lambda(C)$ and $m_2(C) - \mu(C)$ are traceless.

EXAMPLE 2.29. Let $V = M_n(\mathbb{R})$ denote the $n \times n$ matrices and let $SL_n(\mathbb{R})$ act by conjugation. This is the adjoint action of $GL_n(\mathbb{R})$ acting on its Lie algebra. Given the usual inner products, from the trace form, for $C \in V$ the moment map is $m(C) = CC^t - C^tC$.

Observe that the Lie algebra \mathfrak{g} acts on V by $X(C) = XC - CX$ for $X \in \mathfrak{p}$ and $C \in V$. We compute $\langle m(C), X \rangle = \langle X(C), C \rangle = \text{trace}(XC - CX)C^t = \text{trace}X(CC^t - C^tC) = \langle X, CC^t - C^tC \rangle$. The assertion follows as $CC^t - C^tC$ is symmetric, traceless and hence belongs to \mathfrak{p} .

4. Comparison of Real and Complex Cases

Most of algebraic geometry and Geometric Invariant Theory has been worked out exclusively for fields which are algebraically closed. We are interested in the real category and will exploit all the work that has already been done over \mathbb{C} . We use and refer the reader to [Whi57] as our main reference for real algebraic varieties.

Recall that our representation $\rho : G \rightarrow GL(V)$ is the restriction of a representation of $G^\mathbb{C}$. The following is proposition 2.3 of [BHC62] and section 8 of [RS90]. Originally this was stated as a comparison between $G^\mathbb{C}(\mathbb{R})_0$ -orbits and $G^\mathbb{C}$ -orbits, however, it can be restated as a comparison between G and $G^\mathbb{C}$ orbits, for any G satisfying $G^\mathbb{C}(\mathbb{R})_0 \subseteq G \subseteq G^\mathbb{C}(\mathbb{R})$. This is true as $G^\mathbb{C}(\mathbb{R})_0$ has finite index in G . For more details see Proposition 1.15. (Even though the contents of the following theorem are contained in the aforementioned proposition, the theorem, as stated, is referenced later in the text.)

THEOREM 2.30. Let $v \in V$, then $G^\mathbb{C} \cdot v \cap V = \bigcup_{i=1}^m X_i$ where each X_i is a G -orbit. Moreover, $G^\mathbb{C} \cdot v$ is closed in $V^\mathbb{C}$ if and only if $G \cdot v$ is closed in V .

EXAMPLE 2.31. Consider the adjoint action of $SL_2(\mathbb{R})$ on $\mathfrak{sl}_2(\mathbb{R})$. The points $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ lie on different $SL_2(\mathbb{R})$ -orbits but lie on the same $SL_2(\mathbb{C})$ -orbit.

Remark. We thank Dima Arynkin for pointing out this example to us. Additionally, we observe that there are two Hausdorff open sets $\det > 0$ and $\det < 0$ which are $SL_2(\mathbb{R})$ invariant and which are not connected via $SL_2(\mathbb{C})$ -orbits. This is a very interesting phenomenon as it is well-known that generic points from these two open sets have diffeomorphic $SL_2(\mathbb{C})$ -orbits, cf. Section 1.3 and the stabilizer in general position.

PROOF. First we show that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ lie on different $SL_2(\mathbb{R})$ -orbits but lie on the same $SL_2(\mathbb{C})$ -orbit. Observe that these vectors are both minimal for the action of $G = SL_2(\mathbb{R})$, cf. Example 2.29. If they were to lie on the same $SL_2(\mathbb{R})$ orbit, then they would hence lie on the same $K = SO(2)$ orbit by Theorem 2.21. However, the elements of K commute with both vectors. Thus K fixes both vectors and they cannot lie on the same K orbit. Hence they cannot lie on the same G orbit.

To see that these vectors lie on the same $SL_2(\mathbb{C})$ one uses the element $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ to conjugate between them. Lastly, we observe that $SL_2(\mathbb{C})$ preserves the determinant function, and hence this group does not connect the disjoint open sets $\det > 0$ and $\det < 0$ of $\mathfrak{sl}_2(\mathbb{R})$. \square

EXAMPLE 2.32 (Sylvester's Theorem). *Let $G = GL_k(\mathbb{R})$ act on $V = \text{Sym}_k(\mathbb{R})$, the symmetric $k \times k$ matrices, via $g \cdot M = gMg^t$.*

It is well-known (Sylvester's Theorem) that for $M \in V$ there exists $g \in G$ such that gMg^t is a diagonal matrix consisting of 1's, 0's, and -1's along the diagonal. This form is unique up to reordering. If p, n, z are the number of positive, negative, and zero eigenvalues of $M \in V$, then p, n, z are constant along G -orbits and equal the number of 1's, -1's, and 0's in the diagonal, respectively. However, over the complex numbers given $M \in V^{\mathbb{C}}$ there exists $g \in GL_k(\mathbb{C})$ such that gMg^t is diagonal with 1's and 0's along the diagonal; here the number of 1's is $p + n$ and the number of 0's is z . Hence all $M \in V^{\mathbb{C}}$ with the same number of nonzero eigenvalues lie on the same $GL_k(\mathbb{C})$ orbit.

If we choose a generic matrix M , which is nonsingular, then there are exactly k real orbits that comprise $GL_k(\mathbb{C}) \cdot M \cap V$. These are the real matrices with p positive and $k - p$ negative eigenvalues for $1 \leq p \leq k$.

Orbits in Projective space. Since our groups act linearly on vectors spaces we can consider the induced actions on projective space $G \curvearrowright \mathbb{P}V$ and $G^{\mathbb{C}} \curvearrowright \mathbb{R}\mathbb{P}V^{\mathbb{C}}$. The next result extends Proposition 1.15 (f).

LEMMA 2.33. *For $v \in V$, $G^{\mathbb{C}} \cdot [v] \cap \overline{G \cdot [v]} = G \cdot [v]$ in $\mathbb{R}\mathbb{P}V^{\mathbb{C}}$.*

PROOF. The actions of $\mathbb{R}^* \times G$ and G on $\mathbb{P}V$ are the same; moreover, $(\mathbb{R}^* \times G)^{\mathbb{C}} = \mathbb{C}^* \times G^{\mathbb{C}}$. Given $v \in V$ take $g_n \in G$ and $g \in G^{\mathbb{C}}$ such that $[g_nv] \rightarrow [gv]$ in $\mathbb{P}V$. Then we want to show $[gv] \in G \cdot [v]$. Now take $r_n, r \in \mathbb{R}$ such that r_ng_nv, rgv have unit length in $V^{\mathbb{C}}$. We can assume $r_ng_nv \rightarrow rgv$ by passing to $-r$ and a subsequence if necessary. Then $r_ng_nv \rightarrow rgv \in \mathbb{C}^* \times G^{\mathbb{C}} \cdot v \cap \overline{\mathbb{R}^* \times G \cdot v}$. Therefore, $rgv \in \mathbb{R}^* \times G \cdot v$ using Proposition 1.15(f) and our result follows. \square

CHAPTER 3

M-function and Stability of Representations

We begin this chapter by recalling the classical and well-known theorems for determining when one has generically closed orbits. Then we present the real M -function, show how this is a generalization of the Hilbert-Mumford criteria to real groups, and we show consistency for complex groups. That is, our new criteria when applied to complex semi-simple groups gives precisely the Hilbert-Mumford criteria. Much of the work in this section is joint with P. Eberlein [EJ].

DEFINITION 3.1. *Let G be a real, resp. complex, semi-simple algebraic group and $\rho : G \rightarrow GL(V)$ a linear rational representation where V is a real, resp. complex, vector space. We say that ρ is a stable, semi-stable, or good semi-stable representation if it contains such a point, see Definitions 2.9 and 3.2.*

Often we will simply say that a representation is ‘good’ if it is either stable or good semi-stable; that is, if it contains a Zariski open set of closed orbits.

Remark. For complex groups, if there is one point which is stable, semi-stable, or good semi-stable, then there exists a Zariski open set of such of points (see [New78, page 74]). For real groups this is true for semi-stable and good semi-stable (cf. Proposition 1.15 and [New78, page 74]). We observe that if there exist stable points in the complex representation, then there exist real points which are (complex) stable, in fact there exists a Zariski open set of such points in V . However, for *real* stable points we can only guarantee the existence of a Hausdorff open set of such points; in general this Hausdorff open set is not Zariski open, cf. Example 3.20.

DEFINITION 3.2. *We say that a vector $v \in V$ or its orbit $G \cdot v$ is (real) stable if $G \cdot v$ is closed and the isotropy subgroup G_v is compact.*

In the sequel stable will always denote real stable unless explicitly stated otherwise.

Remark. From this definition if a representation $\rho : G \rightarrow GL(V)$ is real stable, then the complex representation $\rho : G^{\mathbb{C}} \rightarrow GL(V^{\mathbb{C}})$ won’t necessarily be complex stable. However, the complex representation will be good semi-stable. Moreover, we choose this terminology as it is consistent with the original definition of stable if we regard a complex group $G^{\mathbb{C}}$ as a real Lie group. For if G is a complex semi-simple group acting on a complex vector space V , $v \in V$ is said to be stable if $G \cdot v$ is closed and G_v is discrete. Since G_v is an affine algebraic group, compact is equivalent to discrete (for complex algebraic groups).

We have two goals. The first is to determine when an orbit $G \cdot v$ is closed. We would especially like to do this locally, that is, with just information about v . The second goal is to determine when we generically have closed orbits. We have already seen that $G \cdot v$ is closed if and only if $G^{\mathbb{C}} \cdot v$ is closed. In this way, we can study complex groups to help work on the first goal. Additionally, since V is Zariski dense in $V^{\mathbb{C}}$, any generic results obtained for $G^{\mathbb{C}}$ acting on $V^{\mathbb{C}}$ immediately translate back to results for G acting on V . So again we can use complex groups to help work on the second goal.

THEOREM 3.3. *Let G be a real semi-simple group acting almost effectively on V ; that is, $\rho : G \rightarrow GL(V)$ has discrete kernel. If all the orbits are closed, then G is compact.*

PROOF. Let G be noncompact. It suffices to show that $0 \in \overline{Gv}$ for some $v \neq 0$. The condition that G act almost effectively is equivalent to the condition that the Lie algebra act effectively; that is, if $X \cdot v = 0$ for $X \in \mathfrak{g}$ and all $v \neq 0$, then $X = 0$. Consider the Cartan decomposition $G = KP$. Since G acts almost effectively, given $X \in \mathfrak{p} \setminus \{0\}$ there exists $v \in V$ such that $Xv \neq 0$. Since X is symmetric, $v = \sum_{\lambda \in \Lambda(X)} v_{\lambda}$, where $\Lambda(X)$ is the set of eigenvalues of X and v_{λ} is the component of v in the λ -eigenspace of X . Pick $\lambda \neq 0$ such that $v_{\lambda} \neq 0$. Then $\exp(tX) \cdot v_{\lambda} = e^{t\lambda} v_{\lambda}$ and letting $t \rightarrow \pm\infty$ we see that v_{λ} is in the null cone. \square

Remark. The condition that G act almost effectively is very natural. For if G did not act effectively, then we could consider the normal subgroup N which acts trivially. Then G and G/N (with the induced action) have the same orbit structure. Note that this theorem is very special to semi-simple groups as it is well-known that for any representation of a unipotent group all orbits are closed, see [PV94].

Let G be a reductive group. It is well-known that if $G \cdot v$ is closed, then the stabilizer G_v is reductive (see [BHC62, Theorem 3.4]). For example, if $w \in G \cdot v$ is minimal, then G_w is self-adjoint by Theorem 2.20 and hence reductive. If $w' \in G \cdot v$, then $G_{w'}$ is conjugate to G_w . The converse is almost true and is a theorem of V.L. Popov [Pop70]. This gives a good criterion in both the real and complex cases. Next we state this general criterion for determining when a generic orbit is closed. Although Popov stated the theorem over \mathbb{C} , it is obviously true for real groups by complexifying all of our objects.

THEOREM 3.4 (Popov). *Let G be a semi-simple algebraic group acting (algebraically) on V . Then generic orbits are closed if and only if the stabilizer in general position is reductive.*

Remark. Popov's proof uses algebraic geometry to obtain his results. We are interested in finding more analytic proofs of these known results. At the moment we do not have a full proof of his result that avoids algebraic geometry. However, our work on the real M -function obtains a criterion for generically closed orbits without using high powered algebraic geometry (see Theorem 3.11).

1. M-function

Until otherwise stated we let G be a real semi-simple algebraic group acting on a real vector space V . Let $\langle \cdot, \cdot \rangle$ be an inner product for which G is self-adjoint and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition compatible

with $\langle \cdot, \cdot \rangle$. Let V_0 and \mathfrak{p}_0 denote the non-zero vectors in V and \mathfrak{p} , respectively. Take $X \in \mathfrak{p}_0$ and let Λ_X denote the set of eigenvalues of X ; for $\mu \in \Lambda_X$ let $V_{\mu, X}$ denote the μ -eigenspace in V . For $v \in V_0$ and $X \in \mathfrak{p}_0$ let $\mu(X, v)$ denote the smallest eigenvalue μ such that the component of v in $V_{\mu, X}$ is non-zero.

The function $\mu : \mathfrak{p}_0 \times V_0 \rightarrow \mathbb{R}$ captures some of the action of G on V and we record two basic properties. The first result is on non-negativity of the μ -function and the second is on semi-continuity.

PROPOSITION 3.5. *Let $(X, v) \in \mathfrak{p}_0 \times V_0$.*

- (a) $\mu(X, v) = 0$ *if and only if the following hold*
 - (i) *The component v_0 of v in $\text{Ker } X$ is nonzero*
 - (ii) *$\exp(tX)v \rightarrow v_0$ as $t \rightarrow -\infty$*
- (b) $\mu(X, v) > 0$ *if and only if $\exp(tX)v \rightarrow 0$ as $t \rightarrow -\infty$. That is, v is in the null cone.*

PROOF. We prove both results simultaneously. Let Λ'_X denote the set of nonzero eigenvalues of X . For $v \in V$ we have $v = v_0 + \sum_{\lambda \in \Lambda'_X} v_\lambda$. By inspection $\mu(X, v) \geq 0$ if and only if $\lambda > 0$ for $v_\lambda \neq 0$ and $\mu(X, v) > 0$ if and only if $v_0 = 0$ and $\lambda > 0$ for $v_\lambda \neq 0$. The assertions of the proposition follow immediately since $\exp(tX)v = v_0 + \sum_{\lambda \in \Lambda'_X} e^{t\lambda} v_\lambda$. \square

PROPOSITION 3.6. *Let $(X, v) \in \mathfrak{p}_0 \times V_0$. Given $\epsilon > 0$ there exist neighborhoods $U \subset V$ and $O \subset \mathfrak{p}$ such that $\mu(X', v') < \mu(X, v) + \epsilon$, for $(X', v') \in U \times O$.*

PROOF. Suppose the proposition is false. Then there would exist an $\epsilon > 0$ and a sequence $(X_n, v_n) \rightarrow (X, v)$ such that $\mu(X_n, v_n) \geq \mu(X, v) + \epsilon$ for all n . By passing to a subsequence we can assume there exists an integer N with the following properties:

- (a) For every n , X_n has N distinct eigenvalues $\{\lambda_1^{(n)}, \dots, \lambda_N^{(n)}\}$ and there exist orthogonal subspaces $\{V_k^{(n)}\}$ such that $V = V_1^{(n)} \oplus \dots \oplus V_N^{(n)}$ with $X_n = \lambda_i^{(n)} \text{Id}$ on $V_i^{(n)}$.
- (b) There exist subspaces V_1, \dots, V_N of V and real numbers $\lambda_1, \dots, \lambda_N$ such that $\lambda_i^{(n)} \rightarrow \lambda_i$ and $V_i^{(n)} \rightarrow V_i$. Since $X_n \rightarrow X$ we see that $V = V_1 \oplus \dots \oplus V_N$ and $X = \lambda_i \text{Id}$ on V_i . Note, these λ_i are eigenvalues of X and might not be distinct.

Choose k so that $\mu(X, v) = \lambda_k$. Then v has a nonzero component in V_k . Thus there is some N_0 such that v_n has a nonzero component in $V_k^{(n)}$ for all $n \geq N_0$. Now we have $\lambda_k^{(n)} \geq \mu(X_n, v_n) \geq \mu(X, v) + \epsilon$ and since $\lambda_k^{(n)} \rightarrow \lambda_k$ we obtain the contradiction $\mu(X, v) = \lambda_k \geq \mu(X, v) + \epsilon$. \square

DEFINITION 3.7. *The function $M : V \rightarrow \mathbb{R}$ is defined by $M(v) = \max\{\mu(X, v) : X \in \mathfrak{p} \text{ \& } |X| = 1\}$*

This function has been considered by A. Marian [Mar01] in this context. A priori one can only define the M function as a supremum, however, Marian has shown that it is a maximum over the unit sphere in \mathfrak{p} . We present some of her results on the basic nature of M .

PROPOSITION 3.8. *The M -function has the following properties:*

- (a) M is constant on G -orbits
- (b) M takes finitely many values
- (c) Let K be a maximal compact subgroup of G with Lie algebra \mathfrak{k} . Let \mathfrak{A} be a maximal abelian subalgebra of \mathfrak{p} . Define $M^{\mathfrak{A}} : V \rightarrow \mathbb{R}$ by $M^{\mathfrak{A}}(v) = \sup\{\mu(X, v) : X \in \mathfrak{A} \text{ \& } |X| = 1\}$. Then $M(v) = \max\{M^{\mathfrak{A}}(kv) : k \in K\}$.

In [EJ] we have refined the study of the M -function and obtained many more useful results concerning the orbit structure of representations of real semi-simple groups. Most importantly, we have derived a new and useful criterion for determining closedness of an orbit using local information.

PROPOSITION 3.9. *Let $v \in V$. There exists an open neighborhood \mathcal{O} of v such that $M(w) \leq M(v)$ for $w \in \mathcal{O}$.*

PROOF. Suppose the proposition is false for some non-zero $v \in V$. Then there exists a sequence $\{v_n\} \subset V$ such that $v_n \rightarrow v$ as $n \rightarrow \infty$ and $M(v_n) > M(v)$ for all n . Since M has only finitely many values we may assume, by passing to a subsequence, that $M(v_n) = c > M(v)$ for some real number c and all n . Choose unit vectors $X_n \in \mathfrak{p}$ such that $c = M(v_n) = \mu(X_n, v_n)$ for all n . By passing to a subsequence we may assume that $X_n \rightarrow X$, a unit vector in \mathfrak{p} , as $n \rightarrow \infty$.

Choose $\epsilon > 0$ such that $c > M(v) + \epsilon$. By Proposition 3.6 there exists N such that $\mu(X_n, v_n) < \mu(X, v) + \epsilon$ for $n \geq N$. Hence $c = M(v_n) = \mu(X_n, v_n) < \mu(X, v) + \epsilon \leq M(v) + \epsilon < c$, which is a contradiction. \square

COROLLARY 3.10. *Let $v \in V$ be such that $M(v) < 0$. Then there is an open neighborhood \mathcal{O} such that $M(w) < 0$ for $w \in \mathcal{O}$.*

The Geometric Significance of $M < 0$.

THEOREM 3.11. *The following conditions are equivalent for a nonzero vector $v \in V$:*

- (a) $M(v) < 0$
- (b) v is stable; that is, the orbit $G \cdot v$ is closed and G_v is compact
- (c) The map $F_v : G \rightarrow [0, \infty)$ is proper, where $F_v(g) = |g(v)|^2$.

Remark. We observe that $M(v) < 0$ if and only if $\mu(X, v) < 0$ for all nonzero $X \in \mathfrak{p}$. This is useful in practice when working with M .

Before proving the theorem, we state a very useful proposition that is interesting in its own right.

PROPOSITION 3.12. *Consider the map $f_v : G \rightarrow V$ defined by $f_v(g) = gv$. The map f_v is proper if and only if $G(v)$ is closed and G_v is compact.*

PROOF. Recall that G is a closed subgroup of $GL(V)$. One direction is clear and we omit the proof. For, if f_v is proper, then clearly $G(v)$ is closed and G_v is compact.

Suppose $G(v)$ is closed, but that f_v is not proper. We will show that G_v cannot be compact. Hence, if $G(v)$ is closed with G_v compact, then f_v is proper.

By assumption, there exists an unbounded sequence $\{g_n\}$ in G such that $\{g_n(v)\}$ is bounded. As G is self-adjoint, we may write $g_n = k_n \exp(X_n)$, where $k_n \in K$ and $X_n \in \mathfrak{p}$ and $|X_n| \rightarrow \infty$ as $n \rightarrow \infty$. Since K is compact, it follows that $\exp(X_n)v \rightarrow w \in V$ by passing to a subsequence if necessary.

Let $Y_n = X_n/|X_n|$, $t_n = |X_n|$, and let $Y_n \rightarrow Y \in \mathfrak{p}$, by passing to a subsequence if necessary. If $f_n(t) = |\exp(tY_n)(v)|^2$ and $f(t) = |\exp(tY)(v)|^2$, then $f_n(t) \rightarrow f(t)$ for each t as $n \rightarrow \infty$. It is proved in Lemma 3.1 of [RS90] that the functions $f_n(t), f(t)$ are convex; that is, $f_n'' \geq 0$ and $f'' \geq 0$. By hypothesis $f_n(t_n) \rightarrow |w|^2$ as $n \rightarrow \infty$. By the convexity of $f_n(t)$, we conclude that $f_n(t) \leq \max\{f_n(0), f_n(t_n)\} \leq |v|^2 + |w|^2 + 1$ if $0 \leq t \leq t_n$ and n is sufficiently large. Hence $f(t) \leq |v|^2 + |w|^2 + 1$ for $t \geq 0$. It follows by convexity that $f(t)$ is nonincreasing on \mathbb{R} .

Let Λ denote the set of nonzero eigenvalues of Y and let $V = V_0 \oplus \sum_{\lambda \in \Lambda} V_\lambda$ be the direct sum decomposition of V into orthogonal eigenspaces of $Y \in \mathfrak{p}$, where $Y = 0$ on V_0 and $Y = \lambda Id$ on V_λ for all $\lambda \in \Lambda$. Write $v = v_0 + \sum v_\lambda$ where $v_0 \in V_0$ and $v_\lambda \in V_\lambda$ for $\lambda \in \Lambda$. Then $\exp(tY)(v) = v_0 + \sum e^{t\lambda} v_\lambda$ and $f(t) = |\exp(tY)(v)|^2 = |v_0|^2 + \sum e^{2t\lambda} |v_\lambda|^2$. By the previous paragraph $\lim_{t \rightarrow \infty} f(t)$ exists, and it follows that $\lambda \in \Lambda$ is negative if $v_\lambda \neq 0$. Thus $\exp(tY)(v) \rightarrow v_0$ as $t \rightarrow \infty$. Moreover, $Y(v_0) = 0$ as $v_0 \in V_0$.

We have shown that $v_0 \in \overline{Gv} = Gv$. Hence there exists $g \in G$ such that $v_0 = gv$. Since $\exp(tY) \subset G_{v_0}$ we see that G_{v_0} and hence G_v is not compact. □

PROOF OF THE THEOREM. We prove (a) implies (b). If $G(v)$ is not closed, then the map $f_v : G \rightarrow V$ is not proper by the proposition above. By the proof of this proposition we know that there exists $Y \in \mathfrak{p}$ such that $\exp(tY)(v) \rightarrow v_0$ as $t \rightarrow \infty$, where $v_0 \in \text{Ker } Y$. Thus $\mu(Y, v) \geq 0$ by Proposition 3.5; for the definition of μ see the remarks preceding Proposition 3.5. Hence $M(v) \geq \mu(Y, v) \geq 0$ which contradicts our hypothesis. Thus $G(v)$ is closed. Moreover, if G_v were not compact, then f_v would not be proper and we would arrive at the same contradiction.

We prove (b) implies (c). If $F_v : G \rightarrow \mathbb{R}$ is not proper, then $f_v : G \rightarrow V$ is also not proper. By the proof of the proposition above, G_v would then not be compact which contradicts our hypothesis.

We prove (c) implies (a). Suppose that $M(v) \geq 0$ and choose a unit vector $Y \in \mathfrak{p}$ such that $\mu(Y, v) = M(v) \geq 0$. Then by Proposition 3.5 $\exp(-tY)(v) \rightarrow v_0$ as $t \rightarrow \infty$, where v_0 is the component of v in $\text{Ker } Y$. Hence $F_v : G \rightarrow [0, \mathbb{R})$ is not proper since $F_v(\exp(-tY)) \rightarrow |v_0|^2$ as $t \rightarrow \infty$. This violates our hypothesis. □

An immediate and useful observation is that stability produces generically closed orbits.

COROLLARY 3.13. *Suppose there exists a stable vector $v \in V$. Then there exists a nonempty Hausdorff open set of stable points, and moreover there exists a nonempty Zariski open set of vectors whose orbits are closed.*

PROOF. Let v be stable, that is $M(v) < 0$. From Proposition 3.9 we see that there exists a Hausdorff open set \mathcal{O} so that $M < 0$ on \mathcal{O} . We know that the set of orbits of maximal dimension is a nonempty Zariski open set in V , and hence it intersects \mathcal{O} . Thus we have the existence of a closed orbit of maximal dimension by Theorem 3.11, and by Proposition 2.8 we know that there exists a Zariski open set of points whose orbit is closed. \square

COROLLARY 3.14. *Let $v \in \mathfrak{M} \subset V$ be a minimal vector. The following are equivalent:*

- (a) $M(v) < 0$
- (b) $G \cdot v$ is closed with G_v compact
- (c) The moment map $m : V \rightarrow \mathfrak{p}$ has maximal rank at v
- (d) If $X(v)=0$ for some $X \in \mathfrak{p}$, then $X = 0$

This corollary is useful in that if we find a minimal vector with one the properties listed above, then we can guarantee the existence of generic closed orbits. For an application of this idea see Section 4 of this Chapter. Moreover, one can actually calculate the dimension of the moduli space $X//G$ using the above information.

PROOF. Assertions (a) and (b) are equivalent by Theorem 3.11 above. For $v, \eta \in V$ and $X \in \mathfrak{p}$ a routine calculation shows that $\langle m_*(\eta_v), X_{m(v)} \rangle = 2 \langle X(v), \eta \rangle$. Hence (c) and (d) are equivalent. We show that (b) and (d) are equivalent. Since v is minimal, $G \cdot v$ is closed by Theorem 2.21. Now minimality implies \mathfrak{g}_v is self-adjoint by Theorem 2.20, and hence $\mathfrak{g}_v = \mathfrak{k}_v \oplus \mathfrak{p}_v$. Thus G_v is compact if and only if $\mathfrak{p}_v = \{0\}$, which proves that (b) and (d) are equivalent. \square

Now we can state our work above and see how it is the analogue of the Hilbert-Mumford criterion, cf. Theorem 2.12.

THEOREM 3.15. *Let G act on V and take $v \in V$. Then*

- (a) $M(v) > 0$ if and only if v is in the null cone
- (b) $M(v) = 0$ if and only if v is semi-stable, but not stable
- (c) $M(v) < 0$ if and only if v is stable

PROOF. The third part of the theorem was proven above. To prove the first we note that $M(v) = \mu(Y, v)$ for some $Y \in \mathfrak{p}_0$, $|Y| = 1$. Now if $M(v) > 0$, Proposition 3.5 shows that $\exp(tY)v \rightarrow 0$ as $t \rightarrow -\infty$. That is, v is in the null cone. If v is contained in the null cone then we know there exists $Y \in \mathfrak{p}$ such that $\exp(tY)v \rightarrow 0$ as $t \rightarrow -\infty$ (see Lemma 2.18). Now, Proposition 3.5 implies $M(v) \geq \mu(Y, v) > 0$. This proves (a); assertion (b) follows immediately from (a) and (c). \square

PROPOSITION 3.16. *Let G act on V and suppose there is a point $v \in V$ such that G_v is compact. Then generic orbits are closed and there is an open set such that $M(v) < 0$.*

Warning! Even though G_v is compact, $G \cdot v$ might not be closed. See Example 4.6.

PROOF. First we show that there is an open set such that G_w is compact. Recall that since G is an algebraic group, the stabilizer G_w is an algebraic group and hence a group with finitely many connected components. Thus, G_w is compact if and only if $(G_w)_0$ is compact, where $(G_w)_0$ is the Hausdorff identity component.

Suppose such an open set did not exist. Then we would have a sequence of $v_n \rightarrow v$ in V such that the groups G_{v_n} are not compact. Let d be a complete Riemannian metric on $\text{End}(V)$. Then G_v being compact it has a diameter, say D . Since each $(G_{v_n})_0$ is arc connected but not compact we can pick $g_n \in (G_{v_n})_0$ such that $\text{dist}(e, g_n) = 2D$. Now $\text{End}(V)$ being a complete metric space, and g_n being a bounded sequence in G , there exists a convergent subsequence converging to some $g \in G$. Passing to this subsequence we have $v = \lim v_n = \lim(g_n v_n) = gv$, which shows that $g \in G_v$. We have a contradiction as $\text{dist}(e, g) = 2 \text{diam}(G_v)$. Thus, there exists some open set $\mathcal{O} \ni v$ such that $(G_w)_0$, and hence G_w , is compact for $w \in \mathcal{O}$.

Let $G^\mathbb{C}$ denote the complexification of G . Recall that $G^\mathbb{C}$ acts on $V^\mathbb{C}$ and has a stabilizer in general position G' , s.g.p. That is, there is an open set $\mathcal{U} \subset V^\mathbb{C}$ such that $G_w^\mathbb{C}$ is isomorphic to G' for $w \in \mathcal{U}$. Since $\mathcal{U} \cap V$ is a Zariski open set of V it intersects \mathcal{O} .

Recall for $v \in V \subset V^\mathbb{C}$ we have $\mathfrak{g}_v^\mathbb{C} = (\mathfrak{g}_v)^\mathbb{C}$ and \mathfrak{g}_v is reductive if and only if $\mathfrak{g}_v^\mathbb{C}$ is reductive. Hence, the s.g.p is reductive since \mathfrak{g}_v is compact and hence reductive for $v \in \mathcal{U} \cap \mathcal{O}$. Now by a theorem of Popov [Pop70] we have that generic orbits are closed.

Let A be a nonempty Zariski open subset of V such that $G \cdot v$ is closed for all $v \in A$. If $v \in A \cap \mathcal{O}$, then $M(v) < 0$ by Theorem 3.11 (b). \square

Remark. At the moment we do not have a proof of this result without using Popov's theorem which relates reductive s.g.p. and generically closed orbits. It would be very interesting to us and worthwhile to find an analytic proof of the above result that avoids the algebraic geometry in Popov's proof.

Closed subgroups in stable representations. If G acts stably on V , then one can say a lot about the induced representations of closed subgroups.

PROPOSITION 3.17. *Let H be a closed subgroup of G and v a G -stable point, cf. Definition 3.2. Then v is H -stable; that is, $H \cdot v$ is closed and H_v is compact.*

PROOF. Let $w \in \overline{H(v)}$ and let $h_n \in H \subset G$ be a sequence such that $h_n(v) \rightarrow w$ as $n \rightarrow \infty$. By (c) of Theorem 3.11 we know that the h_n converge to an element $h \in G$ and since H is closed in G we have $h \in H$. Thus $w = h(v) \in H \cdot v$. That is, $H \cdot v$ is closed.

Now $H_v = H \cap G_v$ is a closed subgroup of a compact group. Thus it is compact. \square

COROLLARY 3.18. *Let H be a semi-simple subgroup of G and v be a G -stable vector. Then v is H -stable.*

PROOF. Recall the well-known result of Mostow that says semi-simple subgroups of semi-simple groups are closed subgroups, see the main theorem in section 6 of [Mos50]. Hence H is closed in G , and we apply the previous result. \square

Remark. The above two results are special to the setting of stable vectors. There do exist representations such that $G(v)$ is closed but $H(v)$ is not closed, where H is an algebraic reductive subgroup, see Example 4.6. However, for ‘generic’ v the above results hold true for reductive subgroups, see Chapter 4.

Suppose we have a representation of G which is stable. We know that the set of generic closed orbits does not have to equal the set of stable points, see Example 3.20. In this example there exists a Hausdorff open set of stable points but there also exists a Hausdorff open set of non-stable, good semi-stable points. The set of generically closed orbits is a Zariski open set with, usually, many Hausdorff components.

QUESTION 3.19. *Suppose G acts on V and v is a stable point. Let $w \in V$ be a $G^{\mathbb{C}}$ conjugate of v , not necessarily one of G . That is, there exists $g \in G^{\mathbb{C}}$ so that $w = gv$. Let H be a closed subgroup of G . We know, from the above, that $H \cdot v$ is closed. Is $H \cdot w$ also closed?*

If H is semi-simple and w a minimal vector then the answer is yes, but in general we do not know. To see this special case, one observes from [RS90, Lemma 8.1] that w will be a minimal vector for $H^{\mathbb{C}}$ and hence $H^{\mathbb{C}}w$ will be closed. But then Proposition 1.15 tells us that Hw is closed.

2. Examples of Stability

Our first example will demonstrate two important aspects of real stability. First, there exist Hausdorff open sets of stable points which are not Zariski open. Second, there are more, in fact many more, real stable representations than complex stable representations. Thus our work truly generalizes the Hilbert-Mumford Criterion.

EXAMPLE 3.20. *Adjoint Representation of $SL_2(\mathbb{R})$.*

Let $G = SL_2(\mathbb{R})$ and $V = \mathfrak{g} = \{A \in M_2(\mathbb{R}) \mid \text{trace} A = 0\}$. The group G acts on V via conjugation. Let \langle, \rangle denote the inner product on V given by $\langle A, B \rangle = \text{trace}(AB^t)$, where B^t denotes the transpose operation in $M_2(\mathbb{R})$. For $g \in G$, the metric adjoint g^* relative to \langle, \rangle corresponds to the usual transpose g^t . Hence G is self-adjoint relative to \langle, \rangle . Moreover, the Cartan involution is the standard one and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where \mathfrak{k} is the set of skew-symmetric matrices and \mathfrak{p} is the set of traceless symmetric matrices.

PROPOSITION 3.21. *Let $\mathcal{O}_1 = \det^{-1}(-\infty, 0)$, $\mathcal{O}_2 = \det^{-1}(0, \infty)$, and $\Sigma = \det^{-1}(0)$, where $\det : M_2(\mathbb{R}) \rightarrow \mathbb{R}$ is the determinant function. Then*

- (a) *The sets $\mathcal{O}_1, \mathcal{O}_2, \Sigma$ are all G -invariant, V is their disjoint union, and the sets \mathcal{O}_i are open in the Hausdorff topology but not Zariski open.*
- (b) *If \mathfrak{M} denotes the minimal set for the action of G on V , then $\mathfrak{M} = \mathfrak{k} \cup \mathfrak{p}$.*

- (c) $G(A)$ is closed in V if $A \in \mathcal{O}_1 \cup \mathcal{O}_2$. The null cone, or set of unstable points, is the set Σ .
(d) $M(A) = 0$ for all $A \in \mathcal{O}_1$, $M(A) = -\sqrt{2}$ for all $A \in \mathcal{O}_2$, and $M(A) = \sqrt{2}$ for all $A \in \Sigma$.

Proof. We omit the proof for (a) as it is clear.

Proof of (b). By Example 2.29 we know that $A \in \mathfrak{M}$ if and only if $AA^t = A^tA$. Since $A \in M_2(\mathbb{R})$ it is easy to show by hand that $A \in \mathfrak{M}$ if and only if A is either skew-symmetric or symmetric.

Proof of (c). Recall that $G(A)$ is closed if and only if $G(A) \cap \mathfrak{M}$ is nonempty. Part (c) now follows from part (b) and the next result.

LEMMA 3.22. *Standard forms*

- (a) If $A \in \mathcal{O}_1$, then there exists $g \in G$ such that $g(A) = gAg^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \in \mathfrak{p}$, where $\lambda = \sqrt{|\det A|}$.
(b) If $A \in \mathcal{O}_2$, then there exists $g \in G$ such that $g(A) = gAg^{-1} = \pm \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \in \mathfrak{k}$, where $\lambda = \sqrt{\det A}$.
(c) If $A \in \Sigma$, then there exists a sequence $\{g_n\}$ in G such that $g_n(A) = \begin{pmatrix} 0 & \lambda_n \\ 0 & 0 \end{pmatrix}$, where $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof of Lemma. For $A \in V = \mathfrak{g}$ we recall that the characteristic polynomial of A acting in the usual way on \mathbb{R}^2 is given by $c_A(x) = x^2 + \det A$.

(a) If $A \in \mathcal{O}_1$, then A has eigenvalues $\pm\lambda$, where $\lambda = \sqrt{|\det A|}$. Let $\{v_1, v_2\}$ be a positively oriented basis of \mathbb{R}^2 such that $A(v_1) = \lambda v_1$ and $A(v_2) = -\lambda v_2$. Let $g \in GL_2(\mathbb{R})$ be an element with $\det g > 0$ such that $g(v_i) = e_i$, where $\{e_1, e_2\}$ is the standard basis of \mathbb{R}^2 . Write $g = ch$, where $\det h = 1$ and $c > 0$. Then $h(A) = hAh^{-1} = gAg^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \in \mathfrak{p}$.

(b) If $A \in \mathcal{O}_2$, then A has eigenvalues $\pm i\lambda$, where $\lambda = \sqrt{\det A}$. Let $v_1, v_2 \in \mathbb{R}^2$ be vectors such that $A(v_1 + iv_2) = i\lambda(v_1 + iv_2)$. Then the v_i are linearly independent and $A(v_1) = -\lambda v_2$ and $A(v_2) = \lambda v_1$. Hence A has matrix $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ relative to the basis $\{v_1, v_2\}$ of \mathbb{R}^2 . Depending on the orientation, choose $g \in GL_2^+(\mathbb{R})$ such that $g(v_1) = e_1$ and $g(v_2) = \pm e_2$. In either case, choose $c > 0$ and $h \in G$ such that $g = ch$. Then we obtain $hAh^{-1} = gAg^{-1} = \pm \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$.

(c) If $A \in \Sigma$, then $A^2 = 0$. As above, we can arrange for A to have the matrix $hAh^{-1} = \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix}$, for

some $h \in G$, relative to the correct choice of basis. Now consider $h_n = \begin{pmatrix} 1/n & 0 \\ 0 & n \end{pmatrix} \in G$. Then conjugation

by $h_n h$ yields $\begin{pmatrix} 0 & \pm n^{-2} \\ 0 & 0 \end{pmatrix}$. This produces the claimed result and the lemma is proven.

We finish the proof of the proposition by showing part (d). Let $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $\{H_0, X, Y\}$ is a basis of \mathfrak{g} such that $[H_0, X] = 2X$, $[H_0, Y] = -2Y$, $[X, Y] = H_0$. The space \mathfrak{p} is 2-dimensional and the 1-dimensional maximal compact subgroup $K \approx S^1$ acts transitively on the spheres (circles) of fixed length vectors in \mathfrak{p} . If $H \in \mathfrak{p}$, then H has eigenvalues $\pm\lambda$ for some real number λ , and $|H|^2 = \text{trace}(H^2) = 2\lambda^2$. It follows that H is a unit vector in \mathfrak{p} if and only if H has eigenvalues $\pm\sqrt{2}/2$. In particular, if H is any unit vector in \mathfrak{p} , then there exists $k \in K$ such that $kHk^{-1} = H_0/\sqrt{2}$.

We show that $M(A) = \sqrt{2}$ if $A \in \Sigma$. The argument in the proof of c) of the lemma above shows that for any $A \in \Sigma$ there exists $g \in G$ and $\lambda \in \mathbb{R}$ such that $gAg^{-1} = \lambda X$. Hence $M(A) = M(\lambda X) = M(X)$ and it suffices to prove that $M(X) = \sqrt{2}$.

Note that $\mu(H_0, X) = 2$ since $[H_0, X] = 2X$. Hence $\mu(H_0/\sqrt{2}, X) = \sqrt{2}$. Now let H be an arbitrary unit vector in \mathfrak{p} and let $k \in K$ be an element such that $kHk^{-1} = H_0/\sqrt{2}$. Choose a real number θ such that $k = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. Then $kXk^{-1} = -\sin \theta \cos \theta H_0 + \cos^2 \theta X - \sin^2 \theta Y$. If $\sin \theta \neq 0$, then $\mu(H, X) = \mu(kHk^{-1}, kXk^{-1}) = \mu(H_0/\sqrt{2}, kXk^{-1}) = -\sqrt{2}$. If $\sin \theta = 0$ then $k = \pm Id$ and in this case $\mu(H, X) = \mu(H_0/\sqrt{2}, X) = \sqrt{2}$. Thus, $M(X) = \sqrt{2}$.

Next we show that $M(A) = -\sqrt{2}$ for all $A \in \mathcal{O}_2$. For $A \in \mathcal{O}_2$ we write $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = aH_0 + bX + cY$ for suitable real numbers a, b, c . By hypothesis $a^2 + bc = -\det A < 0$, and hence b, c are always nonzero. It follows by inspection that $\mu(H_0, A) = -2$ and hence $\mu(H_0/\sqrt{2}, A) = -\sqrt{2}$. If H is a unit vector in \mathfrak{p} , then $kHk^{-1} = H_0/\sqrt{2}$ for some $k \in K$ and $\mu(H, A) = \mu(H_0/\sqrt{2}, kAk^{-1}) = -\sqrt{2}$. Hence $M(A) = -\sqrt{2}$.

Lastly we show $M(A) = 0$ for $A \in \mathcal{O}_1$. Since A has eigenvalues $\pm\lambda$ there exists $g \in G$ with $gAg^{-1} = \lambda H_0$. Hence $M(A) = M(gAg^{-1}) = M(\lambda H_0) = M(H_0)$. It suffices to prove that $M(H_0) = 0$. Note that $\mu(H_0/\sqrt{2}, H_0) = 0$ as $H_0 \in \text{Ker } H_0$. If H is any unit vector in \mathfrak{p} then choose $k = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in K$ such that $kHk^{-1} = H_0/\sqrt{2}$. Then $\mu(H, H_0) = \mu(kHk^{-1}, kH_0k^{-1}) = \mu(H_0/\sqrt{2}, \cos(2\theta)H_0 + \sin(2\theta)X + \sin(2\theta)Y)$. If $\sin(2\theta) \neq 0$, then $\mu(H, H_0) = -\sqrt{2}$. If $\sin(2\theta) = 0$, then $kH_0k^{-1} = \pm H_0$, and $\mu(H, H_0) = \pm\mu(H_0/\sqrt{2}, H_0) = 0$. Hence $M(H_0) = \max\{\mu(H, H_0) : H \in \mathfrak{p}, |H| = 1\} = 0$.

EXAMPLE 3.23. *Let V be a representation of $SL_2(\mathbb{R})$. If V contains no trivial factors and $\dim V \geq 3$ then the representation is stable.*

This result follows from a much more general construction called the Index Method. See Propositions 3.31 and 3.32 and also Section 4 of [EJ].

EXAMPLE 3.24. *Adjoint representations.*

All stable adjoint representations have been classified in [EJ]. There it is shown that “most” adjoint representations are stable. We do not present the details, rather we refer the reader there.

EXAMPLE 3.25. *The action of $SL(q, \mathbb{R}) \times SL_3\mathbb{R}$ on $\mathfrak{so}(q, \mathbb{R})^3$ for $q \neq 3, 6$ is stable. This follows from Section 3.4. Here the geometry of nilmanifolds is employed to obtain new examples of minimal vectors and stability.*

Remark. For the stability of case $q = 6$, see Example 2 of [EJ, Appendix 2]. It is known that the case $q = 3$ is not stable for more general reasons, see Theorem 3.44.

Inheritance of Stability. We finish this section with some methods to build stable representations via summing and tensoring representations.

PROPOSITION 3.26. *Let $V = \oplus V_i$ be a direct sum of G -representations. If one summand is G -stable (cf. Definition 3.2), then the whole sum is G -stable.*

PROOF. It suffices to prove this for the sum of two representations V, W . Take $v \in V$ which is stable. Then $(v, 0) \in V \oplus W$ has a closed G -orbit and compact stabilizer. Hence, $(v, 0)$ is a stable point in $V \oplus W$. \square

PROPOSITION 3.27. *Consider two G -stable representations V, W . Then $V \otimes W$ is also stable.*

PROOF. Let $v \in V$ and $w \in W$ be stable points. We show that the vector $v \otimes w$ is a stable point in $V \otimes W$.

Since v, w are stable points we have $M(v) < 0$ and $M(w) < 0$. Choose $X \in \mathfrak{p}$ with $|X| = 1$ so that $M(v \otimes w) = \mu(X, v \otimes w)$. The X -eigenspace decomposition for $v \otimes w$ is $\sum v_\lambda \otimes w_\mu$, where $v = \sum v_\lambda$ and $w = \sum w_\mu$ are the X -eigenspace decompositions of these vectors. Since $X(v_\lambda \otimes w_\mu) = (Xv_\lambda) \otimes w_\mu + v_\lambda \otimes (Xw_\mu)$, we see that $v_\lambda \otimes w_\mu$ has eigenvalue $\lambda + \mu$. Now $\mu(X, v) \leq M(v) < 0$ and $\mu(X, w) \leq M(w) < 0$, and hence there exists $\lambda < 0$ and $\mu < 0$ with v_λ and w_μ both nonzero. It follows that $M(v \otimes w) = \mu(X, v \otimes w) \leq \lambda + \mu < 0$. Thus $v \otimes w$ is a stable point. \square

The next proposition follows the tradition in representation theory of semi-simple groups to understand representations of G by understanding the representations of its simple factors. If $G = G_1 \times \cdots \times G_k$ is a semi-simple group, then an irreducible representation of G is $V_1 \otimes \cdots \otimes V_k$, where V_i is an irreducible representation of G_i .

PROPOSITION 3.28. *Let V_1 and V_2 be stable representations of G_1 and G_2 , respectively. Then $V_1 \otimes V_2$ is a stable representation of $G_1 \times G_2$.*

Remark. Notice this does not follow from the proposition above. However, the proof is analogous so we omit the details.

3. The Index Method

In this section we are interested in the following question.

QUESTION 3.29. *Given two representations V, W of G , when is $V \otimes W$ a ‘good’ representation? (cf. Definition 3.1)*

The Index Method gives a partial answer to this question. As Proposition 3.27 shows, if V and W are G -stable, then so is $V \otimes W$. However, there are many representations $V \otimes W$ which are G -stable with V, W ill-behaved; e.g., cf. the tables of [KL87].

For a nonzero element $X \in \mathfrak{p}$ let $I_G(X)$ denote the largest dimension of a subspace W of V on which X is negative definite. Let $I_G(V) = \min\{I_G(X) : 0 \neq X \in \mathfrak{p}\}$. We call $I_G(V)$ the *index* of G acting on V . Observe that since G is semi-simple, every element of \mathfrak{p} has trace zero. Hence, for nontrivial representations V , each $X \in \mathfrak{p}$ has a negative eigenvalue which implies $I_G(V) \geq 1$.

At the moment, the definition of the index seems to depend on our choice of inner product under which G is self-adjoint. The following proposition shows that this is not so.

PROPOSITION 3.30. *The index of G acting on V does not depend on the choice of G -compatible inner product \langle, \rangle .*

PROOF. Let \langle, \rangle_1 and \langle, \rangle_2 be two G -compatible inner products on V , and $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ and $\mathfrak{g} = \mathfrak{k}_2 \oplus \mathfrak{p}_2$ denote the corresponding Cartan decompositions. It is well-known that there exists $g \in G$ such that $\mathfrak{k}_2 = \text{Ad}(g)\mathfrak{k}_1$ and $\mathfrak{p}_2 = \text{Ad}(g)\mathfrak{p}_1$; see, e.g., Theorem 7.2 of Chapter III in [Hel01]. Since X and $\text{Ad}(g)X$ acting on V have the same eigenvalues, it follows that $I_G^1(X) = I_G^2(\text{Ad}(g)X)$. Hence $I_G^1(V) = I_G^2(V)$. \square

PROPOSITION 3.31. *Let K denote a maximal compact subgroup of G . If $I_G(V) > \dim K$, then $\{v \in V : M(v) < 0\}$ is an open set of V with full measure in V .*

PROOF. We carry out the proof in several steps.

(1) Weight space decomposition of V .

Let \langle, \rangle be an inner product relative to which G is self-adjoint. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of \mathfrak{g} defined by the Cartan involution $\theta : g \rightarrow (g^t)^{-1}$. Fix a maximal abelian subspace \mathfrak{A} of \mathfrak{p} . It is well-known that every maximal abelian subspace of \mathfrak{p} has the form $\text{Ad}(k)\mathfrak{A}$ for some $k \in K$, and every element of \mathfrak{p} lies in some maximal abelian subspace of \mathfrak{p} . The elements of \mathfrak{p} are symmetric with respect to \langle, \rangle , and hence \mathfrak{A} is a commuting family of symmetric linear maps on V .

For $\lambda \in \mathfrak{A}^*$ let $V_\lambda = \{v \in V : X(v) = \lambda(X)v \text{ for all } X \in \mathfrak{A}\}$. If $\Lambda = \{\lambda \in \mathfrak{A}^* : V_\lambda \neq 0\}$, then Λ is a finite set, called the *weights* of the representation, and we obtain the *weight space decomposition*

$$V = V_0 + \sum_{\lambda \in \Lambda} V_\lambda$$

where $V_0 = \bigcap_{X \in \mathfrak{A}} \text{Ker } X$.

(2) The subspaces V_X^+ and V_X^-

For a nonzero element $X \in \mathfrak{A}$, we let $\Lambda_X^+ = \{\lambda \in \Lambda : \lambda(X) > 0\}$ and $\Lambda_X^- = \{\lambda \in \Lambda : \lambda(X) < 0\}$. We define $V_X^+ = V_0 \oplus \sum_{\lambda \in \Lambda_X^+} V_\lambda$ and $V_X^- = \sum_{\lambda \in \Lambda_X^-} V_\lambda$. The following assertions follow routinely from the definitions:

- (a) $\mu(X, v) \geq 0$ for some nonzero $X \in \mathfrak{A}$ if and only if $v \in V_X^+$.
- (b) $I_G(X) = \dim V_X^-$.
- (c) $V = V_X^+ \oplus V_X^-$.

(3) There exists a finite set of nonzero vectors $\{X_1, \dots, X_n\} \subset \mathfrak{A}$ such that for every nonzero $X \in \mathfrak{A}$ there exists $1 \leq i \leq N$ such that $V_X^+ = V_{X_i}^+$.

Since Λ is a finite set, the number of distinct subsets $\{\Lambda_X^+ : 0 \neq X \in \mathfrak{A}\}$ is also finite. Choose nonzero elements $\{X_1, \dots, X_N\} \subset \mathfrak{A}$ such that for every nonzero $X \in \mathfrak{p}$ there exists $1 \leq i \leq N$ such that $\Lambda_X^+ = \Lambda_{X_i}^+$. This is the desired set.

(4) $\{v \in V : M(v) \geq 0\} = \bigcup_{i=1}^N K(V_{X_i}^+)$, where $\{X_1, \dots, X_N\}$ are chosen as in (3).

By (2) it follows that $M(v) \geq 0$ for all $v \in V_{X_i}^+$, $1 \leq i \leq N$. From the G -invariance of M we conclude that $M(v) \geq 0$ for all $v \in \bigcup_{i=1}^N K(V_{X_i}^+)$. Conversely, let v be a nonzero vector in V such that $M(v) \geq 0$. Let X be a unit vector in \mathfrak{p} such that $\mu(X, v) = M(v) \geq 0$. Choose $k \in K$ such that $Y = \text{Ad}(k)X \in \mathfrak{A}$. Then $\mu(Y, kv) = \mu(X, v) \geq 0$. By (2) and (3) it follows that $kv \in V_Y^+ = V_{X_i}^+$ for some i . Hence, $v \in K(V_{X_i}^+)$, which completes the proof of (4).

We now complete the proof of the proposition. By hypothesis and (2), we obtain $\dim K < I_G(V) \leq I_G(X) = \dim V_X^- = \dim V - \dim V_X^+$ for all nonzero elements X of \mathfrak{A} . For $1 \leq i \leq N$ we define $\varphi_i : K \times V_{X_i}^+ \rightarrow V$ by $\varphi_i(k, v) = kv$. Note that $\dim(K \times V_{X_i}^+) = \dim K + \dim V_{X_i}^+ < \dim V$ for every i , and hence $KV_{X_i}^+ = \varphi_i(K \times V_{X_i}^+)$ has measure zero in V . Hence $\{v \in V : M(v) \geq 0\}$ has measure zero in V by (4). \square

PROPOSITION 3.32 (Additivity of the Index). *Let $\{V_1, \dots, V_N\}$ be G -modules, and consider the G -module $V = \oplus V_i$. Then $I_G(V) \geq \sum_{i=1}^N I_G(V_i)$.*

PROOF. Let $X \in \mathfrak{A}$ be a nonzero element. Using the notation and discussion of (2) above, it is easy to see that $V_X^- = \sum_{i=1}^N (V_i)_X^-$ and $I_G^V(X) = \sum I_G^{V_i}(X) \geq \sum I_G(V_i)$. If $X \in \mathfrak{p}$ is any nonzero element, then

$Y = \text{Ad}(k)X \in \mathfrak{A}$ for some $k \in K$. It follows that $I_G^V(X) = I_G^V(Y)$ since X and Y have the same eigenvalues on V . Hence $I_G(V) = \min\{I_G^V(X) : 0 \neq X \in \mathfrak{p}\} = \min\{I_G^V(X) : 0 \neq X \in \mathfrak{A}\} \geq \sum I_G(V_i)$. \square

COROLLARY 3.33. *Let V be a G -module that is the direct sum of $p > \dim K$ nontrivial submodules. Then $\{v \in V : M(v) < 0\}$ is an open set of full measure*

PROOF. The proof follows from the simple observation that the index of each summand V_i is at least 1. Now applying the additivity of the index we are finished. \square

PROPOSITION 3.34 (Multiplicativity of the Index). *Let V, W be G -modules. Then $I_G(V \otimes W) \geq I_G(V) \cdot I_G(W)$.*

PROOF. If $0 \neq X \in \mathfrak{A}$, then X is negative definite on $V_X^- \otimes W_X^-$. Hence $I_G^{V \otimes W}(X) \geq (\dim V_X^-) \cdot (\dim W_X^-) = I_G^V(X) \cdot I_G^W(X) \geq I_G(V) \cdot I_G(W)$. If $0 \neq X \in \mathfrak{p}$ and $Y = \text{Ad}(k)(X) \in \mathfrak{A}$ for $k \in K$, then $I_G^{V \otimes W}(X) = I_G^{V \otimes W}(Y) \geq I_G(V) \cdot I_G(W)$. \square

4. Stable Points with Geometric Significance and the Nilalgebras Attached to $SU(2)$ Representations

In later chapters we will be concerned with a very particular class of representations and determining when such representations have generically closed orbits. In that particular setting, stable points in the representation space correspond to certain metric nilmanifolds with interesting geometric structures, see Chapter 7.

Consider the action of $SL(q, \mathbb{R})$ on $\mathfrak{so}(q, \mathbb{R})$ where $g(A) = gAg^t$, and the standard action of $SL(p, \mathbb{R})$ on \mathbb{R}^p . Then $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ acts on $\mathfrak{so}(q, \mathbb{R})^p \simeq \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$, cf. Example 2.28. There is a Zariski open set of $V_{pq} = \mathfrak{so}(q, \mathbb{R})^p$ that corresponds to two-step metric nilmanifolds. In this setting, moving along the $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ -orbit amounts to varying the metric on the underlying nilalgebra. The minimal vectors of this representation correspond to the so-called optimal metrics. See Chapter 7 for more details on the relationship between real GIT and the geometry of two-step nilmanifolds.

Once at a minimal vector, one would like to determine how generic it is; e.g., is the orbit of this point maximal dimensional. We can try to compute whether or not M is negative at the minimal vector. To determine this, one just needs to compute \mathfrak{p}_v and show that it equals 0, cf. Corollary 3.14. For the representation above, the stabilizer corresponds to derivations that preserve \mathcal{V} and \mathfrak{Z} , where \mathfrak{Z} is the center of our two-step nilalgebra \mathfrak{N} and \mathcal{V} is the metric complement of \mathfrak{Z} . In light of this, to show \mathfrak{p}_v equals zero, one needs to show that the metric nilalgebra, corresponding to the point v , admits no symmetric derivations other than $(r \text{ Id}|_{\mathcal{V}}, 2r \text{ Id}|_{\mathfrak{Z}})$.

Given a compact semi-simple group, one can construct a metric two-step nilmanifold which is optimal, see [EH96]. The main result of this section is that all two-step nilmanifolds arising from irreducible representations of $\mathfrak{su}(2)$ have no non-trivial symmetric derivations, except the adjoint representation; that is,

for the cases of interest the set of symmetric derivations is a 1 dimensional vector space. Let $v \in V_{3,q}$, with $q = 2k + 1$ or $4k$, denote such an algebra. We show that $\mathfrak{p}_v = \{0\}$ and conclude that $M(v) < 0$. Corollary 3.13 together with Proposition 7.9 now imply that for each $q = 2k + 1, 4k \geq 4$ there exists an open, dense set of algebras in $V_{3,q}$ that admit optimal metrics. Moreover, we provide another construction to show that the above result holds for all types $(3,q)$, $q \neq 3, 6$. Note, the two types missed by our construction coincide with the same two types (for $p=3$) in the exceptional list constructed by Eberlein [Ebe03]. However, this does not necessarily keep us from getting a dense set of optimal metrics (which in fact happens).

Metric two-step nilalgebras. We begin by constructing two-step nilpotent Lie algebras attached to representations of compact Lie groups. In [EH96] it is shown that these special nilpotent Lie algebras admit a canonical metric called an optimal metric, cf. Chapter 7.

Let G be a compact Lie group and $j : G \rightarrow GL(\mathcal{V})$ be a representation of G . Let \mathfrak{g} denote the Lie algebra of G . We have an induced representation of \mathfrak{g} on \mathcal{V} ; we denote both representations by j . The group G acts on \mathfrak{g} via the adjoint action. Endow \mathcal{V} with a G -invariant inner product $\langle \cdot, \cdot \rangle$ and endow \mathfrak{g} with an Ad G -invariant inner product $\langle \langle \cdot, \cdot \rangle \rangle$; this is possible by the compactness of G . Note, the Ad G -invariant inner product on \mathfrak{g} is unique up to scaling if \mathfrak{g} is simple. We construct a two-step metric nilpotent Lie algebra \mathfrak{N} so that $[\mathfrak{N}, \mathfrak{N}] = \mathfrak{g}$ is contained in the center of \mathfrak{N} and \mathcal{V} is a complement of $[\mathfrak{N}, \mathfrak{N}]$. As a vector space we endow $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{g}$ with the metric $\langle \cdot, \cdot \rangle$ which corresponds to $\langle \cdot, \cdot \rangle$ on \mathcal{V} and $\langle \langle \cdot, \cdot \rangle \rangle$ on \mathfrak{g} . The bracket relations on \mathfrak{N} are defined implicitly as follows.

$$\begin{aligned} \mathfrak{g} &\subset \text{center of } \mathfrak{N} \\ \langle [X, Y], Z \rangle &= \langle j(Z)X, Y \rangle \quad \text{for } X, Y \in \mathcal{V} \text{ and } Z \in \mathfrak{g}. \end{aligned}$$

Observe that $j(Z)$ is skew-symmetric as our inner product $\langle \cdot, \cdot \rangle$ on \mathcal{V} is G -invariant; hence, the bracket $[\cdot, \cdot]$ is skew-symmetric. By construction $[\mathfrak{N}, \mathfrak{N}] \subset \mathfrak{Z}(\mathfrak{N})$, the center of \mathfrak{N} , and the bracket satisfies the Jacobi condition trivially; thus $[\cdot, \cdot]$ defines a Lie algebra structure on \mathfrak{N} . Moreover, this automatically makes the Lie algebra \mathfrak{N} into a two-step nilpotent Lie algebra. There is a more general construction of two-step nilpotent Lie algebras that includes this one, cf. Chapter 7.

We denote a metric algebra by a pair $\{\mathfrak{N}, \langle \cdot, \cdot \rangle\}$, where $\langle \cdot, \cdot \rangle$ is an inner product on \mathfrak{N} . Let \mathcal{V} denote the orthogonal complement of $[\mathfrak{N}, \mathfrak{N}]$ in \mathfrak{N} . Consider the following linear map $j : [\mathfrak{N}, \mathfrak{N}] \rightarrow \mathfrak{so}(\mathcal{V})$ defined by

$$(3.1) \quad \langle [X, Y], Z \rangle = \langle j(Z)X, Y \rangle \quad \text{for } X, Y \in \mathcal{V} \text{ and } Z \in [\mathfrak{N}, \mathfrak{N}].$$

In the case that our two-step nilalgebra is constructed from a representation of a compact semi-simple group, this j map would be a Lie algebra homomorphism.

Facts about compact semisimple Lie algebras. This information comes from [Ebe05] where complete descriptions of real weight space decompositions have been given for real representations of real Lie algebras. Our notation is consistent with Eberlein's and we refer the reader to that paper for definitions; this paper may be found on Eberlein's website at www.math.unc.edu.

DEFINITION 3.35. *Let G be a compact Lie group. A nonzero element X of the Lie algebra \mathfrak{g} is regular if $\dim Z(X) \leq \dim Z(Y)$ for all nonzero $Y \in \mathfrak{g}$, where $Z(X)$ denotes the centralizer of X in \mathfrak{g} .*

If $X \in \mathfrak{g}$ is regular, then $Z(X)$ is a maximal abelian subalgebra of \mathfrak{g} .

Root Space Decomposition of Compact Semisimple Lie Algebras. We say that a finite dimensional real Lie algebra \mathfrak{g}_0 is *compact* and *semisimple* if the Killing form B_0 of \mathfrak{g}_0 is negative definite. It is known that any connected Lie group G_0 with Lie algebra \mathfrak{g}_0 must be compact.

Let \mathfrak{g}_0 be a compact, semisimple Lie algebra. Let $\mathfrak{g} = \mathfrak{g}_0^\mathbb{C}$ and let $J_0 : \mathfrak{g} \rightarrow \mathfrak{g}$ denote the conjugation determined by \mathfrak{g}_0 . If \mathfrak{h}_0 is a maximal abelian subspace of \mathfrak{g}_0 , then $\mathfrak{h} = \mathfrak{h}_0^\mathbb{C}$ is a Cartan subalgebra of \mathfrak{g} , and we have the *root space decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\beta \in \Phi} \mathfrak{g}_\beta$, where each \mathfrak{g}_β is 1-dimensional over \mathbb{C} and $\Phi \subset \text{Hom}(\mathfrak{h}, \mathbb{C})$ is a finite set of *roots* determined by \mathfrak{h} . We obtain the *real root space decomposition* $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \sum_{\beta \in \Phi^+} \mathfrak{g}_{0,\beta}$, where $\mathfrak{g}_{0,\beta} = (\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}) \cap \mathfrak{g}_0$ is 2-dimensional over \mathbb{R} . If X_β spans \mathfrak{g}_β , then $\mathfrak{g}_{0,\beta}$ has a natural basis $\{A_\beta, B_\beta\}$, where $A_\beta = X_\beta + J_0(X_\beta)$ and $B_\beta = i(X_\beta - J_0(X_\beta))$.

If $H_\beta \in \mathfrak{h}_0$ is chosen appropriately, then

$$\begin{aligned} [H_\beta, A_\beta] &= 2B_\beta \\ [H_\beta, B_\beta] &= -2A_\beta \\ [A_\beta, B_\beta] &= -2B(X_\beta, J_0(X_\beta))H_\beta \end{aligned} \tag{3.2}$$

where B denotes the Killing form of \mathfrak{g} .

Weight Space Decompositions. Let \mathfrak{g}_0 , $\mathfrak{g} = \mathfrak{g}_0^\mathbb{C}$, \mathfrak{h}_0 , and $\mathfrak{h} = \mathfrak{h}_0^\mathbb{C} \subseteq \mathfrak{g}$ be as above.

If U is a finite dimensional \mathfrak{g}_0 -module, then $V = U^\mathbb{C}$ is a finite dimensional \mathfrak{g} -module and we have the *weight space decomposition* $V = V_0 \oplus \sum_{\lambda \in \Lambda} V_\lambda$, where for $H \in \mathfrak{h}$, $H \equiv 0$ on V_0 and $H = \lambda(H)Id$ on V_λ . Here $\Lambda \subseteq \text{Hom}(\mathfrak{h}, \mathbb{C})$ is a finite set of *weights* determined by \mathfrak{h} . If $U = \mathfrak{g}_0$ is the adjoint representation of \mathfrak{g}_0 , then $\Lambda = \Phi$ and the weight space decomposition is the root space decomposition. In general $-\lambda \in \Lambda$ if $\lambda \in \Lambda$ since (V, \mathfrak{g}) is the complexification of (U, \mathfrak{g}_0) . If Λ^+ is a subset of Λ that contains exactly one of $\{\lambda, -\lambda\}$ for each $\lambda \in \Lambda$, then we obtain the *real weight space decomposition* $U = U_0 \oplus \sum_{\lambda \in \Lambda^+} U_\lambda$, where $U_0 = V_0 \cap U$ and $U_\lambda = (V_\lambda \oplus V_{-\lambda}) \cap U$ for $\lambda \in \Lambda$. (Note that $U_\lambda = U_{-\lambda}$). Moreover, we obtain

$$\begin{aligned} H_\beta &: U_\lambda \rightarrow U_\lambda \\ A_\beta, B_\beta &: U_\lambda \rightarrow U_{\lambda+\beta} \oplus U_{\lambda-\beta} \end{aligned} \tag{3.3}$$

The following facts from Eberlein [Ebe05], Propositions 6.3 and 7.2, are useful.

FACT 3.36. a) $U_\lambda = \{u \in U \mid H_0^2 u = -(i\lambda)^2(H_0)u, \forall H_0 \in \mathfrak{h}_0\}$
b) $\dim U_\lambda = 2$ if $\lambda \neq 0$
c) A_β and B_β are non-singular on U_λ , when $\lambda(H_\beta) \neq 0$.
Moreover, $A_\beta(v)$ and $B_\beta(v)$ are linearly independent
for $v \in U_\lambda$.

Derivations of \mathfrak{N} . Recall the definition of a derivation of a Lie algebra. An endomorphism $D \in \text{End}(\mathfrak{N})$ is said to be a *derivation* of the Lie algebra \mathfrak{N} if $D[X, Y] = [DX, Y] + [X, DY]$. We let $\text{Symmder}(\mathfrak{N})$ denote the vector space of symmetric derivations of \mathfrak{N} . Let $D \in \text{Symmder}(\mathfrak{N})$; then D preserves the commutator $[\mathfrak{N}, \mathfrak{N}]$ and hence it also preserves the orthogonal complement \mathcal{V} . Thus for a symmetric derivation, we can write $D : \mathfrak{N} \rightarrow \mathfrak{N}$ as

$$\begin{aligned} D_1 &= D|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V} \\ D_2 &= D|_{[\mathfrak{N}, \mathfrak{N}]} : [\mathfrak{N}, \mathfrak{N}] \rightarrow [\mathfrak{N}, \mathfrak{N}] \end{aligned}$$

Facts about $\mathfrak{su}(2)$. If $\mathfrak{g}_0 = \mathfrak{su}(2)$, a 3-dimensional real simple Lie algebra, then $\mathfrak{g} = \mathfrak{g}_0^\mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$ has only one positive root $\beta \in \Phi^+$ relative to a 1-dimensional Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0^\mathbb{C}$, where \mathfrak{h}_0 is any 1-dimensional subspace of \mathfrak{g}_0 . If $G_0 = SU(2)$, then the adjoint action of G_0 on \mathfrak{g}_0 is transitive on the 1-dimensional subspaces \mathfrak{h}_0 of \mathfrak{g}_0 . Given \mathfrak{h}_0 we obtain a basis $\{H_\beta, A_\beta, B_\beta\}$ of $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{g}_{0,\beta}$ that satisfies the bracket relations in Equation (3.2) above. We fix this basis for the remainder of this discussion.

If H_0 is any nonzero element of $\mathfrak{su}(2)$ then $\mathfrak{h} = \mathbb{C} - \text{span}\{H_0\}$ is a Cartan subalgebra for $\mathfrak{sl}(2, \mathbb{C})$. If U is an $\mathfrak{su}(2)$ module and $V = U^\mathbb{C}$, then by the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ the weight spaces V_λ , $\lambda \in \Lambda$, are 1-dimensional and $V_0 = \{0\}$ or 1-dimensional depending on whether $\dim_\mathbb{C} V = \dim_\mathbb{R} U$ is even or odd, respectively.

All weights are of the form $\lambda = k\beta$, $k \in \mathbb{Z} - \{0\}$. See [Hum81, § 13.1, Table 1].

In the sequel we let \mathcal{V} denote the representation space for \mathfrak{g}_0 , so that $\mathcal{V}_0 = V_0 \cap \mathcal{V}$ and $\mathcal{V}_\lambda = (V_\lambda \oplus V_{-\lambda}) \cap \mathcal{V}$, where $V = \mathcal{V}^\mathbb{C}$.

FACT 3.37. *Every element of $\mathfrak{su}(2)$ is regular.*

Main Theorem.

THEOREM. *Let $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{su}(2)$ be constructed as above where \mathcal{V} is any irreducible representation space other than the adjoint representation space $\mathfrak{su}(2)$. Then the symmetric derivations of \mathfrak{N} are 1 dimensional; that is, if $D = (D_1, D_2) \in \text{Symmder}(\mathfrak{N})$ then $(D_1, D_2) = (rId, 2rId)$ for some $r \in \mathbb{R}$.*

This theorem is stated as Theorem 3.43. At the end of the section we demonstrate the value of such a theorem. In short, any other metric two-step nilalgebra ‘sufficiently close’ to $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{su}(2)$ will admit a so-called optimal metric. Before proving this theorem we need some technical preliminaries.

Rewriting the derivation condition for a symmetric derivation D on $\mathfrak{N} = \mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$ we obtain

$$j(D_2 Z) = j(Z)D_1 + D_1 j(Z) \text{ for all } Z \in [\mathfrak{N}, \mathfrak{N}]$$

Next pick an orthogonal basis $\{X_i\}$ of $[\mathfrak{N}, \mathfrak{N}]$ such that D_2 is diagonal. Suppressing the j map, we have

$$(3.4) \quad D_1 X_i + X_i D_1 = a_i X_i$$

where $D_2 = \text{diag}\{a_i\}$ relative to $\{X_i\}$.

LEMMA 3.38. *In the case that \mathfrak{N} is built from an irreducible $\mathfrak{su}(2)$ representation, other than the adjoint representation, we have $a_1 = a_2 = a_3$.*

As every element $X \in \mathfrak{su}(2)$ is regular (cf. Fact above), we can take any $\mathfrak{h}_0 = \mathbb{R} - \text{span}\langle X_i \rangle$ to be a maximal abelian subalgebra whose complexification is a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$.

SUBLEMMA 3.39. *The following elements of $\mathfrak{su}(2)$ are mutually orthogonal $H_\beta \perp A_\beta \perp B_\beta$.*

As the $\text{Ad } SU(2)$ -invariant inner product on $\mathfrak{su}(2)$ is unique up to scaling, we may assume that we are working with (a multiple of) $-B_0$, where B_0 denotes the (negative definite) Killing form on $\mathfrak{su}(2)$; that is, $\langle Z_1, Z_2 \rangle = -\text{tr}(\text{ad } Z_1 \circ \text{ad } Z_2)$.

From Equation (3.2) we compute $|H_\beta|^2 = 8$ and $|A_\beta|^2 = |B_\beta|^2 = -8B(X_\beta, J_0(X_\beta))$ where B denotes the Killing form of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

PROOF. We will only show that $H_\beta \perp A_\beta$ as $H_\beta \perp B_\beta$ is similar. Consider the inner product $\langle H_\beta, A_\beta \rangle = -B_0(H_\beta, A_\beta) = -\text{trace}(\text{ad } H_\beta \circ \text{ad } A_\beta)$. By Equation (3.2) the matrix of $\text{ad } H_\beta \circ \text{ad } A_\beta$ relative to $\{H_\beta, A_\beta, B_\beta\}$ has all diagonal entries zero; hence the matrix is traceless.

Next we show $A_\beta \perp B_\beta$. Consider the inner product $\langle A_\beta, B_\beta \rangle = -B_0(A_\beta, B_\beta) = -\text{trace}(\text{ad } A_\beta \circ \text{ad } B_\beta)$. As above, using Equation (3.2), the matrix of $\text{ad } A_\beta \circ \text{ad } B_\beta$ relative to $\{H_\beta, A_\beta, B_\beta\}$ has all diagonal entries zero; hence the matrix is traceless. \square

Focusing on $\mathfrak{h}_0 = \mathbb{R} - \text{span}\langle X_1 \rangle$ we see that $X_1 \perp X_2, X_3$ as these are eigenvectors of the symmetric derivation D , cf. Equation (3.4). This in turn tells us that $X_2, X_3 \in \mathbb{R} - \text{span}\langle A_\beta, B_\beta \rangle$.

SUBLEMMA 3.40. *In Lemma 3.38 we may assume that $X_2 = A_\beta$ and $X_3 = B_\beta$ (up to scaling).*

PROOF. Consider the action of $SU(2)$ on \mathfrak{N} . Here we have $k.(v, Z) = (kv, kZk^{-1})$; that is, $SU(2)$ is acting on \mathcal{V} in the usual way (according to the irreducible representation at hand) and on $\mathfrak{Z} = \mathfrak{su}(2)$ by the Adjoint action. Since the j -map here is a representation, this action of $SU(2)$ is by automorphisms of \mathfrak{N} ; moreover, this action is by isometries of $\langle \cdot, \cdot \rangle$. Thus we have an induced action of $SU(2)$ on $\text{Der}(\mathfrak{N})$ given by $(k \cdot D) = kDk^{-1}$ where $k \in SU(2)$, $D \in \text{Der}(\mathfrak{N})$, and $v \in \mathfrak{N}$.

Recall that $\text{Symmder}(\mathfrak{N})$ denotes the symmetric derivations of $\{\mathfrak{N}, \langle \cdot, \cdot \rangle\}$. Since $SU(2)$ acts by isometries, $k \cdot D = kDk^{-1}$ is symmetric if $D \in \text{Symmder}(\mathfrak{N})$.

Observe that $k \cdot D$ has the same eigenvalues as D . Thus, to prove the Lemma, we just need show that there exists $k \in SU(2)$ such that

$$\begin{aligned} k.X_1 &= X_1 \\ k.X_2 &= A_\beta \\ k.X_3 &= B_\beta \end{aligned}$$

We use $k = \exp(tX_1)$ for some t . Notice that X_2, X_3 are orthogonal, contained in $\text{span}\langle A_\beta, B_\beta \rangle$ and $k = \exp(tX_1)$ is just rotation in this plane. Multiply X_2, X_3 by suitable constants so that $|X_2| = |X_3| = |A_\beta| = |B_\beta|$. Thus we can find such a t so that the equations above hold. Since the derivation $k.D$ has the same eigenvalues as D , the eigenvectors contained in $[\mathfrak{N}, \mathfrak{N}]$ have the same eigenvalue for $k \cdot D$ if and only if they do so for D . This proves the Lemma. \square

SUBLEMMA 3.41. *The following is a list of useful properties:*

- a) D_1 preserves \mathcal{V}_λ for all $\lambda \in \Lambda$
- b) Let μ_i be an eigenvalue of D_1 with eigenvector $v_i \in \mathcal{V}_{\lambda_i}$, $\lambda_i \neq 0$.
 - i) Then $X_1 v_i$ is also an eigenvector of D_1 in \mathcal{V}_{λ_i} with eigenvalue $a_1 - \mu_i$
 - ii) If $j = 2, 3$ and $X_j v_i \neq 0$, then $X_j v_i$ is an eigenvector of D_1 in $\mathcal{V}_{\lambda_i - \beta} \oplus \mathcal{V}_{\lambda_i + \beta}$ with eigenvalue $a_j - \mu_i$

PROOF. (a) It follows immediately from Equation (3.4) that $X_i^2 D_1 = D_1 X_i^2$ for $i = 1, 2, 3$. The characterization of \mathcal{V}_λ for $\mathfrak{h}_0 = \mathbb{R} - \text{span}\{X_1\}$ from Fact 3.36(a) now implies part (a) of Proposition 3.41.

Remark. Assertion (a) holds for any maximal abelian subspace $\mathfrak{h}_0 = \mathbb{R} - \text{span}\{X\}$ such that $X \in \mathfrak{su}(2)$ is an eigenvector of D_2 . We record this observation now as it will be useful later.

We will prove parts (b)(i) and (b)(ii) at the same time. However, even though these have the same proof, it is worth while for us to list them separately in the Proposition. Applying both sides of Equation (3.4) to v_i , we have $D_1 X_j v_i = (a_j - \mu_i) X_j v_i$. \square

PROOF OF LEMMA 3.38. From now on λ will denote the highest weight of our representation of $\mathfrak{sl}(2, \mathbb{C})$ on $V = \mathcal{V}^\mathbb{C}$. Also, recall that we are considering all irreducible representations except the adjoint representation; that is, $\lambda - \beta \neq 0$. Since λ is the highest weight, $V_{\lambda + \beta} = \{0\}$ and hence $\mathcal{V}_{\lambda + \beta} = (V_{\lambda + \beta} \oplus V_{-(\lambda + \beta)}) \cap \mathcal{V} = \{0\}$. This implies $X_2, X_3 : \mathcal{V}_\lambda \rightarrow \mathcal{V}_{\lambda - \beta}$ by Sublemma 3.40 and Equation (3.3).

Let $v \in \mathcal{V}_\lambda$ be an eigenvector of D_1 with eigenvalue μ . By Fact 3.36(c) and Sublemma 3.41 (b)(ii) we see that the following are also eigenvalues of D_1 whose nonzero eigenvectors live in $\mathcal{V}_{\lambda-\beta}$:

$$\begin{aligned} a_2 - \mu &\leftrightarrow X_2 v \\ a_3 - \mu &\leftrightarrow X_3 v \\ a_1 - (a_2 - \mu) &\leftrightarrow X_1 X_2 v \\ a_1 - (a_3 - \mu) &\leftrightarrow X_1 X_3 v \end{aligned}$$

Using the fact that the dimension of $\mathcal{V}_{\lambda-\beta}$ is 2 since $\lambda - \beta \neq 0$ (cf. Fact 3.36(b)), we will show that only two possibilities happen.

LEMMA 3.42. *Either $a_2 = a_3$ or $a_1 = \frac{1}{2}(a_2 + a_3)$.*

PROOF. In the above list of eigenvalues, there can only be 2 distinct numbers in the list of 4 given, as the dimension of $\mathcal{V}_{\lambda-\beta}$ is 2. Suppose $a_2 \neq a_3$. Manipulating the small number of choices in pairing these numbers reveals

$$(3.5) \quad 2\mu = a_2 + a_3 - a_1$$

As the a_i are fixed and μ is any eigenvalue of D_1 on V_λ , we see that

$$(3.6) \quad D_1 \equiv \mu Id \text{ on } \mathcal{V}_\lambda$$

Since μ and $a_1 - \mu$ are eigenvalues for D_1 on V_λ by Sublemma 3.41 (b)(i) we have $a_1 - \mu = \mu$, or $a_1 = 2\mu$. This and the equation above now imply the lemma. Namely, $a_1 = \frac{1}{2}(a_2 + a_3)$. \square

To complete the proof of Lemma 3.38 we recall that any of the X_i can generate a real Cartan subalgebra $\mathfrak{h}_0 = \mathbb{R} - \text{span}\{X_i\}$. Using this observation and the lemma above, we have the following list of possibilities:

$$\begin{aligned} &\text{either } a_2 = a_3 \quad \text{or} \quad a_1 = \frac{1}{2}(a_2 + a_3) \\ &\text{and either } a_1 = a_2 \quad \text{or} \quad a_3 = \frac{1}{2}(a_1 + a_2) \end{aligned}$$

Any four of the above combinations tells us that $a_1 = a_2 = a_3$, as desired. \square

This proves Lemma 3.38 and we have achieved half of our goal, i.e. $D_2 = aId$.

Symmetric Derivations of nilmanifolds arising from representations of compact semi-simple groups.

THEOREM 3.43. *Let $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{su}(2)$ be constructed as above where \mathcal{V} is any irreducible representation other than the adjoint representation. Then the symmetric derivations of \mathfrak{N} are 1 dimensional. That is, if $D = (D_1, D_2) \in \text{Symmder}(\mathfrak{N})$ then $(D_1, D_2) = (rId, 2rId)$ for some $r \in \mathbb{R}$.*

THEOREM 3.44. *Let $\mathfrak{N} = V \oplus \mathfrak{so}(n)$ be the two-step nilmanifold constructed from usual representation of $SO(n)$ acting on $V = \mathbb{R}^n$. Then $\text{Symmder}(\mathfrak{N}) \simeq \text{Symm}(n)$, the symmetric $n \times n$ matrices.*

Remark. The correspondence \simeq in the statement of Theorem 3.44 is explained in the proof. Theorem 3.44 is why Theorem 3.43 fails in the case that we have the adjoint representation for $\mathfrak{su}(2)$; that is, the case when $q = 3$.

PROOF OF THEOREM 3.43. Lemma 3.38 reduces to the case that D_2 is a multiple of the identity. Moreover, in this case every $X \in \mathfrak{su}(2)$ is an eigenvector of D_2 and hence D_1 preserves the weight space decomposition with respect to any maximal abelian subspace $\mathfrak{h}_0 = \mathbb{R} - \text{span}\{X\} \subset \mathfrak{g}_0 = \mathfrak{su}(2)$ (cf. the remark following Sublemma 3.41). If \mathcal{V}_λ is the highest weight space with respect to \mathfrak{h}_0 , then $k \cdot \mathcal{V}_\lambda$ is the highest weight space with respect to $k \cdot \mathfrak{h}_0$ for every $k \in SU(2)$.

Fix our choice of \mathfrak{h}_0 for the moment. By Lemma 3.38 and Equation (3.6) we may let $D_2 = 2r \text{Id}$ and let $D_1 = \mu \text{Id}$ on \mathcal{V}_λ , where λ is the highest weight space with respect to the maximal abelian subalgebra \mathfrak{h}_0 . Then Equations (3.5) and (3.6) imply $\mu = r$ since $a_1 = a_2 = a_3 = 2r$. Observe that since r is independent of our choice of \mathfrak{h}_0 we see that

$$(3.7) \quad D_1 = r \text{Id} \text{ on } k \cdot \mathcal{V}_\lambda \quad \text{for all } k \in SU(2)$$

This follows as $k \cdot \mathcal{V}_\lambda$ is the highest weight space corresponding to the maximal abelian subalgebra $\text{Ad}(k)\mathfrak{h}_0$.

Fix \mathfrak{h}_0 and hence a highest weight space \mathcal{V}_λ . Pick some nonzero $v \in \mathcal{V}_\lambda$. As the representation \mathcal{V} is an irreducible $SU(2)$ -module, the set $\{k \cdot v | k \in SU(2)\}$ spans the vector space \mathcal{V} . Thus $D_1 = r \text{Id}$ by Equation (3.7) and we have shown

$$(D_1, D_2) = (r \text{Id}, 2r \text{Id})$$

□

Remark. The proof for $\mathfrak{g}_0 = \mathfrak{su}(2)$ used the facts that every element X of \mathfrak{g}_0 is regular and every nonzero weight space is 2-dimensional for every irreducible \mathfrak{g}_0 -module. There are no analogues of these facts for an arbitrary compact semisimple Lie algebra \mathfrak{g}_0 , and the extension of Theorem 3.43 to $\mathfrak{g}_0 \neq \mathfrak{su}(2)$ becomes quite a challenge.

PROOF OF THEOREM 3.44. First we explain the correspondence. Let $X \in \text{Symm}(n)$, the symmetric $n \times n$ matrices. We associate to X an endomorphism of \mathfrak{N} and show this to be a derivation. Define $D = (D_1, D_2)$ by $D_1(v) = Xv$ for $v \in \mathcal{V} = \mathbb{R}^n$ and $D_2(Z) = X \cdot Z = XZ + ZX$ for $Z \in \mathfrak{Z} = \mathfrak{so}(n)$. We endow \mathfrak{N} with the inner product that makes the standard basis of $\mathcal{V} = \mathbb{R}^n$ orthonormal and is the negative trace form on $\mathfrak{Z} = \mathfrak{so}(n)$. Observe that D is symmetric with respect to this inner product on \mathfrak{N} .

Recall from the discussion preceding Equation (3.4) that D is a symmetric derivation if it satisfies $j(D_2 Z) = D_1 j(Z) + j(Z) D_1$; here the representation j is the inclusion map $j : \mathfrak{so}(n) \hookrightarrow \mathfrak{gl}(n)$. By definition $j(D_2 Z) = j(X \cdot Z) = j(XZ + ZX) = XZ + ZX = Xj(Z) + j(Z)X = D_1 j(Z) + j(Z) D_1$ and hence D

is a symmetric derivation. To see that every symmetric derivation is of this form, let $D = (D_1, D_2)$ be a symmetric derivation; here $D_1 \in \text{Symm}(\mathcal{V}) = \text{Symm}(n)$ and $D_2 \in \text{Symm}(\mathfrak{z})$. As D satisfies $j(D_2 Z) = D_1 j(Z) + j(Z) D_1$ and the inclusion map j has no kernel, we see that $D_2(Z) = D_1 Z + Z D_1 = D_1 \cdot Z$ as claimed. \square

The relationship between metric two-step nilpotent Lie algebras and points in the representation space $SL(p, \mathbb{R}) \times SL(q, \mathbb{R}) \curvearrowright \mathfrak{so}(q, \mathbb{R})^p$ is explained in Chapter 7. We state the following theorem here for the reader familiar with that relationship.

THEOREM 3.45. *Consider the action of $SL(p, \mathbb{R}) \times SL(q, \mathbb{R})$ on $\mathfrak{so}(q, \mathbb{R})^p$. Given $C = (C^1, \dots, C^p) \in \mathfrak{so}(q, \mathbb{R})^p$ with $\{C^i\}$ linearly independent, we can associate to C a metric two-step nilpotent Lie algebra with structure matrices C . This metric two-step nilalgebra is denoted by $\mathbb{R}^{p+q}(C)$ (cf. Chapter 7). The set $\mathfrak{p}_C = \{X \in \mathfrak{p} \mid X \cdot C = 0\}$ corresponds to the traceless symmetric derivations of $\mathbb{R}^{p+q}(C)$ (cf. Proposition 7.6). Thus, in the event that $\mathbb{R}^{p+q}(C)$ is optimal and there are no traceless symmetric derivations of $\mathbb{R}^{p+q}(C)$, we have that $M(C) < 0$ (cf. Corollary 3.14). Hence there exists a dense open set of algebras admitting optimal metrics (cf. Corollary 3.13 and Proposition 7.9).*

Remark. This statement can be improved upon using Popov's results, cf. Theorem 3.4, and the works of Knop-Littelmann, cf. [KL87].

Additionally, one can extend the arguments of Theorem 3.43 to a wider class of representations of $\mathfrak{su}(2)$ which are reducible. For example the results of Theorem 3.43 hold for a representation $V^{\lambda_1} \oplus V^{\lambda_2}$ when $|\lambda_1 - \lambda_2| \geq 2\beta$. In this way we can show that for all types $(3, q)$, with $q \neq 3, 6, 8$, there exists a dense open set of points in $\mathfrak{so}(q, \mathbb{R})^3$ whose corresponding metric nilalgebras admit optimal metrics. We leave the proof of this extension of Theorem 3.43 to the reader.

Good Representations and Homogeneous Spaces

This chapter is written to stand alone. However, it contains useful examples that are referenced elsewhere in the thesis.

Recall that if F is a complex reductive affine algebraic group acting on a complex affine variety X , there exists a “good quotient” $X//F$ from Geometric Invariant Theory (GIT). Here $X//F$ is an affine variety together with a quotient morphism $\pi : X \rightarrow X//F$ which is a regular map between varieties. The variety $X//F$ has as its ring of regular functions $\mathbb{C}[X//F] = \mathbb{C}[X]^F$, the F -invariant polynomials on X . Moreover, the quotient map is the morphism corresponding to the injection $\mathbb{C}[X]^F \hookrightarrow \mathbb{C}[X]$. See [New78] for a detailed introduction to Geometric Invariant Theory and quotients.

Good quotients are categorical quotients (see [New78, Chapter 3]). As a consequence they possess the following universal property which will be needed later. Let $\phi : X \rightarrow Z$ be a morphism which is constant on F -orbits. Then there exists a unique morphism $\varphi : X//F \rightarrow Z$ such that $\varphi \circ \pi = \phi$.

Let G be a complex reductive affine algebraic group. Let F, H be algebraic reductive subgroups. The homogeneous space G/F has a natural, transitive left action of G on it. We will consider the induced action of H on G/F .

The group F acts on G via $f \cdot g = gf^{-1}$. This gives a left action of F on G such that every orbit is closed. In this way the GIT quotient $G//F$ is a parameter space; that is, every G -orbit is closed. If one considers the analytic topologies on G and $G//F$ one readily sees that $G//F$ and G/F (with the usual Hausdorff quotient topology) are homeomorphic. In this way we endow G/F with a Zariski topology. Here and in later discussion we identify the coset space G/F with the variety $G//F$. Moreover, it will be shown that the natural G -action on G/F is algebraic.

Hereafter a property of a space will be called generic if it occurs on a nonempty Zariski open set. Our main result is the following.

The following theorem and its corollaries are true for both real and complex algebraic groups.

THEOREM 4.1. *Consider the induced action of H on G/F , then generic H -orbits are closed in G/F ; that is, there is a nonempty Zariski open set of G/F such that the H -orbit of any point in this open set is closed.*

COROLLARY 4.2. *Let G, H, F be as above. If H is normal in G , then all orbits of H are closed in G/F . Consequently, if G acts on V and the orbit Gv is closed, then Hv is also closed.*

COROLLARY 4.3. *Let G be a reductive algebraic group. If H, F are generic reductive subgroups, then $H \cap F$ is also reductive. More precisely, take any two reductive subgroups H, F of G . Then $H \cap gFg^{-1}$ is reductive for generic $g \in G$.*

Remark. The word generic cannot be replaced by all. We show this in Example 4.7. Theorem 4.1 is proven first for complex groups then deduced for real groups. It is not known at this time, to the author, whether or not these results hold true for more general algebraic groups. Our proof exploits Weyl's Unitarian Trick.

Before proving this theorem, we present some corollaries to demonstrate its value. Proofs of these results have been placed at the end.

COROLLARY 4.4. *Let G be a reductive group acting linearly on V . Let H be a reductive subgroup of G . If G has generically closed orbits then H does also. Moreover, each closed G -orbit is stratified by H -orbits which are generically closed.*

We say that a representation V of G is *good* if generic G -orbits in V are closed.

COROLLARY 4.5. *Let G be a reductive group, and let V and W be good G -representations, that is, generic G -orbits are closed. Then $V \oplus W$ is also a good G -representation.*

This corollary is of particular interest as it allows us to build good representations from smaller ones. The idea of building good representations from subrepresentations was also carried out in [EJ, Section 3]. In that setting, the representations of interest are those that have points whose real Mumford numerical function is negative. The results of the current paper generalize some of those results.

EXAMPLE 4.6 (non-closed orbits of smaller groups). *Consider the 3 – dim irreducible representation V_3 of $SL_2\mathbb{R}$ acting on the homogeneous degree 2 forms of 2 variables.*

Here $g \in SL_2$ acts on $f \in V_3$ by $(g \cdot f)(x, y) = f(g^{-1}(x, y))$. Letting $\{xy, x^2, y^2\}$ be an orthonormal basis, one can easily show that $v = xy$ is a minimal vector. Thus $SL_2 \cdot v$ is closed in V_3 . So for any $g \in SL_2$, gv also has a closed SL_2 orbit and thus (v, gv) has a closed $SL_2 \times SL_2$ orbit in $V_3 \oplus V_3$. Consider $g = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and the point $w = (v, gv) = (xy, xy + y^2) \in V_3 \times V_3$. Let $H = SL_2$ with the diagonal embedding in $SL_2 \times SL_2$. We claim that $H \cdot w$ is not closed. To see this, observe that by means of the diagonal group $\text{diag}\{\lambda, \lambda^{-1}\} \subset H$ the point $(v, v) \in \overline{Hw}$. Hence Hw is not closed.

EXAMPLE 4.7 (non-reductive intersection and non-closed orbit). *There exist semi-simple G and reductive subgroups H, F such that $H \cap F$ is not reductive. Additionally, we demonstrate a representation V of G so that $G \cdot x$ is closed but $H \cdot x$ is not closed, for some $x \in V$.*

Recall the following well-known fact. Let G be a reductive affine algebraic group acting on an affine variety. If the orbit $G \cdot x$ is closed then G_x is reductive, see [BHC62, Theorem 3.5] or [RS90, Theorem 4.3]. We will choose $F = G_x$ for a particular $x \in V$. Then $H \cap F = H \cap G_x = H_x$. Once it is shown that H_x is not reductive, the orbit $H \cdot x$ cannot be closed by the fact stated above.

Consider $G = SL_6(\mathbb{C})$ acting on $V = \bigwedge^2 \mathbb{C}^6 \simeq \mathfrak{so}(6, \mathbb{C})$. This is the usual action and is described as follows. For $M \in \mathfrak{so}(6, \mathbb{C})$ and $g \in SL_6(\mathbb{C})$, the action is defined as $g \cdot M = gMg^t$. The subgroup $H = SL_2(\mathbb{C})$ is imbedded as the upper left 2×2 block.

Let $v \in V$ be the block diagonal matrix consisting of the blocks $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ along the diagonal. That is, $v = \begin{bmatrix} J & & \\ & J & \\ & & J \end{bmatrix}$. Given the standard inner product (from the trace form) on V , the vector v is a so-called minimal vector as $v^2 = -Id$, thus $G \cdot v$ is closed. See [EJ, Example 1] for details and more information on minimal vectors; see also [RS90]. Consider $x = g \cdot v$ where

$$g = \begin{bmatrix} 1 & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & 1 & & & \\ & & & Id_{3 \times 3} & & \end{bmatrix}$$

Since G_v is reductive, $G_x = gG_vg^{-1}$ is also reductive. One can compute H_x to show that $H_x \simeq \mathbb{C}_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$. This group is clearly not reductive and we have the desired example.

1. Technical Lemmas

We recall the definition of varieties and morphisms which are defined over \mathbb{R} . This is the setting that we will primarily work in. See [Bor91, §§11–14] or [Mar91, Chapter 1, 0.10] for more information on varieties and k -structures on varieties.

DEFINITION 4.8 (Real points of affine subvarieties). *An affine subvariety M of \mathbb{C}^n is the zero set of a collection of polynomials on \mathbb{C}^n . The variety M is said to be defined over \mathbb{R} if M is the zero set of polynomials whose coefficients are real. Thus $\mathbb{C}[M] = \mathbb{R}[M] \otimes_{\mathbb{R}} \mathbb{C}$. The real points of M are defined as the set $M(\mathbb{R}) = M \cap \mathbb{R}^n$; we call such a set a real variety.*

DEFINITION 4.9 (\mathbb{R} -structures). *Given an abstract affine variety X , one defines a \mathbb{R} -structure on X by means of an isomorphism $\alpha : X \rightarrow M$. A morphism $f : X \rightarrow Y$ of \mathbb{R} -varieties is said to be defined over \mathbb{R}*

if the comorphism $f^* : \mathbb{C}[Y] \rightarrow \mathbb{C}[X]$ satisfies $f^*(\mathbb{R}[Y]) \subset \mathbb{R}[X]$. Additionally, we define the real points of X to be $X(\mathbb{R}) = \alpha^{-1}(M(\mathbb{R}))$.

Remark. Let $M \subset \mathbb{C}^m$ and $N \subset \mathbb{C}^n$ be subvarieties defined over \mathbb{R} . Then $f : M \rightarrow N$ being defined over \mathbb{R} implies $f(M(\mathbb{R})) \subset N(\mathbb{R})$. To obtain the converse one needs M to have an additional property that we call the (RC) property, see Definition 4.10. We state the converse after defining this property.

We observe that a variety can be endowed with many different real structures.

DEFINITION 4.10. *[RC - property] Let X be a complex variety defined over \mathbb{R} . We say that X has the (RC) property (real-complexified) if the real points $X(\mathbb{R})$ are Zariski dense in X .*

This scenario arises precisely if one begins with a real variety $Z \subset \mathbb{R}^n$ and considers the Zariski closure $\bar{Z} \subset \mathbb{C}^n$. Here \bar{Z} has the (RC) property; see [Whi57] for an introduction to real varieties and their complexifications.

PROPOSITION 4.11. *Let $M \subset \mathbb{C}^m$ and $N \subset \mathbb{C}^n$ be subvarieties defined over \mathbb{R} . Assume that M has the (RC) property. Then $f : M \rightarrow N$ being defined over \mathbb{R} is equivalent to $f(M(\mathbb{R})) \subset N(\mathbb{R})$.*

This result is useful but not needed in our proofs; we postpone the proof of this proposition till the end. One immediately sees that the same holds more generally for abstract affine varieties with \mathbb{R} -structures. That is, let X, Y be complex affine varieties defined over \mathbb{R} and $f : X \rightarrow Y$ a morphism. Assume that X has the (RC) property. Then f is defined over \mathbb{R} if and only if $f(X(\mathbb{R})) \subset Y(\mathbb{R})$. Often we will simply say that $f : X \rightarrow Y$ is defined over \mathbb{R} , or f is an \mathbb{R} -morphism, when both varieties and the morphism are defined over \mathbb{R} .

LEMMA 4.12. *Let X be an (abstract) complex affine variety and G a complex algebraic reductive group acting on X , with all defined over \mathbb{R} . Then $G(\mathbb{R})$ acts on $X(\mathbb{R})$ and for $x \in X(\mathbb{R})$ the orbit $G(\mathbb{R}) \cdot x$ is Hausdorff closed in $X(\mathbb{R})$ if and only if $G \cdot x$ is Zariski closed in X .*

Remark. It is well-known that $G \cdot x$ is Hausdorff closed if and only if it is Zariski closed, see [Bor91]. Notice that the above situation arises when we have a real algebraic group acting on a real algebraic variety. This lemma has been proven for linear G actions, see [BHC62, Proposition 2.3] and [RS90]; we reduce to this case.

PROOF. Let G and X be as above. It is well-known that there exists a complex vector space V (defined over \mathbb{R}), a closed \mathbb{R} -imbedding $i : X \hookrightarrow V$, and a representation $T : G \rightarrow GL(V)$ defined over \mathbb{R} such that $i(gx) = T(g)i(x)$ for all $g \in G, x \in X$. See [Bor91, I.1.12] for the construction of such an imbedding.

As all of our objects are defined over \mathbb{R} we see that $G(\mathbb{R})$ acts on $X(\mathbb{R})$, $i(X(\mathbb{R})) \subset V(\mathbb{R})$, and $T : G(\mathbb{R}) \rightarrow GL(V(\mathbb{R}))$ is a real linear representation of $G(\mathbb{R})$, cf. the remark before Definition 4.10.

Now take $x \in X(\mathbb{R})$. We have the following set of equivalences

$$\begin{aligned}
G(\mathbb{R})x & \text{ is closed in } X(\mathbb{R}) \\
i(G(\mathbb{R})x) & \text{ is closed in } V(\mathbb{R}), \text{ as } i \text{ is a closed } \mathbb{R}\text{-imbedding} \\
T(G(\mathbb{R}))i(x) & \text{ is closed in } V(\mathbb{R}) \\
T(G)i(x) & \text{ is closed in } V \text{ by [BHC62, RS90]} \\
i(Gx) & \text{ is closed in } V \\
Gx & \text{ is closed in } X
\end{aligned}$$

□

Richardson and Slodowy [RS90] have shown the following

PROPOSITION 4.13. *Let G be a reductive algebraic group acting on X so that G , X , and the action are defined over \mathbb{R} and consider the quotient morphism $\pi : X \rightarrow X//G$. Then π is defined over \mathbb{R} and $\pi(X(\mathbb{R})) \subset (X//G)(\mathbb{R})$ is Hausdorff closed.*

In general one cannot expect $\pi(X(\mathbb{R}))$ to be all the real points $(X//G)(\mathbb{R})$. However, we make the following simple observation.

LEMMA 4.14. *If X has the (RC) property, then so does $X//G$. In fact $\pi(X(\mathbb{R}))$ is Zariski dense in $X//G$.*

The first statement is proven in [RS90] and the second statement is a special case of a more general statement: Let $f : X \rightarrow Y$ be a regular map and Z a Zariski dense set of X , then $f(Z)$ is Zariski dense in $f(X)$.

PROPOSITION 4.15. *Let G be a reductive algebraic group defined over \mathbb{R} and H, F algebraic reductive subgroups defined over \mathbb{R} . Then the action of H on G/F is defined over \mathbb{R} .*

Before presenting the proof of this proposition we state the following useful lemma.

LEMMA 4.16. *Let $H \times F$ act on a variety X , where H, F, X , and the actions are defined over \mathbb{R} . Then there is a unique H action on $X//F$ defined over \mathbb{R} which makes Diagram A (below) commute.*

PROOF OF LEMMA. Since $H \times F$ acts on X we can consider the F action on $H \times X$. We claim that the map $\pi_1 = id \times \pi_2 : H \times X \rightarrow H \times (X//F)$ is a good quotient; where $\pi_2 : X \rightarrow X//F$ is a good quotient. Here $X//F$ is the variety whose ring of regular functions is $\mathbb{C}[X]^F$, the F -invariant polynomials of $\mathbb{C}[X]$. For a detailed introduction to quotients, see [New78, Chapter 3].

To show that $H \times (X//F)$ is the desired quotient, we will show that $\mathbb{C}[H \times (X//F)] = \mathbb{C}[H \times X]^F$ and that the comorphism $(id \times \pi_2)^*$ is the inclusion map. Recall that $\pi_2^* : \mathbb{C}[X//F] = \mathbb{C}[X]^F \hookrightarrow \mathbb{C}[X]$ is the inclusion map.

There is a natural identification between $\mathbb{C}[H \times X]$ and $\mathbb{C}[H] \otimes \mathbb{C}[X]$ defined by $\sum p_i(h)q_i(x) \mapsto (\sum p_i \otimes q_i)(h, x)$. Under this identification, $\mathbb{C}[H \times X]^F \simeq \mathbb{C}[H] \otimes \mathbb{C}[X]^F$ where $\mathbb{C}[H \times X]^F, \mathbb{C}[X]^F$ denote the F -invariant polynomials in $\mathbb{C}[H \times X], \mathbb{C}[X]$, respectively. The map $id \times \pi_2 : H \times X \rightarrow H \times (X//F)$ corresponds to a comorphism $(id \times \pi_2)^* : \mathbb{C}[H \times (X//F)] \rightarrow \mathbb{C}[H \times X]$ and under the natural identification described above, the map $(id \times \pi_2)^*$ corresponds to $id^* \otimes \pi_2^* : \mathbb{C}[H] \otimes \mathbb{C}[X//F] \rightarrow \mathbb{C}[H] \otimes \mathbb{C}[X]$. This map $id^* \otimes \pi_2^*$ is the inclusion map and is an isomorphism onto $\mathbb{C}[H] \otimes \pi_2^*(\mathbb{C}[X//F]) = \mathbb{C}[H] \otimes \mathbb{C}[X]^F$. Thus, $(id \times \pi_2)^*$ is the inclusion map and maps $\mathbb{C}[H \times (X//F)]$ isomorphically onto $\mathbb{C}[H \times X]^F$. We have shown the following.

$$H \times (X//F) \simeq (H \times X)//F$$

Consider the following diagram. Let m_1 denote the morphism corresponding to H -action on X . Since $\pi_2 \circ m_1$ is constant on F -orbits, by the discussion in the introduction there exists a unique map m_2 which factors and makes the diagram commute.

$$(A) \quad \begin{array}{ccc} H \times X & \xrightarrow{m_1} & X \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ H \times (X//F) & \xrightarrow{m_2} & X//F \end{array}$$

where $\pi_1 = id \times \pi_2$ is the quotient of the F action on $H \times X$, $f \cdot (h, x) = (h, f \cdot x)$. Equivalently, for $h \in H$ and a closed orbit $F \cdot x \subset X$, $h(Fx) = F(hx)$ is a closed F -orbit.

We know that m_1, π_1, π_2 are defined over \mathbb{R} and that $m_2 \circ \pi_1 = \pi_2 \circ m_1$ is defined over \mathbb{R} . From this we wish to show m_2 is also defined over \mathbb{R} . Since $\pi_2^*(\mathbb{R}[X//F]) = \mathbb{R}[X]^F$ we have

$$\pi_1^* \circ m_2^* = m_1^* \circ \pi_2^* : \mathbb{R}[X//F] \rightarrow \mathbb{R}[H \times X]^F$$

Since $\pi_1^* : \mathbb{C}[H \times (X//F)] \rightarrow \mathbb{C}[H \times X]^F$ and $\pi_1^* : \mathbb{R}[H \times (X//F)] \rightarrow \mathbb{R}[H \times X]^F$ are isomorphisms, we have

$$m_2^*(\mathbb{R}[X//F]) \subset \pi_1^*{}^{-1}(\mathbb{R}[H \times X]^F) = \mathbb{R}[H \times X//F]$$

Thus, $m_2 : H \times (X//F) \rightarrow X//F$ is defined over \mathbb{R} , or equivalently, the H action on $X//F$ defined by m_2 is defined over \mathbb{R} . The uniqueness of the H -action on $X//F$ is equivalent to the uniqueness of the map m_2 in Diagram A. \square

PROOF OF THE PROPOSITION. Once it is shown that the G -action on G/F is defined over \mathbb{R} , it will be clear that the H -action is also defined over \mathbb{R} . We apply Lemma 4.16 in the setting that G is a reductive group, $H = G$, $X = G$, and F is a reductive subgroup of G .

Since G is an algebraic group defined over \mathbb{R} , the action $G \times F$ on G defined by $(h, f) \cdot g = hgf^{-1}$ is defined over \mathbb{R} , where $h, g \in G$, $f \in F$. Recall that G/F is the GIT quotient $G//F$ under the F action listed above (notice all the orbits are closed, hence the usual topological quotient coincides with the algebraic quotient). The unique H -action described in Lemma 4.16 is precisely the standard action of G on G/F . Thus we have shown that the usual action of G on G/F is algebraic and defined over \mathbb{R} . \square

2. Transitioning between the Real and Complex Settings: Proof of Theorem 4.1

First we remark on how one obtains Theorem 4.1 for real algebraic groups once it is known for complex groups. Let G, H, F be the same as in Theorem 4.1 but with real groups instead of complex groups. Let $G^{\mathbb{C}}$ denote the Zariski closure of G in $GL(n, \mathbb{C})$. It follows that G is the set of real points of $G^{\mathbb{C}}$, and we call $G^{\mathbb{C}}$ the *algebraic complexification* of G . Likewise, H, F are the real points of their complexifications $H^{\mathbb{C}}, F^{\mathbb{C}}$. Here all of our objects have the (RC)-property.

Consider the G -equivariant imbedding $i : G/F \rightarrow G^{\mathbb{C}}/F^{\mathbb{C}}$, defined by $i : gF \mapsto gF^{\mathbb{C}}$, and the quotient $\pi : G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}/F^{\mathbb{C}}$. Note that i is injective since $G \cap F^{\mathbb{C}} = F$. We view G/F as a subset of $G^{\mathbb{C}}/F^{\mathbb{C}}$ via i and we note that $i(G/F) = \pi(G)$.

As $G/F \simeq \pi(G)$ and $G = G^{\mathbb{C}}(\mathbb{R})$, we see that $G/F \subset (G^{\mathbb{C}}/F^{\mathbb{C}})(\mathbb{R})$ and is Zariski dense in $G^{\mathbb{C}}/F^{\mathbb{C}}$ (see Proposition 4.13 and Lemma 4.14). Moreover, assuming the theorem is true in the complex setting, there exists a Zariski open set $\mathcal{O} \subset G^{\mathbb{C}}/F^{\mathbb{C}}$ such every point in \mathcal{O} has a closed $H^{\mathbb{C}}$ orbit. G/F being Zariski dense intersects \mathcal{O} and so, by Lemma 4.12, we see that all points of $G/F \cap \mathcal{O}$ have closed H -orbits in G/F . This proves Theorem 4.1 in the real case.

To prove the theorem for complex groups, we take advantage of certain real group actions. Let G be a complex reductive group and U a maximal compact subgroup. We can realize U as the fixed points of a Cartan involution θ . Moreover, there exists a real structure on G so that U is the set of real points of G (see [BHC62, Remark 3.4]). Observe that G has the (RC) property as G is the complexification of its compact real form U .

LEMMA 4.17. *We may assume that H, F from our main theorem are θ -stable.*

PROOF. It is well-known that there exist conjugations $g_1 H g_1^{-1}$ and $g_2 F g_2^{-1}$ so that these conjugates are θ -stable, see [BHC62] or [Mos55]. So to prove the lemma, we just need to show that the theorem holds for H, F if and only if it holds for conjugates of these groups.

Observe that G/F and $G/(g_2 F g_2^{-1})$ are isomorphic as varieties via conjugation by $g_2 : gF \mapsto (g_2 g g_2^{-1})(g_2 F g_2^{-1})$. We denote this map by $C(g_2)$. Also observe that G acts via left translation on G/F by variety isomorphisms. Thus the left translate of a closed set in G/F is again a closed set G/F . For $k \in G$ we have $(g_1 H g_1^{-1})k(g_2 F g_2^{-1})$ is closed in $G/(g_2 F g_2^{-1})$ if and only if $C(g_2)^{-1} \cdot ((g_1 H g_1^{-1})k(g_2 F g_2^{-1}))$ is closed in G/F . But $C(g_2)^{-1} \cdot ((g_1 H g_1^{-1})k(g_2 F g_2^{-1})) = g_2^{-1} g_1 H g_1^{-1} k g_2 F$ and this is closed if and only if $H g_1^{-1} k g_2 F$ is closed in G/F .

Thus the $g_1 H g_1^{-1}$ -orbit of $k(g_2 F g_2^{-1})$ is closed in $G/(g_2 F g_2^{-1})$ if and only if the H -orbit of $g_1^{-1} k g_2 F$ is closed in G/F .

□

LEMMA 4.18 (Weyl's Unitarian Trick). *Let G be a complex reductive group and U a maximal compact subgroup. Then U is Zariski dense in G .*

PROOF. This statement and its proof are well-known; we include the proof for completeness.

As U is a maximal compact subgroup, U intersects each topological component of G , see [Mos55, Section 3]. Denote by G_0 and U_0 the Hausdorff identity components of G and U , respectively. Writing $G = \bigcup_{i=1}^n u_i G_0$, where $u_i \in U$ for all i , we have $U = \bigcup_{i=1}^n u_i U'$, where $U' = U \cap G_0$. To show that U is Zariski dense in G it suffices to show that U_0 is Zariski dense in G_0 since $U_0 \subset U'$.

Denote by \overline{U}_0 the Zariski closure of U_0 . This is a complex algebraic group as U_0 is a group, cf. [Bor91]. Since G_0 is an algebraic group we have $\overline{U}_0 \subset G_0$ and we have the following inclusions of Lie algebras $LU_0 \subset L\overline{U}_0 \subset LG_0$. Lastly, since LU_0 is a compact form for LG_0 we see that $LU_0 \otimes_{\mathbb{R}} \mathbb{C} \subset LG_0 = LU_0 \otimes_{\mathbb{R}} \mathbb{C}$. Hence the connected subgroup \overline{U}_0 of G_0 has the same Lie algebra as G_0 and they are equal as G_0 is also connected. □

We continue the proof of Theorem 4.1. Now that H, F are θ -stable, and $U = \text{Fix}(\theta)$, we know that their maximal compact subgroups $U_H = U \cap H, U_F = U \cap F$ are contained in U . Moreover, since $U = G(\mathbb{R})$, the compact subgroups U_H, U_F are the real points of the algebraic groups H, F . Observe that H, F have the (RC) property as their maximal compact subgroups are the real points. For a proof of the following useful fact in the complex setting see [New78]. For an extension to the real setting see Section 2.1.

PROPOSITION 4.19. *Let G be a real or complex linear reductive algebraic group acting on an affine variety X . If there exists a closed orbit of maximal dimension, then there is a Zariski open set of such orbits.*

PROOF OF THEOREM 4.1. We apply the above proposition to the action of H on the affine variety G/F . Note that G/F is affine as G is reductive and F is reductive (see [BHC62, Theorem 3.5]).

As the F -action on G is defined over \mathbb{R} , the quotient G/F is defined over \mathbb{R} . Since our objects have property (RC) the image of the real points of G is dense in G/F by Lemma 4.14; that is, $U/U_F \subset G/F$ is dense. Here, as before, we are identifying U/U_F with the image of U under the quotient $G \rightarrow G/F$.

Moreover, Proposition 4.15 shows that the H -action on G/F is defined over \mathbb{R} . If we let \mathcal{O} denote the set of maximal dimension H -orbits in G/F , then $\mathcal{O} \cap (U/U_F)$ is non-empty. However, U_H is the set of real points for H and every U_H orbit in U/U_F is closed (since they are all compact). Therefore, every point in $\mathcal{O} \cap (U/U_F)$ has a closed H -orbit by Lemma 4.12 and we have found a closed H -orbit of maximal dimension.

Applying Proposition 4.19 we see that generic H -orbits are closed. □

3. Proofs of the Corollaries

We note that the proofs of all results below, except for Proposition 4.11, are valid in the real and complex cases simultaneously.

PROOF COROLLARY 4.2. The theorem above provides some point $kF \in G/F$ which has a closed H -orbit. Take $g \in G$ and consider the point $gkF \in G/F$. The H -orbit of this point is $(H \cdot gk)F = (g g^{-1}H gk)F = (g Hk)F$ which is closed as the G action on G/F is by variety isomorphisms. Hence every H -orbit in G/F is closed as G acts transitively on G/F .

We prove the second statement of the corollary using Corollary 4.4 (which is proven below). In the proof of this corollary it is shown that if $G \cdot v$ is closed in V , then there exists $g \in G$ such that Hgv is closed in V . But now $Hgv = gHv$ by the normality of H . Moreover, gHv is closed in V if and only if Hv is closed in V as G acts by isomorphisms of the vector space. This proves the second part of the proposition. \square

PROOF OF COROLLARY 4.4. We prove the second statement first. Take $v \in V$ such that $G \cdot v$ is closed. It is well-known that G_v is reductive, see, e.g., [RS90, Theorem 4.3] or [BHC62, Theorem 3.5]. The orbit $G \cdot v$ is G -equivariantly isomorphic to the affine variety G/G_v . Thus the H -orbit $H \cdot gv \subset G \cdot v \subset V$ corresponds to $H \cdot gG_v \subset G/G_v$ and for generic g these H -orbits are closed by Theorem 4.1. This proves the second statement.

For the first statement, let $\mathcal{O} = \{ v \in V \mid \dim H \cdot v \text{ is maximal} \}$ and let $\mathcal{U} = \{ v \in V \mid G \cdot v \text{ is closed} \}$. The set \mathcal{O} is a nonempty Zariski open set and by hypothesis \mathcal{U} contains a nonempty Zariski open set. Pick $w \in \mathcal{O} \cap \mathcal{U}$. For generic $g \in G$, the orbit $H \cdot gw$ is closed by the argument of the previous paragraph. Moreover, $gw \in \mathcal{O} \cap \mathcal{U}$ for generic $g \in G$. Thus there exists some point gw which has a closed H -orbit of maximal dimension. Therefore by Proposition 4.19 generic H -orbits in V are closed. \square

PROOF OF COROLLARY 4.5. Take $v \in V$ and $w \in W$ which both have closed G -orbits. Then the $G \times G$ orbit of (v, w) is closed in $V \oplus W$. Now consider the diagonal imbedding of G in $G \times G$. In this way, G acts on $V \oplus W$ and since generic $G \times G$ -orbits in $V \oplus W$ are closed, we see that generic G -orbits in $V \oplus W$ are also closed by Corollary 4.4. \square

PROOF OF COROLLARY 4.3. Let G be a reductive group and let H, F be reductive subgroups. There exists a representation V of G such that the reductive subgroup F can be realized as the stabilizer of a point $v \in V$ and such that the orbit $G \cdot v$ is closed, see [BHC62, Proposition 2.4].

By Corollary 4.4, we know that $H \cdot gv$ is closed for generic $g \in G$. Thus H_{gv} is reductive for generic $g \in G$. But $H_{gv} = H \cap G_{gv} = H \cap gG_v g^{-1} = H \cap gFg^{-1}$ and we have the desired result. \square

PROOF OF PROPOSITION 4.11. First we remark on the direction that does not require M to have the (RC) property; that is, if $f : M \rightarrow N$ is defined over \mathbb{R} then $f(M(\mathbb{R})) \subset N(\mathbb{R})$. To see this direction write $f = (f_1, \dots, f_n)$, where $f_i : \mathbb{C}^m \rightarrow \mathbb{C}$. These component functions are precisely $f_i = f^*(\pi_i)$ where π_i is projection from \mathbb{C}^n to the i -th coordinate. Since this projection is defined over \mathbb{R} we see that the f_i take real values when evaluated at real points. That is, $f(M \cap \mathbb{R}^m) \subset N \cap \mathbb{R}^n$.

Now assume M has the (RC) property and let $f : M \rightarrow N$ be a morphism of varieties such that $f(M(\mathbb{R})) \subset N(\mathbb{R})$. We will show $f^*(\mathbb{R}[N]) \subset \mathbb{R}[M]$; that is, f is defined over \mathbb{R} .

We can describe the polynomial f by its coordinate functions, $f = (f_1, \dots, f_n)$ where $f_i : \mathbb{C}^m \rightarrow \mathbb{C}$ and $f_i|_{M \cap \mathbb{R}^m} \rightarrow \mathbb{R}$. Let \bar{f}_i denote the polynomial whose coefficients are the complex conjugates of those of f_i , then we have $\frac{1}{2}(f_i + \bar{f}_i) = f_i$ on the set $M \cap \mathbb{R}^m$. M having the (RC) property means precisely that $M \cap \mathbb{R}^m$ is Zariski dense in M , thus $\frac{1}{2}(f_i + \bar{f}_i) = f_i$ on M . If we define $P = \frac{1}{2}(f + \bar{f})$ then P has real coefficients and restricted to M equals f .

Take $g \in \mathbb{R}[N]$, then $f^*(g) \in \mathbb{C}[M]$ and $f^*(g) = g \circ f = g \circ P$ on M . Since g and P have real coefficients, so does their composition. That is, $f^*(g) \in \mathbb{R}[M]$. □

Distinguished Orbits of Reductive Groups

We prove a generalization and give a new proof of a theorem of Borel-Harish-Chandra on closed orbits of linear actions of reductive groups. Consider a real reductive algebraic group G acting linearly and rationally on V . G can be viewed as the real points of a complex reductive group $G^{\mathbb{C}}$ which acts on $V^{\mathbb{C}} := V \otimes \mathbb{C}$. In [BHC62] it was shown that $G^{\mathbb{C}} \cdot v \cap V$ is a finite union of G -orbits; moreover, $G^{\mathbb{C}} \cdot v$ is closed if and only if $G \cdot v$ is closed, see [RS90]. We show that the same result holds not just for closed orbits but for the so-called distinguished orbits. An orbit is called distinguished if it contains a critical point of the norm squared of the moment map on projective space. Our main result compares the complex and real settings to show $G \cdot v$ is distinguished if and only if $G^{\mathbb{C}} \cdot v$ is distinguished.

In addition, we show that if an orbit is distinguished, then under the negative gradient flow of the norm squared of the moment map the entire G -orbit collapses to a single K -orbit. This result holds in both the complex and real settings.

1. Introduction

An analytical approach to finding closed orbits in the complex setting was developed by Kempf-Ness [KN78] and extended to the real setting by Richardson-Slodowy [RS90]. From their perspective, the closed orbits are those that contain the zeros of the so-called moment map. However, one can consider more generally critical points of this moment map on projective space. Work on the moment map in the complex setting has been done by Ness [NM84] and Kirwan [Kir84]. Following those works, the real moment map was explored in [Mar01] and [EJ].

Consider a real linear reductive group G acting linearly on V . There is a complex linear reductive group $G^{\mathbb{C}}$ such that G is a finite index subgroup of the real points of $G^{\mathbb{C}}$; moreover, $G^{\mathbb{C}}$ acts on the complexification $V^{\mathbb{C}}$ of V . The linear action of G , respectively $G^{\mathbb{C}}$, extends to an action on real projective space $\mathbb{P}V$, respectively complex projective space $\mathbb{C}\mathbb{P}(V^{\mathbb{C}})$. For $v \in V$, we call an orbit $G \cdot v$, or $G \cdot [v]$, *distinguished* if the orbit $G \cdot [v]$ in real projective space contains a critical point of $\|m\|^2$, the norm square of the real moment map. Similarly, for $v \in V^{\mathbb{C}}$, we call an orbit $G^{\mathbb{C}} \cdot v$, or $G^{\mathbb{C}} \cdot \pi[v]$, *distinguished* if the orbit $G^{\mathbb{C}} \cdot \pi[v]$ in complex projective space contains a critical point of $\|\mu^*\|^2$, the norm square of the complex moment map. Here $\pi : \mathbb{R}P V^{\mathbb{C}} \rightarrow \mathbb{C}\mathbb{P}(V^{\mathbb{C}})$ is the natural projection. Our main theorems are

Theorem 5.7 *Given $G \curvearrowright V$, $G^{\mathbb{C}} \curvearrowright V^{\mathbb{C}}$, and $[v] \in \mathbb{P}V$ we have*

$G \cdot [v]$ is a distinguished orbit in $\mathbb{P}V$ if and only if $G^{\mathbb{C}} \cdot \pi[v]$ is a distinguished orbit in $\mathbb{C}\mathbb{P}(V^{\mathbb{C}})$.

Here $\pi : \mathbb{P}V \subseteq \mathbb{R}\mathbb{P}V^{\mathbb{C}} \rightarrow \mathbb{C}\mathbb{P}(V^{\mathbb{C}})$ is the usual projection.

Theorem 5.9 For $x \in \mathbb{C}\mathbb{P}(V^{\mathbb{C}})$, suppose $G^{\mathbb{C}} \cdot x \subseteq \mathbb{C}\mathbb{P}(V^{\mathbb{C}})$ contains a critical point of $\|\mu^*\|^2$. If $z \in \mathfrak{C} \subseteq \mathbb{C}\mathbb{P}(V^{\mathbb{C}})$ is such a critical point, then $\mathfrak{C} \cap G^{\mathbb{C}} \cdot x = U \cdot z$. Moreover, $U \cdot z = \bigcup_{g \in G^{\mathbb{C}}} \omega(gx)$.

Theorem 5.10 For $x \in \mathbb{P}V$, suppose $G \cdot x \subseteq \mathbb{P}V$ contains a critical point of $\|m\|^2$. If $z \in \mathfrak{C}_{\mathbb{R}} \subseteq \mathbb{P}V$ is such a critical point, then $\mathfrak{C}_{\mathbb{R}} \cap G \cdot x = K \cdot z$. Moreover, $K \cdot z = \bigcup_{g \in G} \omega(gx)$.

Here μ^* is the moment map for the action of $G^{\mathbb{C}}$ on $\mathbb{C}\mathbb{P}(V^{\mathbb{C}})$ and \mathfrak{C} is the set of critical points of $\|\mu^*\|^2$ in $\mathbb{C}\mathbb{P}(V^{\mathbb{C}})$, while m is the moment map for the action of G on $\mathbb{P}V$ and $\mathfrak{C}_{\mathbb{R}}$ is the set of critical points of $\|m\|^2$ in $\mathbb{P}V$. The fact that $\mathfrak{C} \cap G^{\mathbb{C}} \cdot x = U \cdot z$ was proven in [NM84] in the complex setting; the fact that $\mathfrak{C}_{\mathbb{R}} \cap G \cdot x = K \cdot z$ was proven in [Mar01] in the real setting. The fact that the orbit collapses under the negative gradient flow of $\|\mu^*\|^2$, respectively $\|m\|^2$, to a single U -orbit, respectively K -orbit, is our new contribution.

The value of Theorem 5.7 is as follows. Since $G^{\mathbb{C}} \cdot v \cap V$ is a finite union of G -orbits, if we can show that one of these G -orbits is distinguished then all of them are. This has been applied to the problem of finding generic 2-step nilpotent Lie groups which admit soliton metrics. See Chapters 6 and 7 for more information on the soliton problem.

2. Notation and Technical Preliminaries

Our goal is to study closed reductive subgroups G of $GL(E)$ which are more or less algebraic. Here E is a real vector space and we denote its complexification by $E^{\mathbb{C}} = E \otimes \mathbb{C}$. We call a subgroup H of $GL(E)$ a *real algebraic group* if H is the zero set of polynomials on $GL(E)$ with real coefficients; that is, polynomials in $\mathbb{R}[GL(E)]$. For basic definitions and results on real algebraic groups see Section 1.2.

We say that a group $G \subset GL(E)$ is a *real linear reductive group* if G is a finite index subgroup of a real algebraic reductive group H ; that is, G satisfies $H_0 \subseteq G \subseteq H$, where H_0 is the Hausdorff identity component of H . It is well-known that there exists a complex (algebraic) reductive group $G^{\mathbb{C}}$ defined over \mathbb{R} such that G is Zariski dense in $G^{\mathbb{C}}$ and is a finite index subgroup of the real points $G^{\mathbb{C}}(\mathbb{R}) := G^{\mathbb{C}} \cap GL(E)$ of $G^{\mathbb{C}}$; that is, $G^{\mathbb{C}}(\mathbb{R})_0 \subseteq G \subseteq G^{\mathbb{C}}(\mathbb{R})$. See Proposition 1.5 for the construction of such a complex group.

It is an important observation that each component of $G^{\mathbb{C}}$ intersects G . The importance of this observation is made clear in the following proposition where certain inner products on real vector spaces are extended to their complexifications.

Let V be a real vector space and denote its complexification by $V^{\mathbb{C}} = V \otimes \mathbb{C}$. We will consider representations $\rho : G \rightarrow GL(V)$ that are the restrictions of morphisms $\rho^{\mathbb{C}} : G^{\mathbb{C}} \rightarrow GL(V^{\mathbb{C}})$ of algebraic groups. We will call such a representation a *rational representation* of G (cf. Section 1.3). Note: We will denote the induced Lie algebra representation by the same letter.

We recall the following results from Propositions 2.16 and 2.17.

PROPOSITION. *Let G be defined as above and $\rho : G \rightarrow GL(V)$ a rational representation, then*

- (a) *There exists a K -invariant inner product on V such that G is self-adjoint; hence, the Lie algebra $L(G) = \mathfrak{g}$ is also self-adjoint. That is, there exist Cartan involutions θ, θ_1 on $G, \rho(G)$, respectively, such that $\rho \circ \theta = \theta_1 \circ \rho$.*
- (b) *There exist decompositions of G and \mathfrak{g} , called Cartan decompositions, so that $G = KP$ as a product of manifolds and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here $K = \{g \in G \mid \theta(g) = g\}$ is a maximal compact subgroup of G , $\mathfrak{k} = L(K) = \{X \in \mathfrak{g} \mid \theta(X) = X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$, and $P = \exp(\mathfrak{p})$. Moreover, there exists an $\text{Ad}K$ -invariant inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on \mathfrak{g} so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is orthogonal.*
- (c) *The inner product $\langle \cdot, \cdot \rangle$ on V is K -invariant, $\rho(X)$ are symmetric transformations for $X \in \mathfrak{p}$, and $\rho(X)$ are skew-symmetric transformations for $X \in \mathfrak{k}$.*

PROPOSITION. *The K -invariant inner product \langle, \rangle on V , described above, extends to a U -invariant inner product S on $V^{\mathbb{C}}$ with a similar list of properties for $G^{\mathbb{C}}$. Here U is a compatible maximal compact subgroup of $G^{\mathbb{C}}$ (cf. Section 2.2). Additionally, the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on \mathfrak{g} extends to an $\text{Ad } U$ -invariant inner product \mathbb{S} on $\mathfrak{g}^{\mathbb{C}}$.*

Recall that the inner products on our complex spaces are said to be *compatible* with the inner products on the underlying real spaces.

Moment maps. We recall the definitions and basic results concerning moment maps. See Chapter 2 for more information.

Real moment maps. Given $G \curvearrowright V$ we define $\tilde{m} : V \rightarrow \mathfrak{p}$ implicitly by

$$\langle\langle \tilde{m}(v), X \rangle\rangle = \langle Xv, v \rangle$$

for all $X \in \mathfrak{p}$. Notice that $\tilde{m}(v)$ is a real homogeneous polynomial of degree 2. Equivalently, we really could define $\tilde{m} : V \rightarrow \mathfrak{g}$; then using K -invariance and $\mathfrak{k} \perp \mathfrak{p}$ we obtain $\tilde{m}(V) \subseteq \mathfrak{p}$.

We can just as well do this for $G^{\mathbb{C}} \curvearrowright V^{\mathbb{C}}$ where we regard $G^{\mathbb{C}}$ as a real Lie group. We use the inner product \mathbb{S} on $V^{\mathbb{C}}$. The (real) moment map for $G^{\mathbb{C}} \curvearrowright V^{\mathbb{C}}$, denoted by $\tilde{n} : V^{\mathbb{C}} \rightarrow \mathfrak{q}$, is defined by

$$\mathbb{S}(\tilde{n}(v), Y) = S(Yv, v)$$

for $Y \in \mathfrak{q}$ and $v \in V^{\mathbb{C}}$.

Since these polynomials are homogeneous, they give rise to well defined maps on (real) projective space. Define

$$\begin{aligned} m : \mathbb{P}V &\rightarrow \mathfrak{p} & n : \mathbb{R}\mathbb{P}V^{\mathbb{C}} &\rightarrow \mathfrak{q} \\ m[v] &= \tilde{m}\left(\frac{v}{|v|}\right) = \frac{\tilde{m}(v)}{|v|^2} & n[w] &= \tilde{n}\left(\frac{w}{|w|}\right) = \frac{\tilde{n}(w)}{|w|^2} \end{aligned}$$

where $|w|^2 = S(w, w)$ and $S = \langle, \rangle$ on V . Since $V \subseteq V^{\mathbb{C}}$ we have $\mathbb{P}V \subseteq \mathbb{R}\mathbb{P}V^{\mathbb{C}}$; this is our main reason for studying the real moment map on $G^{\mathbb{C}}$. The following is Lemma 2.24; this lemma compares these two real moment maps.

LEMMA. *n restricted to $\mathbb{P}V$ equals m .*

Complex moment maps. We choose a notation that is similar to Ness [NM84] as we are following her definitions; the only difference is that we use μ where she uses m . For $v \in V^{\mathbb{C}}$, consider $\rho_v : G^{\mathbb{C}} \rightarrow \mathbb{R}$ defined by $\rho_v(g) = |g \cdot v|^2$, where $|w|^2 = H(w, w) = S(w, w)$. Define a map $\mu : \mathbb{C}\mathbb{P}(V^{\mathbb{C}}) \rightarrow \mathfrak{q}^* = \text{Hom}(\mathfrak{q}, \mathbb{R})$ by $\mu(x) = \frac{d\rho_v(e)}{|v|^2}$, where $v \in V^{\mathbb{C}}$ sits over $x \in \mathbb{C}\mathbb{P}(V^{\mathbb{C}})$, cf. [NM84, section 1]. We define the complex moment map $\mu^* : \mathbb{C}\mathbb{P}(V^{\mathbb{C}}) \rightarrow \mathfrak{q}$ by $\mu = S(\mu^*, \cdot)$. Note, taking the norm square of our complex moment map will give us the norm square of the moment map in Kirwan's setting; in Kirwan's language $i\mu$ would be the moment map [NM84, section 1].

Let π denote the projection $\pi : \mathbb{R}\mathbb{P}V^{\mathbb{C}} \rightarrow \mathbb{C}\mathbb{P}(V^{\mathbb{C}})$. The following is proven in Lemma 2.25.

LEMMA. *The complex and real moment maps for $G^{\mathbb{C}}$ are related by $\mu^* \circ \pi = 2n$*

Remark. Since $\mathbb{P}V$ is not a subspace of $\mathbb{C}\mathbb{P}(V^{\mathbb{C}})$, we use $\mathbb{R}\mathbb{P}V^{\mathbb{C}}$ and the real moment map of $G^{\mathbb{C}}$ to work between the known results of Kirwan and Ness to get information about our real group $G \curvearrowright \mathbb{P}V$.

3. Comparison of Real and Complex Cases

Most of algebraic geometry and Geometric Invariant Theory has been worked out exclusively for fields which are algebraically closed. We are interested in the real category and will exploit all the work that has already been done over \mathbb{C} . We use and refer the reader to Chapters 1 and 2 as our main reference for real algebraic varieties.

Recall that our representation $\rho : G \rightarrow GL(V)$ is the restriction of a representation of $G^{\mathbb{C}}$. The following is proposition 2.3 of [BHC62] and section 8 of [RS90]. Originally this was stated as a comparison between $G^{\mathbb{C}}(\mathbb{R})_0$ -orbits and $G^{\mathbb{C}}$ -orbits, however, it can be restated as a comparison between G and $G^{\mathbb{C}}$ orbits, for any G satisfying $G^{\mathbb{C}}(\mathbb{R})_0 \subseteq G \subseteq G^{\mathbb{C}}(\mathbb{R})$. This is true as $G^{\mathbb{C}}(\mathbb{R})_0$ has finite index in G . (This theorem has already been stated in the text; see Theorem 2.30.)

THEOREM. *Let $v \in V$, then $G^{\mathbb{C}} \cdot v \cap V = \bigcup_{i=1}^m X_i$ where each X_i is a G -orbit. Moreover, $G^{\mathbb{C}} \cdot v$ is closed in $V^{\mathbb{C}}$ if and only if $G \cdot v$ is closed in V .*

Orbits in Projective space. Since our groups act linearly on vectors spaces we can consider the induced actions on projective space $G \curvearrowright \mathbb{P}V$ and $G^{\mathbb{C}} \curvearrowright \mathbb{R}PV^{\mathbb{C}}$.

We recall Lemma 2.33 for later use.

LEMMA. For $v \in V$, $G^{\mathbb{C}} \cdot [v] \cap \overline{G \cdot [v]} = G \cdot [v]$ in $\mathbb{R}PV^{\mathbb{C}}$.

4. Closed and Distinguished Orbits

We use the known results for closed orbits and the moment map to motivate our treatment of the nullcone and distinguished orbits. Below we recall some of the work from Chapter 2. We begin with a theorem of Richardson and Slodowy. To find which orbits are closed, one looks for the infimum of $|g \cdot v|^2$ along the orbit. Such a vector is called a *minimal vector* and it occurs on the orbit precisely when our orbit is closed. Let \mathfrak{M} denote the set of minimal vectors in V . The following is a combination of Theorem 2.21 and Corollary 2.23.

THEOREM 5.1. $G \cdot v$ is closed if and only if there exists $w \in G \cdot v$ such that $\tilde{m}(w) = 0$. Such a vector w is minimal. Moreover, $\mathfrak{M} = \tilde{m}^{-1}(0)$ and $\overline{G \cdot v} \cap \mathfrak{M}$ is a single K -orbit.

Equivalently we could find the zero's of $\|\tilde{m}\|^2$ to find the minimal vectors. Minimal vectors are used to understand the semi-stable points, that is, all the vectors whose orbit closure does not contain zero. In contrast, the null cone is the set of vectors whose orbit closure does contain zero. To study the null cone, we move to projective space. Clearly we cannot use minimal vectors to study the geometry of the null cone, so instead of looking for zeros of $\|m\|^2$ on $\mathbb{P}V$ we look for critical points of $\|m\|^2$.

DEFINITION 5.2. We say that $v \in V$ or $[v] \in \mathbb{P}V$ is distinguished if $\|m\|^2 : \mathbb{P}V \rightarrow \mathbb{R}$ has a critical point at $[v]$. We say that an orbit $G \cdot v$ or $G \cdot [v]$ is distinguished if it contains a distinguished point. Analogously, we define distinguished points and $G^{\mathbb{C}}$ -orbits in $V^{\mathbb{C}}$ and $\mathbb{C}\mathbb{P}(V^{\mathbb{C}})$ using $\|\mu^*\|^2$.

Minimal vectors are distinguished as zero is an absolute minimum of the function $\|m\|^2$. Our goal is to find an analogue of Theorem 2.30 for distinguished orbits. To understand critical points of $\|m\|^2$, we will find a way to relate this function to $\|\mu^*\|^2$ by means of $\|n\|^2$. Recall that $\|\mu^*\|^2$ has been studied extensively in [NM84, Kir84].

Our first observation is that the only closed orbits $G \cdot [v] \subseteq \mathbb{P}V$ occur when $G \cdot [v] = K \cdot [v]$. This is well known, but an elegant and geometric proof is easily obtained using properties of the moment map; see, e.g., [Mar01, Theorem 1]. So our main interest is in the remaining distinguished orbits.

PROPOSITION 5.3. If $[v] \in \mathbb{P}V$, then $\text{grad } \|n\|^2[v] = \text{grad } \|m\|^2[v] \in T_{[v]}G \cdot [v]$. Hence, $\|n\|^2$ has a critical point at $[v] \in \mathbb{P}V \subseteq \mathbb{R}PV^{\mathbb{C}}$ if and only if $\|m\|^2$ does so. Moreover, if $[v] \in \mathbb{P}V$, and $\varphi_t[v]$ is the integral curve of $-\text{grad } \|n\|^2$ starting at $[v]$, then $\varphi_t[v] \in G \cdot [v] \subseteq \mathbb{P}V$ for all t .

Before proving the proposition, we study the gradients of these functions. Let $\phi : G^{\mathbb{C}} \times V^{\mathbb{C}} \rightarrow V^{\mathbb{C}}$ denote the action of $G^{\mathbb{C}}$ on $V^{\mathbb{C}}$, and let $\phi_v : G^{\mathbb{C}} \rightarrow V^{\mathbb{C}}$ denote the induced map for every $v \in V^{\mathbb{C}}$. We define vector

fields on $V^\mathbb{C}$ and $\mathbb{R}PV^\mathbb{C}$ as follows. On $V^\mathbb{C}$ we define

$$\tilde{X}_\alpha(v) := d\phi_v(\alpha) = \left. \frac{d}{dt} \right|_{t=0} \exp t\alpha \cdot v$$

for $\alpha \in \mathfrak{g}^\mathbb{C}$. And on $\mathbb{R}PV^\mathbb{C}$

$$X_\alpha[v] := \pi_* \tilde{X}_\alpha(v)$$

where $\pi : V^\mathbb{C} \rightarrow \mathbb{R}PV^\mathbb{C}$ is projection. Note, this is well defined as our action $G^\mathbb{C} \curvearrowright V^\mathbb{C}$ is linear.

LEMMA 5.4. *For $x \in \mathbb{P}V$, $\text{grad } ||m||^2(x) = 4X_{m(x)}(x)$. For $x \in \mathbb{R}PV^\mathbb{C}$, $\text{grad } ||n||^2(x) = 4X_{n(x)}(x)$.*

Marian proves the first statement for $||m||^2$ on $\mathbb{P}V$, see [Mar01, Lemma 2]. Her proof carries over to obtain the statement for $||n||^2$ on $\mathbb{R}PV^\mathbb{C}$.

PROOF OF PROPOSITION 5.3. The first assertion follows from Lemma 5.4, Lemma 2.24, and the fact that $m[v] \in \mathfrak{p} \subseteq \mathfrak{g}$ for $[v] \in \mathbb{P}V$. The second and third assertions follow immediately from the first. \square

Next we relate the actions of our complex group $G^\mathbb{C}$ on $\mathbb{R}PV^\mathbb{C}$ and $\mathbb{C}\mathbb{P}(V^\mathbb{C})$. By Lemma 2.25 we know $||\mu^* \circ \pi[v]||^2 = 4||n[v]||^2$ for $v \in V \subseteq V^\mathbb{C}$ and $\pi : \mathbb{R}PV^\mathbb{C} \rightarrow \mathbb{C}\mathbb{P}(V^\mathbb{C})$. This shows that $||n||^2$ is not just U -invariant, it is also $U \times \mathbb{C}^*$ -invariant. We wish to relate the actions of $G^\mathbb{C}$ on $\mathbb{R}PV^\mathbb{C}$ and $\mathbb{C}\mathbb{P}(V^\mathbb{C})$ by comparing their gradients from the natural Riemannian structures on these projective spaces.

The Riemannian structures and gradients on projective space. Recall that projective space can be endowed with a natural Riemannian metric so that projection from the vector space is a Riemannian submersion. This natural Riemannian metric is called the Fubini-Study metric and is defined as follows. Take $\zeta_i \in T_{[w]}\mathbb{K}\mathbb{P}(V^\mathbb{C})$, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $\Pi^\mathbb{K} : V^\mathbb{C} \rightarrow \mathbb{K}\mathbb{P}(V^\mathbb{C})$ be the usual projection and take $\xi_i \in T_w V^\mathbb{C}$ such that $\Pi_*^\mathbb{K}(\xi_i) = \zeta_i$. The Fubini-Study metric on $\mathbb{K}\mathbb{P}(V^\mathbb{C})$ is defined by

$$(\zeta_1, \zeta_2) = \frac{(\xi_1, \xi_2)(w, w) - (\xi_1, w)(\xi_2, w)}{(w, w)}$$

One can naturally identify the tangent space $T_{\Pi^\mathbb{K}(w)}\mathbb{K}\mathbb{P}(V^\mathbb{C})$ with the orthogonal compliment of $\mathbb{K} \cdot \text{span} < w >$ in $T_w V^\mathbb{C}$. In our setting, we are using S , the extension of $<, >$ on V , as our inner product on $V^\mathbb{C}$. Using these natural choices of Riemannian structures on $\mathbb{R}PV^\mathbb{C}$ and $\mathbb{C}\mathbb{P}(V^\mathbb{C})$ we see that $\pi : \mathbb{R}PV^\mathbb{C} \rightarrow \mathbb{C}\mathbb{P}(V^\mathbb{C})$ is also a Riemannian submersion.

We are interested in the negative gradient flow of the moment map. Let φ_t denote the negative gradient flow of $||n||^2$ on $\mathbb{R}PV^\mathbb{C}$ and $||\mu^*||^2$ on $\mathbb{C}\mathbb{P}(V^\mathbb{C})$.

DEFINITION 5.5. *The ω -limit set of $\varphi_t(p) \subseteq \mathbb{R}PV^\mathbb{C}$ is the set $\{q \in \mathbb{R}PV^\mathbb{C} \mid \varphi_{t_n}(p) \rightarrow q \text{ for some sequence } t_n \rightarrow \infty \text{ in } \mathbb{R}\}$. We denote this set by $\omega(p)$.*

Analogously, we can define the ω -limit set of $\varphi_t(p) \subseteq \mathbb{CP}(V^\mathbb{C})$ and we denote this set by $\omega(p)$ also. It is easy to see that $\omega(p)$ is invariant under φ_t for all t .

Remark. We observe that points in the ω -limit set of a negative gradient flow are fixed points of the flow, that is, critical points of the given function. In general this is not true for ω -limit points associated to non-gradient flows. We include a brief argument for the reader.

Consider $F : M \rightarrow \mathbb{R}$ and let $\varphi_t(p)$ denote the integral curve of $-\text{grad } F$ starting at $p \in M$. Observe that F is decreasing along $\varphi_t(p)$. Suppose $\omega(p)$ is non-empty. Then we can define $c = \lim_{t \rightarrow \infty} F(\varphi_t(p))$ to obtain $\omega(p) \subseteq F^{-1}(c)$. Thus for $q \in \omega(p)$ we see that $\varphi_t(q) \subseteq F^{-1}(c)$. Hence, $\text{grad } F(q) = 0$. That is, points in the ω -limit set of $-\text{grad } F$ are critical points for F .

PROPOSITION 5.6. *Endow $\mathbb{R}P V^\mathbb{C}$ and $\mathbb{CP}(V^\mathbb{C})$ with the Riemannian metrics so that the projections from $V^\mathbb{C}$ are Riemannian submersions. Then the following are true for $[v] \in \mathbb{R}P V^\mathbb{C}$*

- (a) $4\pi_* \text{grad } ||n||^2[v] = \text{grad } ||\mu^*||^2(\pi[v])$
- (b) $[v] \in \mathbb{R}P V^\mathbb{C}$ is a critical point of $||n||^2$ if and only if $\pi[v] \in \mathbb{CP}(V^\mathbb{C})$ is a critical point of $||\mu^*||^2$.
- (c) $\varphi_t \circ \pi = \pi \circ \varphi_{4t}$, where φ_t denotes the negative gradient flow of $||n||^2$ on $\mathbb{R}P V^\mathbb{C}$ or $||\mu^*||^2$ on $\mathbb{CP}(V^\mathbb{C})$.
- (d) $\pi(\omega([v])) = \omega(\pi[v])$, where $\omega(p)$ denotes the ω -limit set of the negative gradient flow starting from p .

PROOF. Applying Lemma 2.25 we have

$$\begin{aligned} 4 < \text{grad } ||n||^2[v], w_{[v]} > &= 4 \left. \frac{d}{dt} \right|_{t=0} ||n[v + tw]||^2 \\ &= \left. \frac{d}{dt} \right|_{t=0} ||\mu^* \pi[v + tw]||^2 \\ &= < \text{grad } ||\mu^*||^2(\pi[v]), \pi_* w_{[v]} > \end{aligned}$$

Since π_* is a submersion we have that π_* maps the horizontal subspace of $T_{[v]}\mathbb{R}P V^\mathbb{C}$ isometrically onto $T_{\pi[v]}\mathbb{CP}(V^\mathbb{C})$ and part *a.* is proven. Thus if $[v]$ is a critical point for $||n||^2$, then $\pi[v]$ is one for $||\mu^*||^2$. To obtain the reverse direction use the \mathbb{C}^* -invariance of $||n||^2$. This proves part *b.*

Proof of part *c.* Let $[v] \in \mathbb{R}P V^\mathbb{C}$. Consider the curve $\pi \circ \varphi_{4t}[v]$ in $\mathbb{CP}(V^\mathbb{C})$. This curve satisfies the following differential equation

$$\frac{d}{dt} \pi \circ \varphi_{4t}[v] = \pi_* 4(-\text{grad } ||n||^2)(\varphi_{4t}[v]) = -\text{grad } ||\mu^*||^2(\pi \circ \varphi_{4t}[v])$$

That is, the curve $\pi \circ \varphi_{4t}[v]$ is the integral curve of the negative gradient flow of $||\mu^*||^2$ starting at $\pi[v]$. Thus, $\pi \circ \varphi_{4t} = \varphi_t \circ \pi$.

Proof of part *d.* We will show containment in both directions. Take $p \in \omega[v]$, then there exists a sequence of $t_n \rightarrow \infty$ such that $\varphi_{t_n}[v] \rightarrow p$ in $\mathbb{R}P V^\mathbb{C}$. Using part *c.*, we have $\varphi_{t_n/4}(\pi[v]) = \pi \circ \varphi_{t_n}[v] \rightarrow \pi(p)$. That is, $\pi(p) \in \omega(\pi[v])$, or $\omega(\pi[v]) \subseteq \omega(\pi[v])$. To obtain the other direction, take $q \in \omega(\pi[v])$ and $t_n \rightarrow \infty$ so that

$\varphi_{t_n}(\pi[v]) \rightarrow q$ in $\mathbb{CP}(V^\mathbb{C})$. Consider the set $\varphi_{4t_n}[v]$ in $\mathbb{RP}V^\mathbb{C}$. Since $\mathbb{RP}V^\mathbb{C}$ is compact, we can find a limit point of this set and passing to a subsequence we may assume $\varphi_{4t_n}[v] \rightarrow p$. Then $p \in \omega[v]$, $\pi(p) = q$ by (c) and we have shown $q \in \pi(\omega[v])$. That is, $\omega(\pi[v]) \subseteq \pi(\omega[v])$. \square

We finish the section by stating our main theorem and some corollaries.

THEOREM 5.7. *Given $G \curvearrowright V$, $G^\mathbb{C} \curvearrowright V^\mathbb{C}$, and $[v] \in \mathbb{P}V$ we have*

$G \cdot [v]$ is a distinguished orbit in $\mathbb{P}V$ if and only if $G^\mathbb{C} \cdot \pi[v]$ is a distinguished orbit in $\mathbb{CP}(V^\mathbb{C})$.

Here $\pi : \mathbb{P}V \subseteq \mathbb{RP}V^\mathbb{C} \rightarrow \mathbb{CP}(V^\mathbb{C})$ is the usual projection.

Remark. Analysis of the proof of Theorem 5.7 shows the following. Given $v \in V \subseteq V^\mathbb{C}$, the orbits $G \cdot [v] \subseteq \mathbb{P}V$ and $G^\mathbb{C} \cdot \pi[v] \subseteq \mathbb{CP}(V^\mathbb{C})$ being distinguished is equivalent to $G^\mathbb{C} \cdot [v] \subseteq \mathbb{RP}V^\mathbb{C}$ being distinguished using $\|n\|^2$ on $\mathbb{RP}V^\mathbb{C}$.

COROLLARY 5.8. *Suppose we have $v_1, v_2 \in V$ with distinct G -orbits but whose $G^\mathbb{C}$ -orbits are the same. Then $G \cdot [v_1]$ is distinguished if and only if $G \cdot [v_2]$ is distinguished.*

Remark. The phenomenon of two vectors having different real orbits but the same complex orbit happens often. This corollary was a necessary ingredient in the solution to the problem of showing that generic 2-step nilmanifolds admit soliton metrics. See Chapter 7.

5. Proofs of Main Theorems

Here we prove Theorem 5.7 on distinguished orbits. To do this, we first prove a statement for complex moment maps in the complex setting. Then we will relate the complex moment map information to the real moment map for the $G^\mathbb{C}$ action.

Remark. For $x \in \mathbb{CP}(V^\mathbb{C})$, the critical points of $\|\mu^*\|^2$ restricted to $G^\mathbb{C} \cdot x$ are precisely the critical points of $\|\mu^*\|^2$ as a function on $\mathbb{CP}(V^\mathbb{C})$. This is because $\text{grad } \|\mu^*\|^2(x)$ is always tangent to $G^\mathbb{C} \cdot x$. We denote the set of critical points of $\|\mu^*\|^2$ in $\mathbb{CP}(V^\mathbb{C})$ by \mathfrak{C} .

THEOREM 5.9. *For $x \in \mathbb{CP}(V^\mathbb{C})$, suppose $G^\mathbb{C} \cdot x \subseteq \mathbb{CP}(V^\mathbb{C})$ contains a critical point of $\|\mu^*\|^2$. If $z \in \mathfrak{C} \subseteq \mathbb{CP}(V^\mathbb{C})$ is such a critical point, then $\mathfrak{C} \cap G^\mathbb{C} \cdot x = U \cdot z$. Moreover, $U \cdot z = \bigcup_{g \in G^\mathbb{C}} \omega(gx)$.*

Let $\mathfrak{C}_\mathbb{R}$ denote the set of critical points of $\|m\|^2$ on $\mathbb{P}V$. We have a real analogue of the theorem above.

THEOREM 5.10. *For $x \in \mathbb{P}V$, suppose $G \cdot x \subseteq \mathbb{P}V$ contains a critical point of $\|m\|^2$. If $z \in \mathfrak{C}_\mathbb{R} \subseteq \mathbb{P}V$ is such a critical point, then $\mathfrak{C}_\mathbb{R} \cap G \cdot x = K \cdot z$. Moreover, $K \cdot z = \bigcup_{g \in G} \omega(gx)$.*

Before proving Theorems 5.9 and 5.10, we apply Theorem 5.9 to prove Theorem 5.7.

Proof of Theorem 5.7. Suppose first that $G \cdot [v]$ is distinguished. Then $G \cdot [v] = G \cdot [w]$ where $[w]$ is a critical point of $\|m\|^2$. But now Proposition 5.3 implies that $[w]$ is a critical point of $\|n\|^2$ and Proposition 5.6 implies that $\pi[w]$ is a critical point of $\|\mu^*\|^2$; that is, $G^{\mathbb{C}} \cdot \pi[v]$ is distinguished.

Now suppose $G^{\mathbb{C}} \cdot \pi[v]$ is distinguished. Our goal is to show that the orbit $G \cdot [v]$ in $\mathbb{P}V$ contains a critical point of $\|m\|^2$. We will use the $G^{\mathbb{C}}$ action on $\mathbb{R}P V^{\mathbb{C}}$ and the real moment map of this action. As $G^{\mathbb{C}} \cdot \pi[v]$ is distinguished, and $\pi : G^{\mathbb{C}} \cdot [v] \rightarrow G^{\mathbb{C}} \cdot \pi[v]$ is surjective, there exists $w \in G^{\mathbb{C}} \cdot [v]$ such that $\pi[w] \in G^{\mathbb{C}} \cdot \pi[v]$ is a critical point of $\|\mu^*\|^2$.

Apply the negative gradient flow of $\|n\|^2$ in $\mathbb{R}P V^{\mathbb{C}}$ starting at $[v] \in \mathbb{P}V$. By Proposition 5.3 this is the negative gradient flow of $\|m\|^2$ and the ω -limit set $\omega[v] \subseteq \overline{G \cdot [v]}$ consists of critical points of $\|n\|^2$ and $\|m\|^2$ (see the remark following Definition 5.5). By Proposition 5.6 d and Theorem 5.9, we have $\pi(\omega[v]) = \omega(\pi[v]) \subseteq U \cdot \pi[w]$; hence, $\omega[v] \subseteq \pi^{-1}(U \cdot \pi[w]) = \mathbb{C}^* \times U \cdot [w] \subseteq \mathbb{C}^* \times G^{\mathbb{C}} \cdot [v]$. This implies

$$\omega[v] \subseteq \mathbb{C}^* \times G^{\mathbb{C}} \cdot [v] \cap \overline{G \cdot [v]} \subseteq \mathbb{C}^* \times G^{\mathbb{C}} \cdot [v] \cap \overline{\mathbb{R}^* \times G \cdot [v]} = \mathbb{R}^* \times G \cdot [v] = G \cdot [v]$$

by Lemma 2.33 and the fact that $(\mathbb{R}^* \times G)^{\mathbb{C}} = \mathbb{C}^* \times G^{\mathbb{C}}$. Hence $\omega[v]$ consists of critical points of $\|m\|^2$ that lie in $G \cdot [v]$. This proves Theorem 5.7.

Before proving Theorem 5.9, we prove Theorem 5.10. The proof of this theorem is actually embedded in the proof of Theorem 5.7. We present it here.

Proof of 5.10 The fact that $\mathfrak{C}_{\mathbb{R}} \cap G \cdot x$ constitutes a single K -orbit is the content of [Mar01, Theorem 1]. In [Mar01] G is taken to be semi-simple; however, all the results hold for G real reductive with the same proofs, mutatis mutandis. Our original contribution is the second statement of the theorem. We prove it here.

Suppose $G \cdot x \subseteq \mathbb{P}V$ contains a critical point z of $\|m\|^2$. Then the orbit $G^{\mathbb{C}} \cdot \pi(x)$ is distinguished in $\mathbb{C}P(V^{\mathbb{C}})$ by Theorem 5.7. The proof of Theorem 5.7 shows, for $g \in G$, $\omega(gx)$ consists of critical points of $\|m\|^2$ in $G \cdot x$. By Theorem 1 of [Mar01], we have $\omega(gx) \subseteq K \cdot z$. Hence, $\bigcup_{g \in G} \omega(gx) = K \cdot z$, since $\omega(y) = \{y\}$ for all $y \in K \cdot z$.

Lastly we have to prove Theorem 5.9. The first statement is proven in [NM84, Theorem 6.2]. That is, the critical points of $\|\mu^*\|^2$ on a $G^{\mathbb{C}}$ -orbit comprise a single U -orbit. As in Theorem 5.10, our original contribution is the second statement.

The statement that the whole orbit $G^{\mathbb{C}} \cdot x$ flows to one U -orbit $U \cdot z$ is plausible, but is not contained in Kirwan's work [Kir84]. It is a finer statement than the $G^{\mathbb{C}}$ -invariance of Kirwan's stratification of $\mathbb{C}P(V^{\mathbb{C}})$.

There are two problems to be aware of: first, for $g \in G^{\mathbb{C}}$, $\omega(gx)$ might be a set with more than one point and, second, there is no reason to expect that $\omega(gx)$ lies entirely in the orbit $G^{\mathbb{C}} \cdot x$. This proof is just for the complex setting of our complex group $G^{\mathbb{C}} \curvearrowright \mathbb{CP}(V^{\mathbb{C}})$. This is the setting of Kirwan and Ness.

Proof of 5.9 Consider an orbit $G^{\mathbb{C}} \cdot y$ which is distinguished and let $z \in G^{\mathbb{C}} \cdot y$ be a critical point. Let x be any point in $G^{\mathbb{C}} \cdot y$. To show that $\omega(x) \subseteq U \cdot z$, we will first show that the limit set $\omega(x)$ intersects $U \cdot z$ and then show containment. First we need to recall some results from Kirwan's work [Kir84].

We have a smooth stratification of $\mathbb{CP}(V^{\mathbb{C}})$ into strata S_{β} which are $G^{\mathbb{C}}$ -invariant. The strata are determined by a certain decomposition of the critical set \mathfrak{C} of $\|\mu^*\|^2$ in $\mathbb{CP}(V^{\mathbb{C}})$. This critical set is a finite union $\mathfrak{C} = \bigcup_{\beta \in B} C_{\beta}$ where $\|\mu^*\|^2$ takes a constant value on C_{β} and each C_{β} is U -invariant. We will denote this constant value of $\|\mu^*\|^2$ on C_{β} by $M_{\beta} = \|\beta\|^2$; here B is actually a finite set in $\mathfrak{g}^{\mathbb{C}}$ and the norm $\|\cdot\|$ comes from the prescribed inner product on $\mathfrak{g}^{\mathbb{C}}$.

For $\beta \in B$, the stratum S_{β} is defined to be the set of points which flow via the negative gradient flow to the critical set C_{β} , that is, $S_{\beta} = \{x \in \mathbb{CP}(V^{\mathbb{C}}) \mid \omega(x) \subseteq C_{\beta}\}$. In particular, $C_{\beta} \subseteq S_{\beta}$. See section 2 of [Kir84] for a detailed discussion of this Morse Theory approach to Geometric Invariant Theory. If $G^{\mathbb{C}} \cdot y \cap C_{\beta} \neq \emptyset$ then

$$G^{\mathbb{C}} \cdot y \cap C_{\beta} = U \cdot z$$

for $z \in C_{\beta}$, that is, the critical points in a $G^{\mathbb{C}}$ -orbit comprise a single U -orbit, see [NM84, Theorem 6.2]. We show two things. First, if $x \in G^{\mathbb{C}} \cdot z$ is in a neighborhood of $U \cdot z$, then $\omega(x) \subseteq U \cdot z$. Second, this neighborhood of $U \cdot z$ in $G^{\mathbb{C}} \cdot z$ should be the entire orbit; that is, $\omega(x) \subseteq U \cdot z$ for all $x \in G^{\mathbb{C}} \cdot z$. The first is a little more obvious but does rely on the fact that our moment map is a minimally degenerate Morse function, see definition 10.1 of [Kir84]. That fact that $\|\mu^*\|^2$ is a minimally degenerate Morse function can be found in section 4 of [Kir84].

Fix β . We will be interested in $z \in C_{\beta}$ and the orbit $G^{\mathbb{C}} \cdot z$. We define $\mathcal{O}_{\varepsilon} = \{x \in \mathbb{CP}(V^{\mathbb{C}}) \mid \|\mu^*\|^2(x) \in [M_{\beta}, M_{\beta} + \varepsilon)\} \cap S_{\beta}$. This is an open subset of S_{β} that contains $C_{\beta} = \{x \in S_{\beta} \mid \|\mu^*(x)\|^2 = M_{\beta}\}$. We observe that $\mathcal{O}_{\varepsilon}$ is invariant under the forward flow φ_t of $-\text{grad } \|\mu^*\|^2$ as $\|\mu^*\|^2$ decreases along the trajectories $t \rightarrow \varphi_t(x)$. Since $G^{\mathbb{C}} \cdot z$ is a submanifold of $\mathbb{CP}(V^{\mathbb{C}})$, hence also of S_{β} , $\mathcal{O}_{\varepsilon} \cap G^{\mathbb{C}} \cdot z$ is open in $G^{\mathbb{C}} \cdot z$ and contains $U \cdot z$ as C_{β} is U -invariant.

DEFINITION 5.11. We define $\{V_{\varepsilon,i}\}$ to be the collection of connected components of $\mathcal{O}_{\varepsilon} \cap G^{\mathbb{C}} \cdot z$ that intersect $U \cdot z$. We define $V_{\varepsilon} := \bigcup_i V_{\varepsilon,i}$.

Remark. V_{ε} is an open set of $G^{\mathbb{C}} \cdot z$ that contains $U \cdot z$. As U has finitely many components, $U = \bigcup_{i=1}^m \phi_i U_0$ and we can write $V_{\varepsilon} = \bigcup_{i=1}^m V_{\varepsilon,i}$ where $\phi_i U_0(z) \subseteq V_{\varepsilon,i}$. The $V_{\varepsilon,i}$ are connected and open in $G^{\mathbb{C}} \cdot z$ as $\mathcal{O}_{\varepsilon} \cap G^{\mathbb{C}} \cdot z$

is open in $G^{\mathbb{C}} \cdot z$ and $G^{\mathbb{C}} \cdot z$ is locally connected, see [Mun00, Theorem 25.3]. Moreover, since \mathcal{O}_ε and $G^{\mathbb{C}} \cdot z$ are invariant under $\varphi_t, t > 0$, we see that the components $V_{\varepsilon,i}$ are invariant under forward flow, as well.

PROPOSITION 5.12. *There exists $\varepsilon > 0$ such that $\overline{V}_\varepsilon \subseteq G^{\mathbb{C}} \cdot z$. Moreover, $\omega(V_\varepsilon) = U \cdot z$ for small $\varepsilon > 0$.*

PROOF. Before proving this statement, we will show that there exists some open set A containing $U \cdot z$ in $G^{\mathbb{C}} \cdot z$ such that \overline{A} is a compact subset of $G^{\mathbb{C}} \cdot z$. Then we will show that $V_\varepsilon \subseteq A$ for small ε . This would then prove the first assertion of the proposition.

Recall that $G^{\mathbb{C}} = U \exp(iLU)$. If we let B be the open unit ball in iLU then $A = U \exp(B) \cdot z$ has the said property, that is, \overline{A} is a compact subset of $G^{\mathbb{C}} \cdot z$.

LEMMA 5.13. *Either $V_\varepsilon \subseteq A$ or $V_\varepsilon \cap \partial A \neq \emptyset$. For small $\varepsilon > 0$, $V_\varepsilon \subseteq A$.*

This will follow from

LEMMA 5.14. *Either $V_{\varepsilon,i} \subseteq A$ or $V_{\varepsilon,i} \cap \partial A \neq \emptyset$.*

To prove this lemma, suppose $V_{\varepsilon,i} \not\subseteq A$ and $V_{\varepsilon,i} \cap \partial A = \emptyset$. Since $V_{\varepsilon,i} \cap A$ intersects $U \cdot z$, we see that $V_{\varepsilon,i} = (V_{\varepsilon,i} \cap A) \cup (V_{\varepsilon,i} \setminus \overline{A})$; that is, $V_{\varepsilon,i}$ is separated by these disjoint open sets. This contradicts the connectedness of $V_{\varepsilon,i}$ and the lemma is proven.

We continue with the proof of the first lemma. Suppose $V_\varepsilon \not\subseteq A$ for every $\varepsilon > 0$. Then for each ε there exists some point $p_\varepsilon \in V_\varepsilon \cap \partial A$. By definition $\|\mu^*\|^2(p_\varepsilon) \leq M_\beta + \varepsilon$. Letting epsilon go to zero we can find a limit point $p_\infty \in \partial A$ as ∂A is compact. Hence, $p_\infty \in G^{\mathbb{C}} \cdot z - A \subseteq G^{\mathbb{C}} \cdot z - U \cdot z$. Moreover, $\|\mu^*\|^2(p_\infty) = M_\beta$ and we have found a point in $G^{\mathbb{C}} \cdot z$ which is not on $U \cdot z$ but minimizes $\|\mu^*\|$ on $G^{\mathbb{C}} \cdot z$. This is a contradiction since $G^{\mathbb{C}} \cdot z \cap C_\beta = U \cdot z$ by [NM84, Theorem 6.2]. Therefore, $V_\varepsilon \subseteq A$ for small ε . This proves the first lemma and the first claim in the proposition.

To finish the proof of the proposition, we observe that $U \cdot z = \omega(U \cdot z) \subseteq \omega(V_\varepsilon)$ since $U \cdot z \subseteq C_\beta$ and φ_t fixes the points of C_β for all t . Thus we just need to show containment in the other direction. Since the set V_ε is invariant under forward flow and $V_\varepsilon \subseteq G^{\mathbb{C}} \cdot z \subseteq S_\beta$, we see that $\omega(V_\varepsilon) \subseteq \overline{V}_\varepsilon \cap C_\beta \subseteq G^{\mathbb{C}} \cdot z \cap C_\beta = U \cdot z$. \square

DEFINITION 5.15. *Let $\mathcal{O} = \{x \in G^{\mathbb{C}} \cdot z \mid \omega(x) \subseteq U \cdot z\}$.*

LEMMA 5.16. *Consider the set \mathcal{O} defined above. Then $\mathcal{O} = G^{\mathbb{C}} \cdot z$.*

To prove the lemma it suffices to show that \mathcal{O} is open and closed in $G^{\mathbb{C}} \cdot z$ and intersects each component of $G^{\mathbb{C}} \cdot z$. To see that \mathcal{O} intersects each component of $G^{\mathbb{C}} \cdot z$, we observe that \mathcal{O} contains $U \cdot z$ and that each component of $G^{\mathbb{C}}$ intersects U since $G^{\mathbb{C}} = UQ$ and $Q = \exp(\mathfrak{q})$ is contractible, see the remarks before Proposition 2.17. Choose $\varepsilon > 0$ as in Proposition 5.12.

\mathcal{O} is open:

We know for small $\varepsilon > 0$, V_ε is open in $G^\mathbb{C} \cdot z$, contains $U \cdot z$, and V_ε is contained in \mathcal{O} by Proposition 5.12. It suffices to consider $x \in \mathcal{O} \setminus U \cdot z$. Then there exists $t_* > 0$ such that $\varphi_{t_*}(x)$ intersects V_ε , from the definition of \mathcal{O} . But $\varphi_{-t_*} : V_\varepsilon \rightarrow \varphi_{-t_*}(V_\varepsilon)$ is a diffeomorphism of $G^\mathbb{C} \cdot z$ (and also of S_β). Thus, $\varphi_{-t_*}(V_\varepsilon)$ is an open set in $G^\mathbb{C} \cdot z$ containing x , which is contained in \mathcal{O} . Therefore \mathcal{O} is open.

\mathcal{O} is closed:

We will show $\partial\mathcal{O} = \emptyset$; here we mean the boundary of \mathcal{O} in the topological space $G^\mathbb{C} \cdot z$. Take $y_n \in \mathcal{O}$ such that $y_n \rightarrow y \in G^\mathbb{C} \cdot z$. Since $z \in C_\beta \subseteq S_\beta$ and S_β is $G^\mathbb{C}$ -invariant, it follows that $y \in G^\mathbb{C} \cdot z \subseteq S_\beta$ and hence $\omega(y) \subseteq C_\beta$. Thus, there exists $M > 0$ such that $\varphi_M(y) \in \mathcal{O}_\varepsilon$. We will denote the component of $\mathcal{O}_\varepsilon \cap G^\mathbb{C} \cdot z$ containing $\varphi_M(y)$ by $\mathcal{O}_\varepsilon^y$; again, this component is open in $G^\mathbb{C} \cdot z$ as $G^\mathbb{C} \cdot z$ is locally connected. Observe that for $t \geq M$, $\varphi_t(y) \in \mathcal{O}_\varepsilon^y$ and $\varphi_s(\mathcal{O}_\varepsilon^y) \subseteq \mathcal{O}_\varepsilon^y$ for $s \geq 0$ as φ_s leaves $\mathcal{O}_\varepsilon \cap G^\mathbb{C} \cdot z$ invariant for $s \geq 0$. Since φ_t is a diffeomorphism on S_β which preserves $G^\mathbb{C} \cdot z$, $\varphi_M^{-1}(\mathcal{O}_\varepsilon^y)$ is an open set of $G^\mathbb{C} \cdot z$ containing y .

We assert that $\mathcal{O}_\varepsilon^y \cap V_\varepsilon \neq \emptyset$. Since $y_n \in \mathcal{O}$, we know there exists $T_n > 0$ such that $\varphi_{T_n}(y_n) \in V_\varepsilon$, by definition of \mathcal{O} . Additionally, for $t \geq T_n$, $\varphi_t(y_n) \in V_\varepsilon$ by the flow invariance of V_ε .

Pick N such that $y_N \in \varphi_M^{-1}(\mathcal{O}_\varepsilon^y)$, which we can do as $\varphi_M^{-1}(\mathcal{O}_\varepsilon^y)$ is open and $y_n \rightarrow y$. Then we have $\varphi_M(y_N) \in \mathcal{O}_\varepsilon^y$, a single component of $\mathcal{O}_\varepsilon \cap G^\mathbb{C} \cdot z$, and $\varphi_{T_N}(y_N) \in V_\varepsilon$.

If $M \geq T_N$, then $\varphi_M(y_N) = \varphi_{M-T_N}(\varphi_{T_N}(y_N)) \in \varphi_{M-T_N}(V_\varepsilon) \subseteq V_\varepsilon$.

That is, $\varphi_M(y_N) \in \mathcal{O}_\varepsilon^y \cap V_\varepsilon \neq \emptyset$.

If $T_N \geq M$, then $\varphi_{T_N}(y_N) = \varphi_{T_N-M}(\varphi_M(y_N)) \in \varphi_{T_N-M}(\mathcal{O}_\varepsilon^y) \subseteq \mathcal{O}_\varepsilon^y$.

That is, $\varphi_{T_N}(y_N) \in \mathcal{O}_\varepsilon^y \cap V_\varepsilon \neq \emptyset$.

Thus, $\mathcal{O}_\varepsilon^y$ being a connected component of $\mathcal{O}_\varepsilon \cap G^\mathbb{C} \cdot z$ which intersects V_ε , a union of connected components of $\mathcal{O}_\varepsilon \cap G^\mathbb{C} \cdot z$, we have $\mathcal{O}_\varepsilon^y \subseteq V_\varepsilon$. That is, $y \in \mathcal{O}$ since $\varphi_t(y) \in V_\varepsilon$ for $t \geq M$ and $\omega(V_\varepsilon) \subseteq U \cdot z$ by Proposition 5.12. This proves the lemma.

This completes the proof of Theorem 5.9.

Part 2

Structures on Nilpotent Lie Groups

Soliton metrics on nilmanifolds

1. Introduction

The aim of this chapter is to introduce the reader to the general results in regards to existence and non-existence of soliton metrics on nilmanifolds.

Soliton metrics arise in the study of Einstein metrics. Originally they were discovered as special solutions of a particular geometric evolution equation on the space of Riemannian metrics on a fixed differentiable manifold. However, they also arise in the search for Einstein metrics on negatively curved homogeneous manifolds.

It is well known that a homogeneous space of negative curvature is isometric to a solvable Lie group with a left-invariant metric, see [Hei74]. In [Heb98], Heber classifies the (standard) Einstein solvmanifolds. We note that this classification was originally done for the so-called ‘standard’ Einstein solvmanifolds, standard being a technical requirement. While the standard Einstein metrics were shown to be an open set within the set of Einstein metrics on solvable Lie groups, it was not known whether or not an Einstein metric had to be ‘standard’. This question was resolved in the affirmative by Lauret in [Lau07].

Let S be a solvable Lie group with left invariant metric and N its nilradical. The Lie group N is given the geometry of a submanifold and this is a left-invariant metric on N . If the codimension of N in S is 1, we say that S is a *rank 1 solvable extension of N* . It is known that S admits an Einstein metric if and only if S' admits an Einstein metric where S' is a solvable subgroup of S which is a rank one extension of N . This reduction is Proposition 6.11 of [Heb98]. It has been shown that a rank 1 extension S' of N admits an Einstein metric if and only if N admits a so called soliton metric, [Lau01]. See Chapters 1 and Section 3 (below) for the definitions of Einstein and Ricci soliton metrics, respectively.

While searching for soliton metrics on nilmanifolds is interesting in its own right, by finding which nilmanifolds admit soliton metrics we gain insight into which solvmanifolds admit Einstein metrics. Our contribution to this problem is the following.

Theorem 7.25 *A generic two-step nilmanifold admits a soliton metric.*

2. Nilmanifolds and Left-invariant Geometry

Recall the following definition from Section 1.2.

DEFINITION. The lower central series of \mathfrak{N} is a descending series of ideals defined by

$$\begin{aligned} C^1\mathfrak{N} &= \mathfrak{N} \\ C^n\mathfrak{N} &= [\mathfrak{N}, C^{n-1}\mathfrak{N}] \end{aligned}$$

for $n \geq 2$. A Lie algebra \mathfrak{N} is called nilpotent if there exists k such that $C^k\mathfrak{N} = 0$; moreover, we call \mathfrak{N} k -step nilpotent if k is the smallest integer such that $C^k\mathfrak{N} = 0$.

Observe that abelian groups are nilpotent and equal their center. The closest group to being abelian, without actually being abelian, is a two-step nilpotent group. In this case $[\mathfrak{N}, \mathfrak{N}] \subset \mathfrak{Z}$, where \mathfrak{Z} is the center of our two-step nilpotent algebra \mathfrak{N} .

DEFINITION 6.1. A nilmanifold is a homogeneous space such that a nilpotent group of isometries acts transitively on it.

It has been shown that such a manifold is actually a nilpotent Lie group N' which is the quotient of a simply connected nilpotent group N by a discrete central subgroup Z_Γ [Wil82]. By considering the covering $N \rightarrow N/Z_\Gamma$ as a local isometry, studying the left-invariant geometry of N is the same as studying the left-invariant geometry of N' . Therefore we reduce to the case that $N' = N$ is simply connected.

Let N be a simply connected nilpotent Lie group with a left-invariant metric. Let \mathfrak{N} denote the Lie algebra of N . Then a left-invariant metric on N is equivalent to an inner product on \mathfrak{N} denoted by $\langle \cdot, \cdot \rangle$. This is the viewpoint that we will take.

QUESTION 6.2. What are the nilpotent Lie groups that admit left-invariant Einstein metrics?

The answer is none. The following theorem of Milnor [Mil76] demonstrates why these groups do not admit such nice metrics.

THEOREM 6.3. Let (N, g) be a nilpotent Lie group with left-invariant metric. Then there exist directions $v, w \in \mathfrak{N}$ such that $\text{Ric}(v) > 0$ and $\text{Ric}(w) < 0$. Hence, N cannot admit an Einstein metric.

This leaves one asking the following philosophical question.

QUESTION 6.4. Is there a different notion of preferred or distinguished metric that a nilpotent group can admit? If so, what are the nilmanifolds that admit these distinguished metrics?

We argue that the correct notion of distinguished metric for nilmanifolds is the notion of a soliton metric.

3. Soliton Metrics

Soliton metrics arise naturally in the study of normalized Ricci flow on a compact manifold (M, g) . Ricci flow, respectively normalized Ricci flow, is a geometric evolution equation which evolves a given metric g_0

according to the differential equation $\frac{\partial}{\partial t}g = -2ric$, respectively $\frac{\partial}{\partial t}g = -2ric + \frac{2 \, sc(g)}{n}g$, where ric is the Ricci $(2,0)$ tensor of (M, g) and sc is the scalar curvature function. The fixed points of normalized Ricci flow are the Einstein metrics on M . However, one can consider special solutions to these equations which evolve via diffeomorphisms and rescaling; that is, the solution looks like $g(t) = \sigma(t)\psi_t^*g_0$, where $\sigma(t)$ is a scalar function of time, ψ_t are diffeomorphisms, and g_0 is the initial metric that we started with. The idea is that one is just rescaling space and time.

The initial metric g_0 is called a (*homothetic*) *Ricci soliton* of the Ricci flow, resp. normalized Ricci flow, if $g(t) = \sigma(t)\psi_t^*g_0$ is a solution to the Ricci flow, resp. normalized Ricci flow. It is a simple exercise to show that a metric g_0 is a homothetic Ricci soliton for Ricci flow if and only if it is a homothetic Ricci soliton for normalized Ricci flow. For a comprehensive introduction to Ricci flow and Ricci solitons see [CK04].

Consider a nilpotent Lie group N with left-invariant metric g_0 . As g_0 is left invariant, any solution $g(t) = \sigma(t)\psi_t^*g_0$ to Ricci flow will also be left invariant. We call a left-invariant Ricci soliton a *nilsoliton*. For nilmanifolds the following algebraic characterization was given in [Lau01, Proposition 1.1].

PROPOSITION 6.5. *Let (N, g) be a nilpotent group N with left invariant metric g . Then g is a soliton metric if and only if*

$$ric_g = cI + D$$

for some $c \in \mathbb{R}$ and some symmetric $D \in Der(\mathfrak{N})$.

Remark. Note that when the metric g is a soliton metric, the derivation D is symmetric with respect to g since Id and Ric_g are symmetric. Moreover, it can be shown that the eigenvalues of D (up to scaling) lie in \mathbb{N} .

We will take the characterization of nilsolitons in the proposition to be our working definition of a metric being a soliton metric. On two-step nilmanifolds there is a special kind of soliton metric called optimal metric, see Definition 7.7. These special metrics were first discovered in [EH96] and have many strong geometric properties.

We are interested in when our Lie algebra will admit a soliton metric. More precisely, we want to know when our Lie algebra admits an inner product so that the associated left-invariant metric on our Lie group is a soliton metric. To do this, we can think of varying the inner products on our Lie algebra or we can vary the bases of our Lie algebra and declare them to be orthonormal, see the following Section. These yield the same outcome and are only different in perspective. We will adopt the latter view. That is, we will vary our bases and assign the inner product so that the bases are orthonormal. If we can find a basis so that the associated metric algebra is soliton, we say that our algebra **admits a soliton metric**.

Recall from [Lau01] that a simply connected nilpotent Lie group N admits a soliton metric if and only if N is the nilradical of a rank 1 solvable extension S that admits an Einstein metric. See Proposition 6.13.

DEFINITION 6.6. *A nilpotent Lie algebra \mathfrak{N} is called an Einstein nilradical if \mathfrak{N} admits a soliton metric.*

With Lauret's algebraic characterization of soliton metrics many existence and non-existence results have been obtained for the general case of k -step nilpotent groups. We briefly record some of the known facts concerning which nilpotent Lie groups do and do not admit soliton metrics; for proofs and more detailed exposition on the soliton problem for nilmanifolds see [LW] and references therein. Not much attention has been placed exclusively on two-step nilmanifolds.

PROPOSITION 6.7. *(Necessary condition for existence) Let N be a nilpotent Lie group with a left invariant metric. If N admits a soliton metric, then \mathfrak{N} necessarily admits an \mathbb{N} -grading. That is, there exists a decomposition $\mathfrak{N} = \oplus \mathfrak{N}_i$ such that $[\mathfrak{N}_i, \mathfrak{N}_j] \subset \mathfrak{N}_{i+j}$.*

Remark. The existence of an \mathbb{N} -grading on \mathfrak{N} is equivalent to the existence of a symmetric derivation whose eigenvalues are integral. When the algebra admits a soliton metric, one can choose the symmetric derivation that is given in the previous proposition.

PROPOSITION 6.8. *(Non-existence result) Let \mathfrak{N} be a nilpotent algebra. We say that \mathfrak{N} is characteristically nilpotent if $\text{Der}(\mathfrak{N})$ consists only of nilpotent elements. Such an algebra cannot admit a soliton metric as there do not exist any symmetric derivations.*

All metric two-step nilalgebras have a natural \mathbb{N} -grading. Consider $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$ the orthogonal decomposition relative to our metric, where \mathfrak{Z} is the center and \mathcal{V} its orthogonal complement. Then define $D : \mathfrak{N} \rightarrow \mathfrak{N}$

$$D = \begin{cases} Id & \text{on } \mathcal{V} \\ 2Id & \text{on } \mathfrak{Z} \end{cases}$$

This is a derivation which is symmetric and has integer eigenvalues. Naturally the question was asked: Do all two-steps admit nilsolitons? The answer to this question turns about to be no, see the Theorem 6.10 below. However, as our main result shows, most two-step nilmanifolds admit a soliton metric, see Theorem 7.25.

PROPOSITION 6.9. *(Existence result) Every nilpotent Lie group of dimension ≤ 6 admits a soliton metric.*

See [Wil03] for a proof of this fact.

THEOREM 6.10. *(Non-existence result) There exist two-step nilmanifolds which do not admit an Einstein metric.*

In [LW], Lauret and Will constructed two-step nilalgebras that cannot possibly admit a soliton metric. They achieved their results by a finer analysis of the so called “eigenvalue type” of an algebra. In our context, that work can be summarized as follows. See Section 7.1 for the definition of algebras of type (p, q) .

COROLLARY 6.11. *Consider $(p, q) = (m + 2t, m + t + 1)$ for $m \geq 1$ and $t \geq 0$. All but finitely many of these types have nilalgebras that will not admit a soliton metric.*

In contrast, it is easy to see that all algebras of type $(1, q)$ admit soliton metrics. We suspect that this rigidity happens for some other (p, q) types as well.

Nilgeometry and Negative Curvature. The geometry of nilmanifolds is intimately related to that of solvmanifolds. Recall that negatively curved homogeneous spaces are isometric to solvable Lie groups endowed with left invariant metrics, see [Hei74]. The following tight relationship to negative curvature was obtained in [Lau01, Theorem 3.7].

DEFINITION 6.12. *Let S be a solvable Lie group and let \mathfrak{s} denote the Lie algebra of S . If we denote the nilradical of \mathfrak{s} by \mathfrak{N} , then as a vector space we can decompose $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{N}$ where \mathfrak{a} is a complementary vector space to \mathfrak{N} . The Lie group S and the Lie algebra \mathfrak{s} are said to be of rank k if $\dim \mathfrak{a} = k$.*

PROPOSITION 6.13. *Let (N, g) be a nilsoliton. Then there exists a solvmanifold (S, \tilde{g}) such that*

- (i) $\dim S = \dim N + 1$
- (ii) N is the nilradical of S
- (iii) $\tilde{g}|_{\mathfrak{N}} = g$ and (S, \tilde{g}) is Einstein

Such a solvmanifold S is called a rank 1 extension of N . Conversely, if S is a rank 1 Einstein manifold, then the nilradical N is a soliton.

It was known that any standard Einstein solvmanifold could be reduced to a rank 1 Einstein solvmanifold. This was developed in [Heb98, Section 4.5] where the standard Einstein solvmanifolds were classified. We know that a rank 1 solvmanifold S is a standard Einstein solvmanifold if and only if its nilradical is soliton, by the theorem above. However, it was only recently shown that an Einstein solvmanifold had to be standard. See [Lau07]. In this way, by classifying which nilalgebras admit soliton metrics we are able to classify which solvmanifolds admit Einstein metrics.

4. Algebraic Group actions, Moment Maps, and Einstein Nilradicals

Let \mathfrak{N} be a vector space. We are interested in studying the nilpotent Lie algebra brackets and inner products that can be put on \mathfrak{N} . To do this, we consider the following space $\bigwedge^2 \mathfrak{N}^* \otimes \mathfrak{N}$. This is the space of skew-symmetric bilinear forms on \mathfrak{N} . We can further reduce to the set $\mathcal{N} \subset \bigwedge^2 \mathfrak{N}^* \otimes \mathfrak{N}$ which is the real algebraic variety which consists of nilpotent Lie algebra structures. To see that this is a variety, observe that the Jacobi condition and Cartan's criterion for nilpotency are described by polynomials.

Consider an inner product \langle, \rangle on \mathfrak{N} . For $\mu \in \mathcal{N} \subset \bigwedge^2 \mathfrak{N}^* \otimes \mathfrak{N}$, we denote by $(\mathfrak{N}_\mu, \langle, \rangle)$ the nilpotent Lie algebra with bracket structure μ and metric \langle, \rangle ; similarly, we denote by $(N_\mu, \langle, \rangle)$ the simply connected Lie group over \mathfrak{N}_μ with the corresponding left-invariant metric.

The group $GL(\mathfrak{N})$ acts on the variety $\mathcal{N} \subset \bigwedge^2 \mathfrak{N}^* \otimes \mathfrak{N}$. For $\mu \in \mathcal{N}$ and $g \in GL(\mathfrak{N})$ we have

$$g \cdot \mu(X, Y) = g\mu(g^{-1}X, g^{-1}Y)$$

for $X, Y \in \mathfrak{N}$. Lauret [Lau01] has shown that the orbit $GL(\mathfrak{N}) \cdot \mu$ corresponds to the set of metric Lie algebras with underlying bracket structure μ . In this way, one sees that fixing the inner product on \mathfrak{N} while varying the bracket structure is equivalent to fixing the bracket structure while varying the inner product. This perspective has been very fruitful. We use the same philosophy to study the two-step nilpotent Lie algebras, see Chapter 7.

Moreover, Lauret's work shows the following. The fixed inner product on \mathfrak{N} extends naturally to an inner product on $\bigwedge^2 \mathfrak{N}^* \otimes \mathfrak{N}$. In addition, the group $GL(\mathfrak{N})$ is self-adjoint with respect to this inner product. Although not phrased using the language of distinguished orbits, the following is Theorem 4.2 of [Lau01].

THEOREM 6.14 (Lauret). *Let $\mu \in \mathcal{N} \subset \bigwedge^2 \mathfrak{N}^* \otimes \mathfrak{N}$. Then N_μ is an Einstein nilradical if and only if the orbit $GL(\mathfrak{N}) \cdot \mu$ is a distinguished orbit.*

Recall that distinguished orbits are those that attain critical points of the norm squared of the moment map on projective space, cf. Chapter 5.

Two-step Einstein Nilradicals

The goal of this chapter is to show that a generic two-step nilmanifold admits a soliton metric; that is, generic two-step nilmanifolds are Einstein nilradicals. In fact, we show that except for a small class, most two-step nilmanifolds admit so-called optimal metrics. Optimal metrics are soliton metrics with the additional strong property of geodesic flow invariance. Moreover, we use this approach to calculate the dimension of the moduli of soliton metrics up to scaling and isometry around certain generic points. In [Heb98] Heber calculates the dimension of the moduli space around the rank 1 symmetric spaces, until now this was all that was known in regards to the size of the moduli space.

It has been shown that there do exist two-step nilmanifolds that do not admit soliton metrics, see [LW]. Below we motivate why the set of two-step nilmanifolds is a natural setting for our question.

Our proof of the main theorem follows the works of Lauret and Eberlein. The relationship between left-invariant soliton metrics on nilmanifolds and Geometric Invariant Theory was first worked out by Lauret in [Lau05]. Eberlein used the methods and results of Lauret to study the Ricci tensor of a metric two-step nilpotent Lie group in [Ebe07]. We use theorems of Littlemann-Knopf, Elashvili, and Popov to obtain our result.

1. Two-step Nilmanifolds and Their Stratification

Remark. Our use of the term stratification is in the loose sense; that is, we simply mean a decomposition of the space.

In this chapter, N will denote a simply connected, two-step nilpotent Lie group, see Section 1.2 for the definition of nilpotent. We denote the Lie algebra of N by the gothic letter \mathfrak{N} . Two-step nilpotent groups are the closest groups to being abelian without actually being such. In this case we have $[\mathfrak{N}, \mathfrak{N}] \subset \mathfrak{Z}$, where \mathfrak{Z} is the center of our two-step nilpotent algebra \mathfrak{N} .

As stated before, a left-invariant metric on N is equivalent to an inner product on \mathfrak{N} . We denote such an inner product by \langle, \rangle . Let \mathfrak{Z} denote the center of \mathfrak{N} . Then $[\mathfrak{N}, \mathfrak{N}] \subset \mathfrak{Z}$ and we have an orthogonal decomposition $\mathfrak{N} = \mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$. Since $[\mathcal{V}, \mathcal{V}] \subset \mathfrak{Z}$, we can recover all of the algebra information from the j -map defined by

$$\langle j(Z)v, w \rangle = \langle [v, w], Z \rangle$$

For each $Z \in [\mathfrak{N}, \mathfrak{N}]$ the map $j(Z) : \mathcal{V} \rightarrow \mathcal{V}$ is skew-symmetric. Equivalently, one could define $j(Z)v = (ad\ v)^*Z$, where $(ad\ v)^*$ is the metric adjoint of $ad\ v$ relative to the fixed inner product \langle, \rangle on \mathfrak{N} . We have a linear map $j : [\mathfrak{N}, \mathfrak{N}] \rightarrow \mathfrak{so}(\mathcal{V})$. In the event that j has more structure, e.g. is the representation of a compact algebra, much more can be said about the geometry of \mathfrak{N} . For example, this is how the naturally reductive nilmanifolds arise. See [Lau98, Gor85].

Aroldo Kaplan first used the j -map to study the geometry of nilpotent groups of Heisenberg type in [Kap83]. Eberlein then used the j -map to study all two-step nilgroups. The next two results are propositions 2.5 and 2.7 from [Ebe94].

PROPOSITION 7.1. *Let ric denote the $(2,0)$ -ricci tensor and Ric denote the $(1,1)$ -ricci tensor. These tensors are related by $ric(X, Y) = \langle Ric(X), Y \rangle$ for $X, Y \in \mathfrak{N}$. The following are true*

- (i) *$ric(X, Z) = 0$ for $X \in \mathcal{V}$ and $Z \in \mathfrak{Z}$. So Ric leaves \mathcal{V} and \mathfrak{Z} invariant.*
- (ii) *If $\{Z_1, \dots, Z_m\}$ is an orthonormal basis of \mathfrak{Z} , then $Ric|_{\mathcal{V}} = \frac{1}{2} \sum_{k=1}^m j(Z_k)^2$. From this one sees that $Ric|_{\mathcal{V}}$ is negative definite as the $j(Z_k)^2$ have non-positive eigenvalues.*
- (iii) *$ric(Z, Z^*) = -\frac{1}{4} \text{trace}\{j(Z) \circ j(Z^*)\}$. Thus, $Ric|_{\mathfrak{Z}}$ is positive semi-definite. The kernel of Ric in $\mathfrak{N} = \{Z \in \mathfrak{Z} : j(Z) = 0\} = \{Z \in \mathfrak{Z} : Z \text{ is orthogonal to } [\mathfrak{N}, \mathfrak{N}]\}$.*

Remark. If we write \mathfrak{N} as an orthogonal direct sum $\mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$, then the proposition above is modified as follows. Assertion (i) remains true with \mathfrak{Z} replaced by $[\mathfrak{N}, \mathfrak{N}]$. In (ii) $Ric|_{\mathcal{V}}$ is negative semidefinite if $\{Z_1, \dots, Z_k\}$ is an orthonormal basis of $[\mathfrak{N}, \mathfrak{N}]$ and $Ker\ Ric|_{\mathcal{V}}$ is the common kernel of $\{j(Z) : Z \in [\mathfrak{N}, \mathfrak{N}]\}$. In (iii) $Ric|_{[\mathfrak{N}, \mathfrak{N}]}$ is positive definite.

PROPOSITION 7.2. *Let N be a simply connected 2-step nilpotent Lie group with left-invariant metric $\langle \cdot, \cdot \rangle$. Let $\mathcal{E} = \{Z \in \mathfrak{Z} : j(Z) = 0\}$ and let \mathfrak{N}^* denote its orthogonal complement in \mathfrak{N} relative to our inner product $\langle \cdot, \cdot \rangle$. Then*

- (i) *\mathcal{E} and \mathfrak{N}^* are commuting ideals in \mathfrak{N} and N is the direct product of the subgroups $N^* = \exp(\mathfrak{N}^*)$ and $E = \exp(\mathcal{E})$.*
- (ii) *N is isometric to the Riemannian product of the totally geodesic submanifolds N^* , E , and E is the Euclidean de Rham factor.*

To study the geometry of N and classify our two-step nilalgebras it is simpler to strip off this Euclidean de Rham factor. In regards to our goal of studying which nilalgebras admit a soliton metric we will see that this Euclidean de Rham factor is irrelevant (see Proposition 7.13).

Nilalgebras of type (p, q) .

DEFINITION 7.3. *Let $\mathfrak{N} = \mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$ be a two-step nilalgebra. We say that \mathfrak{N} is of type (p, q) if $\dim[\mathfrak{N}, \mathfrak{N}] = p$ and $\dim \mathcal{V} = q$.*

Remark. No inner product has been assigned to \mathfrak{N} at this point and the decomposition above is merely a direct sum. Additionally, for any choice of metric \langle, \rangle , \mathfrak{N} has no Euclidean de Rham factor precisely when $\mathfrak{Z} = [\mathfrak{N}, \mathfrak{N}]$.

Given a metric algebra $\mathfrak{N} = \mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$, take a basis $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ which respects our decomposition. Such a basis is called an *adapted basis*, see [Ebe03] for more details. Consider the structure coefficients defined by

$$[v_i, v_j] = \sum_k C_{ij}^k Z_k$$

If our basis were orthonormal, then we could equivalently say $C_{ij}^k = \langle [v_i, v_j], Z_k \rangle$. Note the skew-symmetry in i, j . The structure coefficients completely determine the bracket structure of our algebra. We can organize these as a p-tuple of matrices $C = (C^1, \dots, C^p) \in \mathfrak{so}(q, \mathbb{R})^p$. These matrices are called the *structure matrices* of \mathfrak{N} determined by the above adapted basis. It is easy to see that the $\{C^i\}$ are linearly independent, and in particular $p \leq D = \dim \mathfrak{so}(q, \mathbb{R}) = \frac{1}{2}q(q-1)$.

The example $\mathbb{R}^{p+q}(C)$.

Conversely, if we were given a p-tuple $C = (C^1, \dots, C^p) \in \mathfrak{so}(q, \mathbb{R})^p$ where the C^i are linearly independent, then we could associate a metric two-step nilpotent algebra of type (p, q) to it. To do this, just use the standard orthonormal basis $\{e_1, \dots, e_q\}$ as our orthonormal basis of $\mathcal{V} = \mathbb{R}^q$ and take $\{e_{q+1}, \dots, e_{q+p}\}$ to be an orthonormal basis of $[\mathfrak{N}, \mathfrak{N}] = \mathbb{R}^p$. Then define the bracket relations on this vector space with inner product as above using our p-tuple C . That is, $[e_i, e_j] = \sum_k C_{ij}^k e_{q+k}$. The metric nilalgebra constructed in this way will be denoted by $\mathbb{R}^{p+q}(C)$. It is easy to check that the structure matrices of the adapted basis $\{e_1, \dots, e_{p+q}\}$ are $\{C^1, \dots, C^p\}$.

Note that $\mathfrak{Z} = [\mathfrak{N}, \mathfrak{N}] \oplus \mathcal{E}$, where $\mathcal{E} \subset \mathcal{V} \simeq \mathbb{R}^q$ is the common kernel of all the structure matrices. Having no Euclidean de Rham factor is equivalent to $\mathfrak{Z} = [\mathfrak{N}, \mathfrak{N}]$, which is a very natural condition. The algebras of type (p, q) with no Euclidean de Rham factor form a Zariski open set in $\mathfrak{so}(q, \mathbb{R})^p$; the relationship between nilalgebras of type (p, q) and points in the space $\mathfrak{so}(q, \mathbb{R})^p$ is described in Proposition 7.4. This open set is always non-empty except in the case $(p, q) = (1, 2k+1)$. In regards to finding those algebras which admit soliton metrics, having a Euclidean de Rham factor is not an obstruction (see Proposition 7.13) and we will usually assume $\mathfrak{Z} = [\mathfrak{N}, \mathfrak{N}]$. The following characterization of two-step nilalgebras was communicated to us by P. Eberlein and is very useful. See also [Ebe07].

PROPOSITION 7.4. *The tuples $C = (C^1, \dots, C^p) \in \mathfrak{so}(q, \mathbb{R})^p$ which correspond to two-step nilalgebras of type (p, q) form a Zariski open set in $V_{pq} := \mathfrak{so}(q, \mathbb{R})^p$. This Zariski open set consists of p-tuples whose $\{C^i\}$ are linearly independent and is denoted by V_{pq}^0 .*

Action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on $\mathfrak{so}(q, \mathbb{R})^p$.

Let $GL(q, \mathbb{R})$ act on $\mathfrak{so}(q, \mathbb{R})^p$ by $g \cdot (C^1, \dots, C^p) = (gC^1g^t, \dots, gC^pg^t)$. Let $GL(p, \mathbb{R})$ act on $\mathfrak{so}(q, \mathbb{R})^p$ by

$h \cdot (C^1, \dots, C^p) = (D^1, \dots, D^p)$, where $D^j = \sum_{i=1}^p h_{ji} C^i$. We identify $\mathfrak{so}(q, \mathbb{R})^p$ with $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ via the isomorphism $(C^1, \dots, C^p) \mapsto \sum_{i=1}^p C^i \otimes e_i$, where $\{e_i\}$ is the standard basis of \mathbb{R}^p .

Consider the following group actions. Let $GL(q, \mathbb{R})$ act on $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ by $g(M \otimes v) = (gMg^t \otimes v)$ and let $GL(p, \mathbb{R})$ act on $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ by $h(M \otimes v) = M \otimes (hv)$, where $g \in GL(q, \mathbb{R})$, $h \in GL(p, \mathbb{R})$, $M \in \mathfrak{so}(q, \mathbb{R})$, $v \in \mathbb{R}^p$ and $GL(p, \mathbb{R})$ acts on \mathbb{R}^p in the usual way. One immediately see that the above isomorphism $\mathfrak{so}(q, \mathbb{R})^p \simeq \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$ is equivariant with respect to the actions of $GL(q, \mathbb{R})$ and $GL(p, \mathbb{R})$. In particular, since the actions of $GL(q, \mathbb{R})$ and $GL(p, \mathbb{R})$ commute on $\mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$, the actions of $GL(q, \mathbb{R})$ and $GL(p, \mathbb{R})$ commute on $\mathfrak{so}(q, \mathbb{R})^p$ and we obtain an action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on both spaces that respects the isomorphism.

Compatible inner product on $\mathfrak{so}(q, \mathbb{R})^p$. Let \langle, \rangle denote the canonical inner product on $\mathfrak{so}(q, \mathbb{R})$ given by $\langle A, B \rangle = -\text{tr}(AB)$, and extend \langle, \rangle to $\mathfrak{so}(q, \mathbb{R})^p$ by defining $\langle (C^1, \dots, C^p), (D^1, \dots, D^p) \rangle = \sum_{i=1}^p \langle C^i, D^i \rangle$. The group $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ is self-adjoint with respect to \langle, \rangle and $K = O(q, \mathbb{R}) \times O(p, \mathbb{R})$ is the fixed group of the corresponding Cartan involution of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$.

Change of basis formulas. Varying the inner products on \mathcal{V} and $[\mathfrak{N}, \mathfrak{N}]$ is equivalent to changing the bases for \mathcal{V} and $[\mathfrak{N}, \mathfrak{N}]$; see Proposition 7.12 for justification of varying the inner products on only these pieces as opposed to all of \mathfrak{N} . Let $\{v'_1, \dots, v'_q, Z'_1, \dots, Z'_p\}$ be another basis of $\mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$. Then there exists $g \in GL(\mathcal{V})$ and $h \in GL([\mathfrak{N}, \mathfrak{N}])$ such that

$$v'_i = \sum_j g_{ij} v_j \quad Z'_k = \sum_l h_{kl} Z_l$$

How do the structure matrices for these different bases compare? Let C' be the structure matrix with respect to the basis $\{v'_1, \dots, v'_q, Z'_1, \dots, Z'_p\}$. That is, $[v'_i, v'_j] = \sum_k C'_{ij}{}^k Z'_k$. Substituting in the old basis written in terms of the new basis we can relate C and C' by

$$\sum_{ts} g_{it} g_{js} C_{ts}^l = \sum_k h_{kl} C'_{ij}{}^k \quad \text{for } 1 \leq l \leq p, 1 \leq i, j \leq q$$

The left hand side of the equation above is $(gC^l g^t)_{ij}$. If $h^t(C') = (D^1, \dots, D^p)$ relative to the action of $GL(p, \mathbb{R})$ on $\mathfrak{so}(q, \mathbb{R})^p$, then the right hand side of the equation is $(D^l)_{ij}$. Hence the equation may be written as $g \cdot C = h^t \cdot C'$ or $C' = (g, (h^t)^{-1}) \cdot C$ relative to the action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on $\mathfrak{so}(q, \mathbb{R})^p$. This is our motivation for studying the action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on $\mathfrak{so}(q, \mathbb{R})^p$. The following is a neat interpretation of the change of basis formula above.

PROPOSITION 7.5. *Isomorphism classes of two-step nilalgebras correspond to $GL(q, \mathbb{R}) \times GL_p(\mathbb{R})$ orbits in $V_{pq}^0 \subset \mathfrak{so}(q, \mathbb{R})^p$.*

The group action listed above immediately gives rise to an action of the Lie algebra $\mathfrak{gl}(q, \mathbb{R}) \times \mathfrak{gl}(p, \mathbb{R})$. Let $(X, Y) \in \mathfrak{gl}(q, \mathbb{R}) \times \mathfrak{gl}(p, \mathbb{R})$ and $C = (C^1, \dots, C^p) = \sum_k C^k \otimes e_k$ then

$$(X, Y) \cdot C = \sum_k X(C^k) \otimes e_k + C^k \otimes Y(e_k) = \sum_k (XC^k + C^k X^t) \otimes e_k + C^k \otimes Y e_k$$

We will be interested later in symmetric derivations. Since such a derivation preserves $[\mathfrak{N}, \mathfrak{N}]$ and hence \mathcal{V} , we record some information about automorphisms and derivations that respect the decomposition $\mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$.

PROPOSITION 7.6. *Let $M = (M_1, M_2) \in M(q, \mathbb{R}) \times M(p, \mathbb{R})$ act on $\mathbb{R}^{p+q}(C)$ in the usual way: for $1 \leq i \leq q$, $M(e_i) = M_1(e_i) = \sum_j (M_1)_{ji} e_j$ and for $1 \leq k \leq p$ $M(e_{q+k}) = M_2(e_{q+k}) = \sum_l (M_2)_{lk} e_{q+l}$. Then*

- (i) *If $M = (M_1, M_2) \in GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ then M is an automorphism of $\mathbb{R}^{p+q}(C)$ if and only if $M_1^t \cdot C = M_2 \cdot C$, with the action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on $\mathfrak{so}(q, \mathbb{R})^p$ defined above.*
- (ii) *If $M = (M_1, M_2) \in M(q, \mathbb{R}) \times M(p, \mathbb{R})$ then M is a derivation of $\mathbb{R}^{p+q}(C)$ if and only if $M_1^t \cdot C = M_2 \cdot C$, with the action of $\mathfrak{gl}(q, \mathbb{R}) \times \mathfrak{gl}(p, \mathbb{R})$ on $\mathfrak{so}(q, \mathbb{R})^p$ defined above.*

The proof is immediate upon writing out the bracket conditions to be a derivation or an automorphism in terms of the natural basis $\{e_i\}$ of $\mathbb{R}^{p+q}(C)$.

2. Soliton Metrics on Nilmanifolds

In Chapter 6 we gave the definition of soliton metrics on nilmanifolds. Recall the following proposition which gives an algebraic characterization of soliton metrics on nilmanifolds.

THEOREM 6.5 *Let (N, g) be a nilpotent group N with left invariant metric g . Then g is a soliton metric if and only if*

$$ric_g = cI + D$$

for some $c \in \mathbb{R}$ and some symmetric $D \in Der(\mathfrak{N})$.

Recall that this is our working definition of a metric being a soliton metric. A special kind of soliton metric is the so called optimal metric, defined below. These special metrics were first discovered in [EH96] and have many strong geometric properties.

DEFINITION 7.7. *Let N be a two-step nilmanifold with inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{N} . Then we say $\langle \cdot, \cdot \rangle$ is an optimal metric if there exist $\lambda, \mu > 0$ such that $Ric = -\lambda Id$ on \mathcal{V} and $Ric = \mu Id$ on $[\mathfrak{N}, \mathfrak{N}]$.*

DEFINITION 7.8. *We will say that $C \in \mathfrak{so}(q, \mathbb{R})^p$ has a property if and only if $\mathbb{R}^{p+q}(C)$ has the said property. For example, we say C is optimal or soliton if $\mathbb{R}^{p+q}(C)$ is, respectively, optimal or soliton. We say that C admits a property if $g \cdot C$ has that property for some $g \in GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$.*

Criterion for optimal metrics. Recall that if \mathfrak{N} is a two-step nilpotent Lie algebra of type (p, q) , then a basis \mathcal{B} of \mathfrak{N} is adapted if it contains a basis of $[\mathfrak{N}, \mathfrak{N}]$. If we write $\mathcal{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$, where $\{Z_1, \dots, Z_p\}$ is a basis of $[\mathfrak{N}, \mathfrak{N}]$, then \mathcal{B} determines structure matrices $\{C^1, \dots, C^p\} \subset \mathfrak{so}(q, \mathbb{R})$ by the bracket relations $[v_i, v_j] = \sum_{k=1}^p C_{ij}^k Z_k$ for $1 \leq i, j \leq q$.

The next result is Proposition 7.4 of [Ebe07].

PROPOSITION 7.9. *Let \mathfrak{N} be a two-step nilpotent Lie algebra of type (p, q) and let $\{C^1, \dots, C^p\}$ be the structure matrices of some adapted basis \mathcal{B} . Then \mathfrak{N} admits a metric \langle, \rangle with optimal Ricci tensor if and only if the $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ orbit of $C = (C^1, \dots, C^p)$ is closed in $V_{pq} = \mathfrak{so}(q, \mathbb{R})^p$.*

CLAIM 7.10. *Optimal metrics are soliton.*

PROOF. We need to show that there exists a symmetric derivation $D \in \text{Der}(\mathfrak{N})$ and $c \in \mathbb{R}$ such that $\text{Ric} = c\text{Id} + D$. Consider the map $D = d \text{Id}$ on \mathcal{V} and $D = 2d \text{Id}$ on \mathfrak{Z} . This is a derivation of any 2-step nilalgebra. Then using $c = -2\lambda - \mu$ and $d = \mu + \lambda$ we have the desired result. \square

Recall that fixing the bracket structure on \mathfrak{N} and varying the inner product is equivalent to fixing the inner product on \mathfrak{N} and varying the bracket structure. We vary the bracket structure via the change of basis action, cf. Chapter 6. Recall that \mathfrak{N} is an *Einstein nilradical* if \mathfrak{N} admits a soliton metric. We can specialize our change of basis action as follows.

Let \mathfrak{s} be the Lie algebra of a solvable Lie group S . In Heber's development of the the classification of (standard) Einstein solvmanifolds, he showed that if \mathfrak{s} admits an Einstein metric then one can change the basis in a very special way to achieve an Einstein metric on \mathfrak{s} , see Proposition 6.8 of [Heb98] for a proof of the following.

THEOREM 7.11. *Suppose that \mathfrak{N} admits a soliton metric. That is, given an inner product \langle, \rangle on \mathfrak{N} , there exists $g \in GL(\mathfrak{N})$ such that $g \cdot \langle, \rangle$ is a soliton metric on \mathfrak{N} . Then we have a decomposition $\mathfrak{N} = \oplus \mathfrak{N}_i$, where the \mathfrak{N}_i are the eigenspaces of a symmetric derivation of \mathfrak{N} . Moreover, we can actually choose $g \in GL_1 \times GL(N_1) \times \dots \times GL(N_k)$ so that $g \cdot \langle, \rangle$ is a soliton metric.*

Now we have a strong motivation to exploit our stratification of two-step nilalgebras. In the two-step case, Heber's theorem translates to the following

PROPOSITION 7.12. *Suppose a two-step nilalgebra \mathfrak{N} admits a soliton metric. That is, there exists $A \in GL(\mathfrak{N})$ such that $A \cdot \langle \cdot, \cdot \rangle = \langle A^{-1} \cdot, A^{-1} \cdot \rangle$ is a soliton metric. Then there exists $B \in GL(\mathcal{V}) \times GL([\mathfrak{N}, \mathfrak{N}])$ such that $B \cdot \langle \cdot, \cdot \rangle = \langle B^{-1} \cdot, B^{-1} \cdot \rangle$ is a soliton metric.*

This proposition is the motivation for studying the change of basis action on just the \mathcal{V} and $[\mathfrak{N}, \mathfrak{N}]$ parts. In what follows we retranslate the soliton condition to the standpoint of the $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ action on V_{pq} . Before continuing we make note of the following

PROPOSITION 7.13. *Let N be a two-step nilgroup and let N^* be defined as above, that is, $N = N^* \times E$ where E is the Euclidean de Rham factor of N . Then N admits a soliton metric if and only if N^* does so.*

PROOF. Recall from theorem 6.5 that g is a soliton metric if and only if $Ric_g = cId + D$ for some $c \in \mathbb{R}$ and some $D \in Der(\mathfrak{N})$. Let g be our metric on N and g^* be the restriction onto N^* . We denote the Ricci tensor for (N, g) by Ric and the Ricci tensor for (N^*, g^*) by Ric^* . Using Propositions 7.1 & 7.2 observe that

$$Ric|_{\mathfrak{N}^*} = Ric^*$$

and

$$Ric|_{\mathcal{E}} = 0$$

Since $Ric = cId + D$ we have $D|_{\mathcal{E}} = -cId$. That is D preserves \mathcal{E} and since it is symmetric it also preserves \mathfrak{N}^* . Moreover, since D is a derivation of \mathfrak{N} we see that $D^* = D|_{\mathfrak{N}^*}$ is a derivation of \mathfrak{N}^* . Hence we have obtained

$$Ric^* = Ric|_{\mathfrak{N}^*} = cId + D|_{\mathfrak{N}^*} = cId + D^*$$

where D^* is a derivation of \mathfrak{N}^* .

□

We would like to work out the criteria for a metric two-step nilalgebra to be a soliton from the perspective of the $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ action on V_{pq} . Since every metric two-step nilalgebra takes the form $\mathbb{R}^{p+q}(C)$ for some C , it is reasonable to state the requirements from this point of view. This will be worked out in Section 7.4 but first recall some basic information about algebraic group actions (cf. Chapters 1 and 2).

3. Algebraic Group Actions and Certain Special Representations

We briefly recall some theorems from real Geometric Invariant Theory (GIT). We use GIT as a tool to study the action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on V_{pq} to obtain some very general results. See Chapter 2 for a more thorough introduction to GIT.

For the above representation we are interested in finding the orbits which are either closed or distinguished (cf. Definition 5.2); moreover, we wish to show that generic orbits are either closed or distinguished. The following theorems motivate our treatment of two-step nilsolitons.

PROPOSITION 2.8 *Consider an algebraic group G which acts linearly on V . Let $G \cdot v$ be a closed orbit of maximal dimension, then the set of closed orbits is dense in V . Alternatively we could say, if there exists a closed generic orbit, then generic orbits are closed, that is, the set of closed orbits contains a nonempty Zariski open set.*

PROPOSITION 2.30 *Consider $v \in V \subset V^{\mathbb{C}}$. Then $G \cdot v$ is closed if and only if $G^{\mathbb{C}} \cdot v$ is closed. Moreover, since V is Zariski dense in $V^{\mathbb{C}}$, V has a Zariski open set of closed orbits if and only if $V^{\mathbb{C}}$ does so.*

THEOREM 2.21 *An orbit $G \cdot v$ is closed if and only if it contains a minimal vector. Moreover, $\mathfrak{M} \cap \overline{G \cdot v} =$ a single K -orbit.*

Remark. We show in Section 7.4 that minimal vectors in a particular setting correspond to metrics on two-step nilmanifolds with nice geometric properties.

THEOREM 3.4 *Let G be a semi-simple group. If the generic stabilizer is a reductive subgroup, then generic orbits are closed.*

We point out that if we have an open set of points whose stabilizers are reductive not all of these points necessarily have closed orbits. For more detailed information see [PV94] and Chapters 2-4. A lot of work was poured into the problem of groups acting linearly on vector spaces, i.e., representations. Since most representations of complex semi-simple groups have trivial generic stabilizers, lists were developed to understand the remaining cases. The following is a subset of the tables listed in [KL87, Ela72].

PROPOSITION 7.14. *Let $1 \leq p \leq \frac{1}{2}q(q-1) = D$. For all pairs (p, q) other than $(1, 2k+1), (2, 2k+1), (D-1, 2k+1), (D-2, 2k+1)$, the generic stabilizer of $SL(q, \mathbb{C}) \times SL(p, \mathbb{C})$ acting on $\left(\bigwedge^2 \mathbb{C}^q\right) \otimes \mathbb{C}^p$ is reductive. Here generic orbits are closed.*

PROOF. To verify this fact, one just consults the lists generated in [KL87] and [Ela72]. In fact, for most of these representations the stabilizer is finite. Knop and Littleman record the groups with representation, up to outer automorphism and casting transformation (defined below in Lemma 7.15), whose generic stabilizer is not finite. We also note that the list in [KL87] picks up some cases that were originally missed in [Ela72].

Our goal is to apply Theorem 3.4 by showing that in the cases listed above the generic stabilizer is reductive. In referencing Tables 2a & 2b of [KL87], we first look for our group type $A_n \times A_m$. Recall $SL_{n+1} = A_n$. If the group doesn't show up in the table, then we know that for our representation the generic stabilizer is finite, which is reductive, and hence generic orbits are closed.

Before continuing with the groups whose generic stabilizer is not finite we need two lemmas.

LEMMA 7.15. *Let $G \times SL(p, \mathbb{C})$ act on $V \otimes \mathbb{C}^p$ where each group acts on the respective factor. Let $n = \dim V$, assume $p < n$, and let $p' = n - p$. If the generic stabilizer of $G \times SL(q, \mathbb{C})$ acting on $V \otimes \mathbb{C}^p$ is reductive, then the generic stabilizer of $G \times SL(p', \mathbb{C})$ acting on $V^* \otimes \mathbb{C}^{p'}$ is also reductive. Here V^* is the dual representation. In fact the stabilizers are isomorphic. This transformation is called the castle transform of $G \times SL(q, \mathbb{C})$ acting on $V \otimes \mathbb{C}^p$.*

Proof of lemma. See Corollary 1 of [Ela72].

LEMMA 7.16. *Let $\rho : G \rightarrow GL(V)$ be a representation of G and let $\sigma \in \text{Aut}(G)$. Then the generic stabilizer of $\rho(G)$ is reductive if and only if the generic stabilizer of $\rho \circ \sigma(G)$ is reductive. Moreover, these stabilizers are isomorphic.*

Proof of lemma. The proof is trivial, but this lemma allows us to find groups and representations in Tables 2a & 2b of [KL87] up to automorphism of the group. We can then apply Theorem 3.4.

CLAIM 7.17. *For the representations given in the proposition above, type (p, q) is equivalent to type $(D - p, q)$ via automorphisms of the group and castling transformation, where $D = \frac{1}{2}p(p - 1) = \dim \mathfrak{so}(q, \mathbb{R})$.*

Proof of claim. To see this we first construct an outer automorphism σ of $G = SL(q, \mathbb{C})$ so that G acting on V^* is equivalent to $\sigma(G)$ acting on V . Once we have this automorphism of $SL(q, \mathbb{C})$, we consider the automorphism $\sigma \times \text{id}$ of $SL(q, \mathbb{C}) \times SL_{D-p}\mathbb{C}$. Now we have the desired composition:

$$V \otimes \mathbb{C}^p \xrightarrow{\text{castle transform}} V^* \otimes \mathbb{C}^{D-p} \xrightarrow{\sigma \times \text{id}} V \otimes \mathbb{C}^{D-p}$$

We finish the proof of the claim. For $f \in V^*$, $g \cdot f(*) = f(g^{-1}*)$. Next we construct the automorphism σ of $SL_q\mathbb{C}$. Recall that we have a symmetric, non-degenerate bilinear form B on $V = \bigwedge^2 \mathbb{C}^q = \mathfrak{so}(q, \mathbb{C})$. $B(v, w) = \text{tr}(v w^t)$ for $v, w \in \mathfrak{so}(q, \mathbb{C})$. For $g \in SL(q, \mathbb{C})$, the adjoint with respect to B corresponds with the usual transpose of g . Define $\sigma \in \text{Aut}(SL(q, \mathbb{C}))$ by $\sigma(g) = (g^t)^{-1}$. Now consider $F : V \rightarrow V^*$ defined by $F(v) = B(v, \cdot)$. This is an isomorphism as B is non-degenerate. It is also easy to check the equivariance of the G -actions, i.e., $F(g \cdot v) = \sigma(g) \cdot F(v)$.

Now Lemmas 7.15 and 7.16 show that if the generic stabilizer is reductive in the (p, q) case, then it is so for the $(D - p, q)$ case. This completes the proof of the claim.

To finish the proof of our proposition, we note that Knop-Littelmann only record one of (p, q) or $(D - p, q)$. The groups $SL(q, \mathbb{C}) \times SL(p, \mathbb{C})$ whose generic stabilizer is not finite correspond to $(p, q) = \{(2, m) \text{ with } m \text{ even}, (3, 4), (3, 5), (3, 6)\}$. Elashvili calculated the generic stabilizer for all of these except $(3, 6)$ in [Ela72]. He found these to be reductive. The case $(3, 6)$ is taken care of in [EJ] and has generically reductive stabilizers. \square

The above proposition is of importance because the complexification of $\mathfrak{so}(q, \mathbb{R})$ is $\bigwedge^2 \mathbb{C}^q$ and our action of $SL(q, \mathbb{R})$ complexifies to the representation of $SL(q, \mathbb{C})$ with highest weight ω_2 .

The remaining exceptional cases also need to be analyzed to understand the metrics that a two-step nilmanifold can admit. In these cases the only minimal vector is the origin. So we move to projective space and study the critical points of the moment map instead of just zeros of the moment map. These points have geometric significance which will be explained below.

The Moment Map on Projective Space. We recall the definition of the moment map on projective space and the definition of distinguished points. See Chapter 5 for more details. Since the moment map $m : V \rightarrow \mathfrak{p}$ is homogeneous of degree 2 we can consider the map $\overline{m} : \mathbb{P}V \rightarrow \mathfrak{p}$ defined by

$$\overline{m}([v]) = \frac{m(v)}{|v|^2}$$

. This map is also called the *moment map*.

DEFINITION 5.2 *The points $v \in V$ and $[v] \in \mathbb{P}V$ are called distinguished points if $[v] \in \mathbb{P}V$ is a critical point of $|\overline{m}|^2$. If $G \cdot [v]$ contains a distinguished point then we say $G \cdot [v]$ and $G \cdot v$ are distinguished orbits.*

LEMMA 7.18. *A point $v \in V$ is distinguished if and only if $m(v) \cdot v = rv$ for some $r \in \mathbb{R}$.*

Remark. Note that $r \geq 0$ since $r|v|^2 = \langle m(v) \cdot v, v \rangle = |m(v)|^2$.

The proof of the lemma is a simple exercise which we leave to the reader. To understand and find distinguished vectors, we will resort to using complex groups. Consider $G \curvearrowright V$ and its complexification $G^{\mathbb{C}} \curvearrowright V^{\mathbb{C}}$. Let $v \in V \subset V^{\mathbb{C}}$. It is known that $(G^{\mathbb{C}} \cdot v) \cap V$ is a finite union of G -orbits (see Proposition 1.15). The following theorem and corollary give a way of finding distinguished vectors in the real setting.

THEOREM 5.7 *Let G be a reductive real algebraic group and $G^{\mathbb{C}}$ its complexification. Then $G \cdot v$ is distinguished if and only if $G^{\mathbb{C}} \cdot v$ is distinguished. Consequently, if $G \cdot v$ is distinguished, then each of the other finite orbits that comprise $G^{\mathbb{C}} \cdot v \cap V$ is also distinguished.*

COROLLARY 7.19. *If an orbit is Hausdorff open in $\mathbb{P}V$ and is distinguished, then generic points in V lie on distinguished orbits.*

PROOF. Our group G will have an open orbit in $\mathbb{P}V$ if and only if $\mathbb{R} \times G$ has an open orbit in V . The property of being a distinguished point is scale invariant, so $G \cdot v$ being a distinguished orbit is equivalent to $\mathbb{R} \times G \cdot v$ being a distinguished orbit.

We recall that $\mathbb{R} \times G \cdot v$ being an open orbit in V is equivalent to $\mathbb{C} \times G^{\mathbb{C}} \cdot v$ being a Zariski open orbit in $V^{\mathbb{C}}$. To see this, note that the real dimension of the first equals the complex dimension of the second. Finally, a complex orbit is open in its Zariski closure, which in this case is $V^{\mathbb{C}}$ since the complex orbit contains a Hausdorff open set (cf. [Bor91, Proposition 1.8]). Since the real orbit is distinguished, we have that the complex orbit is also distinguished by the theorem above. But then $\mathbb{C} \times G^{\mathbb{C}} \cdot v \cap V$ is a union of finitely many $\mathbb{R} \times G$ -orbits and is a Zariski open set of V all of whose points lie on $\mathbb{R} \times G$ -distinguished orbits, again by Theorem 5.7 and Proposition 1.15. Hence we have found a Zariski open set in V of points whose G -orbits are distinguished. \square

This corollary is exceptionally useful for finding distinguished orbits. Recall that a Zariski open set in $V^{\mathbb{C}}$ is Hausdorff connected whereas a Zariski open set in V will often have many disconnected Hausdorff components. Using the theorem above on our real space V , if we can show that all the orbits in just one Hausdorff component are distinguished, then we are guaranteed that this happens for all the Hausdorff components. Hence it happens on a Hausdorff dense set. Our application is

PROPOSITION 7.20. *In the exceptional cases of $(1, 2k+1)$, $(2, 2k+1)$, $(D-1, 2k+1)$, $(D-2, 2k+1)$ the action $SL(2k+1, \mathbb{R}) \times SL(p, \mathbb{R}) \curvearrowright V_{p, 2k+1}$ has generic orbits which are distinguished.*

See Section 5 for details.

4. Soliton Metrics on the Two-step Metric Algebra $\mathbb{R}^{p+q}(C)$

In this section we write out the conditions that the metric two-step nilalgebra $\mathbb{R}^{p+q}(C)$ admits a soliton metric. The following description of solitons may be found in [Ebe07].

The Ric map. We can define a function $Ric : V_{pq} = \mathfrak{so}(q, \mathbb{R})^p \rightarrow Symm(q, \mathbb{R}) \times Symm(p, \mathbb{R})$ that captures all of the information of the Ricci tensor for $\mathbb{R}^{p+q}(C)$. Here $Symm(q, \mathbb{R})$ is the set of symmetric $q \times q$ matrices. Given $C = (C^1, \dots, C^p) \in V_{pq} = \mathfrak{so}(q, \mathbb{R})^p$ we define

$$(7.1) \quad Ric(C) = (Ric_1(C), Ric_2(C))$$

where

$$\begin{aligned} Ric_1(C) &= -2 \sum_{k=1}^p (C^k)^2 \\ Ric_2(C)_{ij} &= -trace(C^i C^j) = \langle C^i, C^j \rangle \end{aligned}$$

Remark. The map $Ric : V_{pq} \rightarrow Symm(q, \mathbb{R}) \times Symm(p, \mathbb{R})$ is also the moment map for the action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on V_{pq} (see Example 2.28). The following proposition justifies the statement above that our map $Ric : V_{pq} = \mathfrak{so}(q, \mathbb{R})^p \rightarrow Symm(q, \mathbb{R}) \times Symm(p, \mathbb{R})$ captures the information of the Ricci tensor for $\mathbb{R}^{p+q}(C)$.

PROPOSITION 7.21. *Let $(\mathfrak{N}, \langle, \rangle)$ be a metric two-step nilalgebra of type (p, q) . Let $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ be an orthonormal adapted basis for \mathfrak{N} with structure element C . Let Ric denote the ricci $(1,1)$ -tensor of N , that is, $ric(X, Y) = \langle Ric(X), Y \rangle$. Then*

- (i) *Ric leaves $[\mathfrak{N}, \mathfrak{N}] = span\{Z_1, \dots, Z_p\}$ and $[\mathfrak{N}, \mathfrak{N}]^{\perp} = span\{v_1, \dots, v_q\}$ invariant*
- (ii) *Ric restricted to $[\mathfrak{N}, \mathfrak{N}]^{\perp}$ has matrix $-\frac{1}{4}Ric_1(C)$ relative to $\{v_1, \dots, v_q\}$*
- (iii) *Ric restricted to $[\mathfrak{N}, \mathfrak{N}]$ has matrix $\frac{1}{4}Ric_2(C)$ relative to $\{Z_1, \dots, Z_p\}$*

We want to relate the map Ric to a group action. Writing out the conditions to be a soliton metric algebra we have

PROPOSITION 7.22. $\mathbb{R}^{p+q}(C)$ is a Ricci soliton if and only if $\text{Ric}(C) \cdot C = \alpha C$ for some $\alpha \in \mathbb{R}$. Here $\text{Ric}(C) \in \text{Symm}(q, \mathbb{R}) \times \text{Symm}(p, \mathbb{R}) \subset M(q, \mathbb{R}) \times M(p, \mathbb{R})$ acts via the Lie algebra action induced by the group action of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ on V_{pq}

COROLLARY 7.23. $\mathbb{R}^{p+q}(C)$ admits a soliton if and only if $SL(q, \mathbb{R}) \times SL(p, \mathbb{R}) \cdot C$ is a distinguished orbit if and only if $GL(q, \mathbb{R}) \times GL(p, \mathbb{R}) \cdot C$ is a distinguished orbit.

PROOF OF THE PROPOSITION. Suppose first that $\text{Ric}(C) \cdot C = \alpha C$. Then we have $\text{Ric}_1(C) \cdot C + \text{Ric}_2(C) \cdot C = \alpha C$ which gives

$$-\frac{1}{4}\text{Ric}_1(C) \cdot C + 2\frac{\alpha}{4}C = \frac{1}{4}\text{Ric}_2(C) \cdot C + \frac{\alpha}{4}C$$

Define $D = (D_1, D_2) \in \text{Symm}(q) \times \text{Symm}(p)$ by $D_1 = -\frac{1}{4}\text{Ric}_1(C) + \frac{\alpha}{4}I_q$ and $D_2 = \frac{1}{4}\text{Ric}_2(C) + \frac{\alpha}{4}I_p$. Then $D = \text{Ric}_{\mathfrak{N}} + \frac{\alpha}{4}Id$ by proposition 7.21, where $\text{Ric}_{\mathfrak{N}}$ is the Ricci (1,1)-tensor. Since $D_1(C) = D_2(C)$, D is a derivation by Proposition 7.6, and thus $\mathbb{R}^{p+q}(C)$ is a soliton.

Conversely, suppose $\mathbb{R}^{p+q}(C)$ is a Ricci soliton and consider the symmetric derivation $D = \text{Ric}_{\mathfrak{N}} + \lambda Id = (-\frac{1}{4}\text{Ric}_1(C) + \lambda Id, \frac{1}{4}\text{Ric}_2(C) + \lambda Id)$ for some λ ; we see that $\text{Ric}(C) \cdot C = 4\lambda C$. \square

PROOF OF THE COROLLARY. It was shown above, Example 2.28, that the moment map of $G = GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ is $m_G = (m_1, m_2) = (\text{Ric}_1, \text{Ric}_2) = \text{Ric}$ and the moment map for $H = SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ is $m_H = (\text{Ric}_1 - \lambda Id, \text{Ric}_2 - \mu Id)$ where $\lambda(C) = \frac{2|C|^2}{q}$ and $\mu(C) = \frac{|C|^2}{p}$. Also, notice that $(r_1 Id, r_2 Id) \cdot C = (2r_1 + r_2)C$. So one has

$$m_H(C) \cdot C = m_G(C) \cdot C + (-2\lambda(C) - \mu(C))C$$

Clearly, C is an eigenvector of $m_H(C)$ if and only if it is so for $m_G(C)$. Now apply Lemma 7.18. \square

Observe that the optimal metrics happen exactly when $m_H(C) = 0$. This is a special case of having a critical point and our orbit is more than just distinguished, it is actually closed. From this discussion we obtain the following.

COROLLARY 7.24. $\mathbb{R}^{p+q}(C)$ has an optimal metric if and only if $m_H(C) = 0$.

More generally we have the following (cf. Proposition 7.4 of [Ebe07]).

PROPOSITION 7.9 Consider a metric two-step nilalgebra \mathfrak{N} and its associated tuple of structure matrices $C_{\mathcal{B}} \in V_{pq}$ for an adapted basis \mathcal{B} of \mathfrak{N} . Then \mathfrak{N} admits an optimal metric if and only if the orbit $SL(q, \mathbb{R}) \times SL(p, \mathbb{R}) \cdot C_{\mathcal{B}}$ is closed in $V_{pq} = \mathfrak{so}(q, \mathbb{R})^p$.

Recall that for a two-step nilpotent Lie algebra of type (p, q) we define $D = \dim \mathfrak{so}(q, \mathbb{R}) = \frac{1}{2}q(q-1)$. Using Proposition 7.14, and the fact that a real group has a closed orbit if and only if its complexification does so,

cf. Proposition 2.30, we see that for types (p, q) other than $(1, 2k+1), (2, 2k+1), (D-1, 2k+1), (D-2, 2k+1)$ a generic orbit is closed. That is, a generic tuple of structure matrices corresponds to a two-step Einstein nilradical.

In the four exceptional cases we will show in Section 5 below that a generic orbit is distinguished but not closed. We use the main result of Section 5. Putting these together we have our main result.

THEOREM 7.25. *A generic two-step nilmanifold is an Einstein nilradical. Moreover, the types (p, q) other than $(1, 2k+1), (2, 2k+1), (D-1, 2k+1), (D-2, 2k+1)$ generically admit optimal metrics.*

Remark. It is very important to note that when we say generic we mean it in the Hausdorff sense. That is, every two-step nilalgebra is the limit of algebras which admit a soliton metric.

Dimension of the moduli spaces. We use the results that are known for the representations of interest to us to calculate the dimension of moduli of Einstein metrics on rank 1 solvmanifolds whose nilradical is two-step. This question was raised in [Heb98]. Here he gave computations that calculated the dimension of the moduli space near the rank 1 symmetric spaces.

Computing the moduli of Einstein metrics on rank 1 solvmanifolds is equivalent to computing the moduli of nilsolitons. In the two-step case, we can give a complete answer near the generic algebras. The dimension of the moduli is the dimension of the open set of smooth points. Since $\mathfrak{M}/K \simeq V//G$, cf. [RS90], the dimension of this open set $= \dim V_{pq} - \dim \text{generic orbit} = \dim V_{pq} - \dim H + \dim H_v$, where $H = SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ and H_v is the generic stabilizer. This finds the dimension of the moduli space up to isometry, but then one needs to subtract 1 more to know the dimension up to isometry and scaling.

Since $\mathfrak{h}_v^{\mathbb{C}} = (\mathfrak{h}_v)^{\mathbb{C}}$ the dimension of our real moduli space equals the complex dimension of the complex moduli space $V_{pq}^{\mathbb{C}}//SL(q, \mathbb{C}) \times SL(p, \mathbb{C})$. All of the information needed to compute this is contained in the lists of Elashvili. Additionally this information was computed by Knop-Littlemann in [KL87]. The following table lists the dimension of the moduli of nilsolitons up to scaling and isometry. Note, the dimension of moduli will be the same for (p, q) and the dual $(D-p, q)$. The information listed below appears also in Propositions A and B of [Ebe03] since these are also the dimensions of the spaces $X(p, q)$ of isomorphism classes of two-step nilpotent Lie algebras of type (p, q) .

Dimension of Moduli about generic points

(p, q) and $(D - p, q)$	dimension
$(1, q)$	0
$(2, 4)$	0
$(2, 2k), k \geq 3$	$k-3$
$(2, 2k + 1)$	0
$(3, 4)$	0
$(3, 5)$	0
$(3, 6)$	2
(D, q)	0
all other (p, q)	$p \frac{1}{2}q(q-1) - (q^2 + p^2 - 2) - 1$

5. The Exceptional Cases

We derive the results for the four exceptional cases here. In each of these cases the group $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ will have an open orbit in $\mathbb{P}V_{pq}$ which says that we have an open set in V_{pq} of points whose orbits are distinguished, cf. Corollary 7.19.

PROPOSITION 7.26. *If $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ has an open orbit in $\mathbb{P}V_{pq}$ then $SL(q, \mathbb{R}) \times SL(D - p, \mathbb{R})$ has an open orbit in $\mathbb{P}V_{D-p, q}$.*

Before beginning the the proof we need the following lemma.

LEMMA 7.27. *The action of $SL(q, \mathbb{R})$ on V_{pq} induces a natural action on the Grassmann $Gr(p, D)$. The group $SL(q, \mathbb{R})$ has an open orbit in $Gr(p, D)$ if and only if $SL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ has an open orbit in V_{pq} .*

PROOF OF THE LEMMA. Consider $W \in Gr(p, D)$ whose $SL(q, \mathbb{R})$ orbit is open and take a basis $\{C_i\}$ of W . Now define $v = \sum C_i \otimes e_i \in V_{pq} = \mathfrak{so}(q, \mathbb{R}) \otimes \mathbb{R}^p$, where $\{e_i\}$ is the usual basis of \mathbb{R}^p . We will show that the $SL(q, \mathbb{R})$ -orbit of W corresponds to the $SL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ orbit of v .

Consider the neighborhood of $v \in V_{pq}$ which consists of $u = \sum D_i \otimes e_i$ such that the D_i span a subspace which is in the $SL(q, \mathbb{R})$ orbit of W . (Note: any vector $w \in V_{pq}$ can be written in the form $\sum E_i \otimes e_i$ where E_i is a skew-symmetric matrix and e_i is the standard basis of \mathbb{R}^p .) This neighborhood is open as $SL(q, \mathbb{R}) \cdot W$ is open in $Gr(p, D)$. Now take $g \in SL(q, \mathbb{R})$ so that $\{g \cdot C_i\}$ and $\{D_i\}$ have the same span. As these are two bases of the same vector space there exists $h \in GL(p, \mathbb{R})$ such that $h(g \cdot C_i) = D_i$. But this says $(g, h) \cdot \sum C_i \otimes e_i = \sum D_i \otimes e_i$. Hence $SL(q, \mathbb{R}) \times GL(p, \mathbb{R}) \cdot v$ is open if $SL(q, \mathbb{R}) \cdot W$ is open. The other direction is trivial. \square

PROOF OF THE PROPOSITION. It is easy to check that if $g(W) = W^* \in Gr(p, D)$, then $\sigma(g)(W^\perp) = (W^*)^\perp \in Gr(D - p, D)$ where $\sigma(g) = (g^t)^{-1}$. Hence if $SL(q, \mathbb{R}) \cdot W$ is open in $Gr(p, D)$ then $SL(q, \mathbb{R}) \cdot W^\perp$

is open in $Gr(D - p, D)$. The orbits of $GL(p, \mathbb{R})$ in $\mathbb{P}V_{pq}$ are the same as the orbits of $G' = \{g \in GL(p, \mathbb{R}) : \det g = 1 \text{ or } -1\}$. Since $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ has index two in $SL(q, \mathbb{R}) \times G'$ the proposition follows from the Lemma. \square

We now work on each of the exceptional cases. Our goal is to construct a generic soliton, aka distinguished point, and argue why we must have an open set of distinguished orbits.

Case $(1, 2k + 1)$: Consider the matrix $A = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \\ & & & & 0 \end{bmatrix}$. This clearly has an open or-

bit in $\mathfrak{so}(2k + 1, \mathbb{R}) = \mathfrak{so}(2k + 1, \mathbb{R}) \otimes \mathbb{R}^1$ as any generic skew-symmetric matrix can be conjugated to this one. Moreover, it is easy to see that this matrix satisfies the soliton condition $Ric(C) \cdot C = rC$ for some $r \in \mathbb{R}$.

Case $(D - 1, 2k + 1)$: In this case we construct $C \in \mathfrak{so}(2k + 1, \mathbb{R})^{D-1}$ whose span will be orthogonal to the soliton A from the $(1, 2k + 1)$ case. First we tackle the issue of genericity.

Since $SL(2k + 1, \mathbb{R}) \times SL(1, \mathbb{R}) \cdot [A]$ is an open orbit in $\mathbb{P}V_{1, 2k+1}$ we have that $SL(2k + 1, \mathbb{R}) \times SL_{D-1} \mathbb{R} \cdot [C]$ is an open orbit in $\mathbb{P}V_{D-1, 2k+1}$, see Proposition 7.26 and its proof. Thus by Corollary 7.19 we see that there exists an open set of $V_{D-1, 2k+1}$ whose points lie on distinguished orbits, i.e., we have an open set of algebras which admit soliton metrics. Next we construct the desired $C \in V_{D-1, 2k+1}$.

Some notation. We denote the usual basis vectors of $\mathfrak{so}(2k + 1, \mathbb{R})$ by $v_{ij} = E_{ij} - E_{ji}$, for $i < j$, where E_{ij} is the matrix with a 1 in the ij -th position and zeros elsewhere. The space $\mathfrak{so}(2k, \mathbb{R})$ sits naturally in $\mathfrak{so}(2k + 1, \mathbb{R})$ as $\mathfrak{so}(2k, \mathbb{R}) = \text{span} - \langle v_{ij} \rangle$ such that $j \neq 2k + 1$; i.e., the upper left $2k \times 2k$ block as in the matrix A above. It is easier to describe $C = (C_1, \dots, C_{D-1})$ by splitting the C_i into two sets. The first set of $2k^2 - k - 1$ elements will consist of an orthogonal basis of $\mathfrak{so}(2k, \mathbb{R}) \cap A^\perp$, all of whose elements are of length $|A| = \sqrt{2k}$, where A is defined in the case above. At first this may seem mysterious; however, we will show that the properties given for Set 1 are enough to compute $Ric(C)$ with the second set defined below.

The second set of C_i (with $2k$ -many elements) will consist of $\{\sqrt{a} v_{i, 2k+1}\}, 1 \leq i \leq 2k$. In matrix form

$$v_{i, 2k+1} = \begin{pmatrix} \vdots & e_i \\ \cdots & \vdots \\ -e_i^t & 0 \end{pmatrix}$$

where e_i is the column vector with 1 in the i -th position and zeros elsewhere.

We claim that $a = \frac{2k^2 - 1}{2k}$ yields a soliton. That is $Ric(C) \cdot C = rC$ for some $r \in \mathbb{R}$ with this choice of a . We compute $Ric_2(C)$ first. Recall that $Ric_2(C)$ is defined by $(Ric_2(C))_{ij} = -tr(C_i C_j)$. Fortunately we constructed *Set 1* and *Set 2* to be orthogonal to each other. Also observe that the C_i in *Set 1* are all

orthogonal and of length $|A| = \sqrt{2k}$ and the C_i in *Set 2* are all orthogonal and of length $\sqrt{2a}$. Thus $Ric_2(C)$ has the convenient form

$$(7.2) \quad Ric_2(C) = diag\{\underbrace{2k, \dots, 2k}_{set1}, \underbrace{2a, \dots, 2a}_{set2}\}$$

Next we compute $Ric_1(C) = -2 \sum C_i^2$ by computing the sum of squares first over *Set 1*, then over *Set 2*, and then adding. Before we compute $-\sum_{set1} C_i^2$, we make the following very useful observation.

LEMMA 7.28. *Let $\{D_1, \dots, D_p\} \subset \mathfrak{so}(q, \mathbb{R})$ be an orthogonal set of vectors, each with length $|D_i| = d$ and let $W = span \langle D_i \rangle$. If $\{E_1, \dots, E_p\} \subset \mathfrak{so}(q, \mathbb{R})$ is any other orthogonal basis of W whose elements have length d , then $\sum D_i^2 = \sum E_i^2$.*

PROOF. This follows immediately from the fact that $kRic_1(C)k^{-1} = Ric_1(C)$ for $k \in SO(p, \mathbb{R})$ and $C \in V_{pq}$. \square

We apply this lemma to $W = \mathfrak{so}(2k, \mathbb{R}) \subset \mathfrak{so}(2k+1, \mathbb{R})$. The C_i in *Set 1* together with A span $\mathfrak{so}(2k, \mathbb{R})$; additionally, these vectors have $|C_i|^2 = |A|^2 = 2k$. Note that $\mathfrak{so}(2k, \mathbb{R})$ has $\{v_{ij} : 1 \leq i < j \leq 2k\}$ as a basis and $|v_{ij}|^2 = 2$ for all i, j . Applying the lemma we have

$$-\sum_{set1} C_i^2 - A^2 = -\sum_{1 \leq i < j \leq 2k} k |v_{ij}|^2 = k \left[\begin{array}{c|c} (2k-1)Id_{2k} & \\ \hline & 0 \end{array} \right]$$

Adding A^2 we have

$$\begin{aligned} -\sum_{set1} C_i^2 &= diag\{2k^2 - k, \dots, 2k^2 - k, 0\} - diag\{1, \dots, 1, 0\} \\ &= (2k^2 - k - 1)diag\{1, \dots, 1, 0\} \end{aligned}$$

Computing the sum of squares over *Set 2* is straight forward and we obtain

$$-\sum_{set2} C_i^2 = a diag\{1, \dots, 1, 2k\}$$

which gives

$$Ric_1(C) = -2 \sum C_i^2 = 2 \left[\begin{array}{c|c} (2k^2 - k - 1 + a)Id_{2k} & \\ \hline & 2ka \end{array} \right]$$

Putting all of our computations together we can easily compute the i -th component of $Ric(C) \cdot C = Ric_1(C) \cdot C + Ric_2(C) \cdot C$. Using Equation (7.2) we see that $Ric_2(C)$ is a diagonal matrix and we have

$(Ric(C) \cdot C)_i = Ric_1(C) \cdot C_i + |C_i|^2 C_i$, where $Ric_1(C)$ acts via $Ric_1(C) \cdot C_i = Ric_1(C)C_i + C_i Ric_1(C)$. For C_i in *Set 1* and C_j in *Set 2* we have

$$\begin{aligned} (Ric(C) \cdot C)_i &= \{2 \cdot 2(2k^2 - k - 1 + a) + 2k\} C_i \\ (Ric(C) \cdot C)_j &= \{(2(2k^2 - k - 1 + a) + 4ak) + 2a\} C_j \end{aligned}$$

We have a soliton when $Ric(C) \cdot C = rC$, for some $r \in \mathbb{R}$. This happens when the above two coefficients are equal and that is precisely when $a = \frac{2k^2 - 1}{2k}$.

Case $(2, 2k + 1)$: We construct $C \in \mathfrak{so}(2k + 1, \mathbb{R})^2$ which has an open $SL(2k + 1, \mathbb{R}) \times SL(2, \mathbb{R})$ orbit and which makes $\mathbb{R}^{2+(2k+1)}(C)$ a soliton. Then by Corollary 7.19 we know that we have a Zariski open set of the vector space which consists of distinguished orbits. Consider the vector $C = (C_1, C_2)$ defined by

$$C_1 = \begin{bmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & & & & \\ & & 0 & a_2 & & \\ & & -a_2 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 & a_k \\ & & & & & -a_k & 0 \\ & & & & & & & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & & & & & & & \\ & 0 & a_k & & & & & \\ & -a_k & 0 & & & & & \\ & & & \ddots & & & & \\ & & & & 0 & a_2 & & \\ & & & & -a_2 & 0 & & \\ & & & & & & 0 & a_1 \\ & & & & & & -a_1 & 0 \end{bmatrix}$$

with $a_i = \sqrt{k + 1 - i}$. This gives a vector which is distinguished and whose orbit is open in the vector space. We omit the details that the group orbit is open. In fact, for any pair of matrices of the above form, if $a_i \neq 0$ for all i , then the orbit is open.

These matrices are orthogonal with the same length $a = (2 \sum a_i^2)^{1/2} = (k(k + 1))^{1/2}$. Hence $Ric_2(C) = a^2 I_2$. To show that C defines a soliton one just needs to verify that $Ric_1(C) = -2(C_1^2 + C_2^2)$ acts like a multiple of the identity. From the choice of the $\{a_i\}$ we obtain $-C_1^2 - C_2^2 = \text{diag}\{k, k + 1, k, k + 1, k, \dots, k + 1, k\}$. Notice the terms along this diagonal alternate between k and $k + 1$. As before we define $v_{ij} = E_{ij} - E_{ji} \in \mathfrak{so}(2k + 1, \mathbb{R})$ for $i < j$. Thus for any $v_{i, i+1}$ we have $Ric_1(C) \cdot v_{i, i+1} = 2(2k + 1)v_{i, i+1}$. Since our chosen C_i have such a special form, i.e., they lie in the span of $\{v_{i, i+1}\}$, we have $Ric_1(C) \cdot C = 2(2k + 1)C$.

Case $(D - 2, 2k + 1)$: Let A_1, A_2 denote the C_i given above in the previous case $(2, 2k + 1)$. We will construct $C \in \mathfrak{so}(2k + 1, \mathbb{R})^{D-2}$ which makes $\mathbb{R}^{(D-2)+2k+1}(C)$ a soliton and so that $\text{span} \langle C \rangle \perp A_1, A_2$. In this way we can guarantee that generic orbits are distinguished by applying Corollary 7.19 and Proposition 7.26. The components of C will be broken up into 4 sets. In what follows we denote the usual basis for $\mathfrak{so}(2k + 1, \mathbb{R})$ by $v_{ij} = E_{ij} - E_{ji}$, $i < j$.

Set 1 consists of an orthogonal basis of $\{span < v_{i,i+1} >\} \cap \{span < A_1, A_2 >\}^\perp$ which have length $= |A_i| = \sqrt{k(k+1)}$.

Set 2 consists of the vectors $\sqrt{a} v_{ij}$, where both i, j are odd.

Set 3 consists of the vectors $\sqrt{b} v_{ij}$, where both i, j are even.

Set 4 consists of the vectors $\sqrt{c} v_{ij}$, where i, j have different parity and $j \neq i + 1$.

These four sets constitute a basis of $span < A_1, A_2 >^\perp$ and so there are precisely $D - 2$ of these C_i .

We claim that the above yields a soliton precisely when $a = \frac{k^3+k^2-1}{2k}, b = \frac{k^3+k^2+1}{2k}, c = \frac{k(k+1)}{2} = \frac{|A_i|^2}{2}$. Note, $a, b, c > 0$ as needed. To see this, first observe that $Ric_2(C)$ is again a diagonal matrix by construction since the vectors $\{v_{ij}\}$ are orthogonal of length $\sqrt{2}$. We show how to compute $Ric_1(C)$ and leave the remaining details to the reader.

In order to compute $Ric_1(C)$ we calculate $-\sum C_i^2$. We will do this for C_i in the different sets, then finish by describing what $Ric_1(C)$ should look like. To calculate $-\sum_{set1} C_i^2$ we observe that *Set 1* along with A_1, A_2 span the same space as $\{v_{i,i+1}\}$. Using Lemma 7.28 we have

$$\begin{aligned}
-\sum_{set1} C_i^2 &= \frac{k(k+1)}{2} (-\sum v_{i,i+1}^2) - (-A_1^2 - A_2^2) \\
&= \frac{k(k+1)}{2} \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & 2 \\ & & & & & 1 \end{bmatrix} - \begin{bmatrix} k & & & & \\ & k+1 & & & \\ & & k & & \\ & & & \ddots & \\ & & & & k+1 \\ & & & & & k \end{bmatrix} \\
&= diag\{\frac{k(k-1)}{2}, k^2-1, k^2, \dots, k^2-1, \frac{k(k-1)}{2}\} \\
\\
-\sum_{set2} C_i^2 &= ak \, diag\{1, 0, 1, \dots, 0, 1\} \\
\\
-\sum_{set3} C_i^2 &= b(k-1) \, diag\{0, 1, 0, \dots, 1, 0\} \\
\\
-\sum_{set4} C_i^2 &= c(k-1) \, diag\{1, 1, 0, 1, \dots, 0, 1, 0, 1, 1\} \\
&\quad + c(k-2) \, diag\{0, 0, 1, 0, \dots, 1, 0, 1, 0, 0\}
\end{aligned}$$

where, in the equation for Set 4, we have a) the first diagonal matrix begins and ends with a pair of 1's and alternates in between, and b) the second diagonal matrix begins and ends with a pair of zeros and alternates in between.

Once all the computations are made $Ric_1(C) = diag\{r_1, r_2, r_1, \dots, r_2, r_1\}$. Because $Ric_1(C)$ and the C_i have such special forms, the calculations are manageable and we readily see that we have constructed a soliton which is also generic. \square

In showing that generic orbits are distinguished for these four exception cases, we really found solitons in the cases $(1, 2k+1)$ and $(2, 2k+1)$ then showed that they had “dual” algebras which also admitted soliton metrics. Recall that two algebras $\mathfrak{N}_1, \mathfrak{N}_2$ are dual if their structure matrices $C_{\mathfrak{N}_i}$ satisfy

$$span < C_{\mathfrak{N}_1} > \perp span < C_{\mathfrak{N}_2} >$$

It is a fact that the dual of an optimal matrix is optimal. This fact is easy to deduce knowing that $\mathfrak{so}(q, \mathbb{R})$ has an orthonormal basis $\{C_i\}$, under the negative Killing form, such that $\sum C_i^2 = -rId$ for some $r > 0$. We would be very interested in an answer to the following question.

QUESTION 7.29. *Is the dual of an Einstein nilradical an Einstein nilradical?*

Constructing new Einstein and Non-Einstein Nilradicals

As the previous chapter demonstrates, finding nilpotent Lie groups that do not admit soliton metrics is a very subtle problem. There it is shown that generic two-step nilalgebras are Einstein nilradicals. In this chapter we construct a new family of examples of non-Einstein nilradicals (see Proposition 8.10) that was not previously known; moreover, we give a general technique for building new Einstein nilradicals from ‘smaller’ ones (see Proposition 8.4 and Theorem 8.5). This new family of non-Einstein nilradicals which is constructed also answers some questions that were posed to me by J. Lauret. I would like to thank him for some very useful conversations.

In this chapter we will construct tuples $C \in \mathfrak{so}(q, \mathbb{R})$ such that $C_i = j(Z_i)$ where $\{Z_i\}$ is an adapted basis of $\mathfrak{Z} \subset \mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$ instead of being an adapted basis of $[\mathfrak{N}, \mathfrak{N}] \subset \mathfrak{N} = \mathcal{V}' \oplus [\mathfrak{N}, \mathfrak{N}]$. All the previous results from Chapter 7 hold when changing to this perspective. Although this technical change is not necessary, it is preferred in this chapter.

1. An Amalgamated Lie Algebra

Consider two metric two-step nilpotent Lie algebras $\mathfrak{N}_1 = \mathcal{V}_1 \oplus \mathfrak{Z}$ and $\mathfrak{N}_2 = \mathcal{V}_2 \oplus \mathfrak{Z}$ whose centers are the same dimension. One can construct a new nilpotent Lie algebra $\mathfrak{N} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathfrak{Z}$ so that the \mathcal{V}_i commute. To do this, one must identify the two different centers via a vector space isomorphism; this choice might change the isomorphism type of the resulting amalgamated Lie algebra.

In addition to constructing a bracket on \mathfrak{N} , we simultaneously endow \mathfrak{N} with a choice of metric. This construction is dependent on the identification (isometry) of the centers of \mathfrak{N}_1 and \mathfrak{N}_2 ; equivalently, the construction is dependent on a choice of orthonormal basis of \mathfrak{Z} . By hypothesis, the inner product on the center of \mathfrak{N}_1 is the same as the inner product on the center of \mathfrak{N}_2 . Endow $\mathfrak{N} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathfrak{Z}$ with the inner product such that this is an orthogonal direct sum and when restricted to each piece corresponds to the original inner products. We define $[\cdot, \cdot]$ on \mathfrak{N} using the following set of relations

$$\begin{aligned} [\mathcal{V}_1, \mathcal{V}_2] &= 0 \\ \langle [v_i, w_i], Z \rangle &= \langle j_i(Z)v_i, w_i \rangle \quad \text{for } Z \in \mathfrak{Z} \text{ and } v_i, w_i \in \mathcal{V}_i, i = 1, 2 \end{aligned}$$

where j_i is the j -map for the metric Lie algebra \mathfrak{N}_i , $i = 1, 2$. Equivalently, the bracket above could be defined via the j -map by

$$j(Z) = \begin{pmatrix} j_1(Z) \\ j_2(Z) \end{pmatrix}$$

Here $j(Z)$ is a block matrix relative to a basis which respects the orthogonal direct sum $\mathcal{V}_1 \oplus \mathcal{V}_2$. This construction is very natural and from the perspective of the j -map says that $j(Z)$ preserves the subspaces \mathcal{V}_i for all $Z \in \mathfrak{Z}$.

Remark. If the adapted basis contains a basis of \mathfrak{Z} (rather than $[\mathfrak{N}, \mathfrak{N}]$), then the structure matrices C^1, \dots, C^p may not be linearly independent. For example, this will happen when we have a nontrivial Euclidean de Rham factor.

QUESTION 8.1. Consider $\mathfrak{N} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathfrak{Z}$. Is \mathfrak{N} an Einstein nilalgebra if and only if both \mathfrak{N}_1 and \mathfrak{N}_2 are so?

We give a full negative answer to this question. There do exist \mathfrak{N}_i which are Einstein nilradicals but \mathfrak{N} is not (see Proposition 8.10). Conversely, there exist an \mathfrak{N}_1 which is a non-Einstein nilradical and an \mathfrak{N}_2 which is an Einstein nilradical such that the constructed \mathfrak{N} is an Einstein nilradical (see Example 8.12).

2. Concatenation of Structure Matrices

Let $A = (A_1, \dots, A_{q_1}) \in \mathfrak{so}(q_1, \mathbb{R})^p$ and $B = (B_1, \dots, B_{q_2}) \in \mathfrak{so}(q_2, \mathbb{R})^p$ be structure matrices associated to \mathfrak{N}_1 and \mathfrak{N}_2 , where $q_i = \dim \mathcal{V}_i$. The \mathfrak{N} constructed above corresponds to the structure matrix $C \in \mathfrak{so}(q, \mathbb{R})^p$ where $q = q_1 + q_2$ and

$$C_i = \begin{pmatrix} A_i \\ B_i \end{pmatrix}$$

We call this process *concatenation*. Denote this process of concatenation by $C = A +_c B$. This definition depends on more than the isomorphism classes of \mathfrak{N}_1 and \mathfrak{N}_2 , it depends on the choice of adapted bases to produce the structure matrices.

DEFINITION 8.2. Let $C \in \mathfrak{so}(q, \mathbb{R})^p$ be a distinguished point of the $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ action (cf. Definition 5.2). We will say that C is $SL(p, \mathbb{R})$ -minimal if $m_2(C) = 0$ where m_2 is the moment map for the $SL(p, \mathbb{R})$ action (cf. Example 2.28).

Remark. Equivalently, $C = (C^1, \dots, C^p)$ is $SL(p, \mathbb{R})$ -minimal if the C^i are mutually orthogonal and all of the same length. There do exist distinguished points which are not $SL(p, \mathbb{R})$ -minimal. See Proposition 8.10 with $k = 3$. Moreover, this example shows the stark difference between distinguished and minimal points. That is, if a point is minimal for $G_1 \times G_2$ then it is so for each G_i on its own. However, an analogous result for distinguished points is not true.

LEMMA 8.3. *Let A be a distinguished $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ point which is $SL(p, \mathbb{R})$ -minimal. Then A is $SL(q, \mathbb{R})$ -distinguished.*

Remark. The proof actually only requires A to be $SL(p, \mathbb{R})$ distinguished; however, we only need the result for the case of $SL(p, \mathbb{R})$ -minimal.

PROOF. Recall that the moment map for the $SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$ action is $m = m_1 + m_2$ where m_1 is the moment map for $SL(q, \mathbb{R})$ and m_2 is the moment map for $SL(p, \mathbb{R})$ (see Proposition 2.26). Then A being distinguished is equivalent to $m(A) \cdot A = aA$ for some $a \in \mathbb{R}$. But if $m_2(A) \cdot A = a_2A$, then $m_1(A) \cdot A = (a - a_2)A$. That is, A is $SL(q, \mathbb{R})$ -distinguished. \square

PROPOSITION 8.4. *Let $A \in \mathfrak{so}(q_1, \mathbb{R})^p$, $B \in \mathfrak{so}(q_2, \mathbb{R})^p$, and $C = A +_c B \in \mathfrak{so}(q_1 + q_2, \mathbb{R})^p$ be the concatenation of A and B . If A, B are distinguished and $SL(p, \mathbb{R})$ -minimal then so is C , after rescaling B .*

Remark. This gives a natural way of constructing new soliton algebras from smaller pieces.

PROOF. We first observe that A being $SL(p, \mathbb{R})$ -minimal is equivalent to $|A_i| = |A_j|$ and $A_i \perp A_j$ for all i, j . Thus, if A and B are $SL(p, \mathbb{R})$ -minimal then the concatenation C automatically is so, since $|C_i|^2 = |A_i|^2 + |B_i|^2$ and $\langle C_i, C_j \rangle = \langle A_i, A_j \rangle + \langle B_i, B_j \rangle$.

By the lemma above, since A and B are $SL(p, \mathbb{R})$ -minimal, we see that $m_1(A) \cdot A = \lambda_a A$ and $m_1(B) = \lambda_b B$. Note that $\lambda_a, \lambda_b \geq 0$ by Lemma 7.18. By rescaling B , we may assume that $\lambda_a = \lambda_b = \lambda \in \mathbb{R}$ since m_1 is a degree 2 homogeneous polynomial and $\lambda_a, \lambda_b \geq 0$. Let $C = \begin{pmatrix} A \\ B \end{pmatrix}$ be the concatenation of A and B . Then

$$m_1(C) = -2 \sum C_i^2 = \begin{pmatrix} -2 \sum A_i^2 & \\ & -2 \sum B_i^2 \end{pmatrix} = \begin{pmatrix} m_1(A) & \\ & m_1(B) \end{pmatrix}$$

and since we rescaled our initial pair, we see that

$$m_1(C) \cdot C = m_1(C)C + Cm_1(C) = \begin{pmatrix} m_1(A) \cdot A & \\ & m_1(B) \cdot B \end{pmatrix} = \lambda C$$

Since the components of C are orthogonal and of the same length, we see that $m_2(C) = 0$. Thus, $m(C) \cdot C = \lambda C$. \square

THEOREM 8.5. *Consider $q_1 \leq q_2$, $D = \frac{1}{2}q_2(q_2 - 1)$, and $1 \leq p \leq D$ with $p \neq D - 1, D - 2$. Let \mathfrak{N}_1 and \mathfrak{N}_2 be generic nilsolitons of types (q_1, p) and (q_2, p) , respectively. Then the concatenation $\mathfrak{N} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathfrak{Z}$ is also a nilsoliton.*

PROOF. See Chapter 7 for details on nilsolitons of type (p, q) . By generic soliton we mean a soliton in the isomorphism class of a generic algebra. Observe for the ‘non-exceptional’ types (p, q) that generic

algebras admit optimal metrics by Theorem 7.25, and optimal metrics are necessarily $SL(p, \mathbb{R})$ -minimal. For the exceptional types (p, q) that satisfy the given constraints on p , generic algebras admit solitons which are $SL(p, \mathbb{R})$ -minimal; see the constructions in Chapter 7 of the exceptional cases $(1, 2k+1)$ and $(2, 2k+1)$. Now apply the proposition above. \square

Remark. See Corollary 8.7 for a worthwhile application. Moreover, this theorem speaks only to the generic setting. We show by construction that not all concatenations of solitons can admit a soliton metric. The following theorem will be very useful in the study of algebras of type $(2, 2k+1)$ but is very valuable in its own right.

We are interested in tuples C which are the concatenation of n -many tuples A^1, \dots, A^n ; that is, each $C_i = \begin{pmatrix} A_i^1 & & \\ & \ddots & \\ & & A_i^n \end{pmatrix}$. Equivalently, all the $C_i \in \mathfrak{so}(q, \mathbb{R})$ simultaneously preserve the same subspaces of \mathbb{R}^q .

Let $V = \mathfrak{so}(q, \mathbb{R})^p$ and W be the subspace of block diagonal tuples of matrices $\begin{pmatrix} \mathfrak{so}(q_1, \mathbb{R}) & & \\ & \ddots & \\ & & \mathfrak{so}(q_n, \mathbb{R}) \end{pmatrix}^p$, where $q = q_1 + \dots + q_n$.

THEOREM 8.6. *Suppose $C \in W$ admits a soliton metric, that is, there exists $g \in G = GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ such that $g \cdot C$ is a soliton. Then there exists $h \in G$ such that $h \cdot C \in W$ is a soliton.*

Recall that if $C \in \mathfrak{so}(q, \mathbb{R})^p$ is a point which is distinguished and defines a type (p, q) algebra then C is a soliton; that is, $\mathbb{R}^{p+q}(C)$ is a soliton metric nilalgebra.

PROOF. This proof relies on the main result of Chapter 5 as follows. Let C be a tuple which admits a soliton metric. Let $\varphi_t(C)$ denote the negative gradient flow of the norm squared of the moment map for $G = GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ starting at C , and let $\omega(C)$ denote the ω -limit set of $\phi_t(C)$. It is known that $\omega(C)$ consists of soliton metrics as this set consists of fixed points of the negative gradient flow; see the remark following Definition 5.5, Lemma 7.18, and Proposition 7.22. However, since C admits a soliton by hypothesis, it follows from Theorem 5.10 that $\omega(C) \subset G \cdot C$.

We assert that $-\text{grad } |m|^2$ is tangent to W at all points of W . Note that $-\text{grad } |m|^2(C) = -4m(C) \cdot C$ by Lemma 5.4 and $m(C) \in \mathfrak{gl}(q_1, \mathbb{R}) \times \dots \times \mathfrak{gl}(q_n, \mathbb{R}) \times \mathfrak{gl}(p, \mathbb{R})$ for all $C \in W$ by inspection. Since $GL(q_1, \mathbb{R}) \times \dots \times GL(q_n, \mathbb{R}) \times GL(p, \mathbb{R})$ leaves W invariant the assertion follows.

Consider $[C] \in \mathbb{P}W \subset \mathbb{P}V$. We have shown that $\varphi_t[C] \subset GL(q_1, \mathbb{R}) \times \dots \times GL(q_n, \mathbb{R}) \times GL(p, \mathbb{R}) \cdot [C] \subset \mathbb{P}W$ for all t as the flow is always tangent to the submanifold $GL(q_1, \mathbb{R}) \times \dots \times GL(q_n, \mathbb{R}) \times GL(p, \mathbb{R}) \cdot [C] \subset \mathbb{P}W \subset \mathbb{P}V$. Therefore, $\omega[C] \subset \mathbb{P}W \cap G \cdot [C]$ as was to be shown. \square

Remark. It is not clear whether or not the algebra $h \cdot C$ above must actually be in the $GL(q_1, \mathbb{R}) \times \cdots \times GL(q_n, \mathbb{R}) \times GL(p, \mathbb{R})$ orbit of C . I have some partial results towards this question and plan to work on it more in the future.

3. Algebras of Type $(2, q)$

When $q = 2k + 1$ the orbits of $GL(q, \mathbb{R}) \times GL(2, \mathbb{R})$ are open in $V = \mathfrak{so}(q, \mathbb{R})^2$. Hence generically there are only finitely many isomorphism classes of type $(2, q)$ algebras, possibly just one. We have shown that such algebras are Einstein nilradical(s), see Section 7.5. From this, we can build more Einstein nilradicals from the work in the previous section. Recall that a two-step nilpotent Lie algebra \mathfrak{N} is called an Einstein nilradical if \mathfrak{N} admits a soliton metric (see Definition 6.6).

COROLLARY 8.7. *Most algebras of type $(2, 2k+1)$ are Einstein nilradicals. More precisely, write $2k+1 = (2l+1) + q$ for positive integers l, q with $q \geq 4$. Consider a block decomposition of structure matrices $C = A +_c B$, where $A \in \mathfrak{so}(2l+1, \mathbb{R})^2$ and $B \in \mathfrak{so}(q, \mathbb{R})^2$. For generic choices of A, B the constructed $C = A +_c B \in \mathfrak{so}(2k+1)^2$ admits a soliton metric.*

Remark. Warning! This does not necessarily hold for the other (p, q) types. For the other types, one would have to show that the Zariski open set \mathcal{O} of ‘generic’ algebras in $\mathfrak{so}(q, \mathbb{R})^p$ constructed in Chapter 7 actually intersects this vector subspace W of block matrices. However, a priori it could happen that $W \subset \mathfrak{so}(q, \mathbb{R})^p - \mathcal{O}$.

Here we show that $W \cap \mathcal{O}$ is nonempty by showing that for a generic element A of $\mathfrak{so}(2l+1, \mathbb{R})^2$ the orbits $GL(2l+1, \mathbb{R}) \times GL(2, \mathbb{R}) \cdot A$ and $GL(2l+1, \mathbb{R}) \cdot A$ are equal. This statement is false for $p \geq 3$.

Additionally, this corollary shows that the word *most* carries much more weight than just the existence of a Zariski open set in $\mathfrak{so}(2k+1, \mathbb{R})^2$. From this one can construct/guarantee the existence of moduli of Einstein nilradicals of type $(2, 2k+1)$, as opposed to the finite set of ‘generic’ algebras of type $(2, 2k+1)$. See Example 8.9. Before proving the corollary, we state the following lemma.

LEMMA 8.8. *Consider the groups $G = GL(2l+1, \mathbb{R}) \times GL(2, \mathbb{R})$ and $H = GL(2l+1, \mathbb{R})$ acting on $\mathfrak{so}(2l+1, \mathbb{R})^2$. For generic $A \in \mathfrak{so}(2l+1, \mathbb{R})^2$ we have $H \cdot A = G \cdot A$*

PROOF OF THE LEMMA. It is a fact, which we omit the proof of, that generic H -orbits in $\mathfrak{so}(2l+1, \mathbb{R})^2$ are open. The proof of this fact amounts to calculating the dimension of the stabilizer at a specific point $A \in \mathfrak{so}(2l+1, \mathbb{R})^2$ to see that $\dim H_A = \dim H - \dim \mathfrak{so}(2l+1, \mathbb{R})^2$. The A that I used is

$$A = \left(\left[\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & \ddots & \\ & & 0 & 1 \\ & & -1 & 0 \\ \hline & & & 0 \end{array} \right], \left[\begin{array}{cc|cc} 0 & & & \\ 0 & 1 & & \\ -1 & 0 & & \\ \hline & & \ddots & \\ & & 0 & 1 \\ & & -1 & 0 \end{array} \right] \right)$$

If $\mathcal{O} = \{A \in \mathfrak{so}(2l+1, \mathbb{R})^2 : H(A) \text{ is open in } \mathfrak{so}(2l+1, \mathbb{R})^2\}$, then \mathcal{O} is invariant under G since H is normal in G . If $A \in \mathcal{O}$, then $G(A)$ is a union of open H orbits, and hence $G(A) = H(A)$ since G is connected. \square

PROOF OF THE COROLLARY. Consider $C = A +_c B \in \mathfrak{so}(2k+1, \mathbb{R})^2$ where $A \in \mathfrak{so}(2l+1, \mathbb{R})^2$ and $B \in \mathfrak{so}(q, \mathbb{R})^2$ are generic. As A is generic the lemma above states that $GL(2l+1, \mathbb{R}) \times GL(2, \mathbb{R}) \cdot A = GL(2l+1, \mathbb{R}) \cdot A$. Moreover, the example constructed in Section 7.5 shows there exists $g \in GL(2l+1, \mathbb{R}) \times GL(2, \mathbb{R})$ such that $g \cdot A$ is soliton and $SL(2, \mathbb{R})$ minimal. Thus there exists $g \in GL(2l+1, \mathbb{R})$ such that $g \cdot A$ is soliton and $SL(2, \mathbb{R})$ minimal.

Now consider $B \in \mathfrak{so}(q, \mathbb{R})^2$, where $q+2l+1 = 2k+1$ and $q \geq 4$. As B is generic, by Theorem 7.25 there exists $h \in GL(q, \mathbb{R}) \times GL(2, \mathbb{R})$ such that $h \cdot B$ is optimal and in particular $SL(2, \mathbb{R})$ -minimal. Hence we have $(g, h) \in GL(2l+1, \mathbb{R}) \times GL(q, \mathbb{R}) \times GL(2, \mathbb{R}) \subset GL(2k+1, \mathbb{R}) \times GL(2, \mathbb{R})$ such that $(g, h) \cdot C = (g \cdot A) +_c (h \cdot B)$ is a soliton by Proposition 8.4, after rescaling $h \cdot B$. \square

EXAMPLE 8.9. Let $A \in \mathfrak{so}(2l+1, \mathbb{R})^2$ be a generic Einstein nilradical and consider all $B \in \mathfrak{so}(2k, \mathbb{R})^2$ which are Einstein nilradicals. For fixed A the dimension of this moduli space of such B is $k-3$ (see Section 7.4). As A and B vary the set of $C = A +_c B$ consists of Einstein nilradicals and the moduli of such C has dimension $k-3$.

Remark. We omit the proof of the claimed results above. To prove the Example, one can reduce to the case of A being a fixed soliton and then show that $A +_c B$ and $A +_c B'$ are isomorphic if and only if B and B' are isomorphic when B, B' are generic. Genericity of B, B' might not be required, however we do not know of a simple proof without using such a fact.

In addition to constructing moduli of Einstein nilradicals of type $(2, 2k+1)$, we can also construct some nilalgebras which are non-Einstein (in fact, we can construct moduli of such nilalgebras); that is, they cannot possibly admit an invariant Ricci soliton metric. To do this we will consider structure matrices based on \mathfrak{Z} instead of based on $[\mathfrak{N}, \mathfrak{N}]$. That is, our structure matrices are $\{j(Z) | Z \in \mathfrak{Z}\}$. We note that if $e = \dim \mathfrak{Z} - \dim [\mathfrak{N}, \mathfrak{N}]$, then e is the dimension of the Euclidean de Rham factor of \mathfrak{N} for any choice of metric \langle, \rangle . See Proposition 2.7 of [Ebe94] and Proposition 1.3 of [Ebe03].

Take B to be (generic) of type $(2, 3)$ with no Euclidean de Rham factor (i.e., $\mathfrak{Z} = [\mathfrak{N}, \mathfrak{N}]$, which is equivalent to the linear independence of $\{B_1, B_2\}$) and $A = (A_1, A_2) \in \mathfrak{so}(2k, \mathbb{R})^2$ such that A_1 and A_2 are linearly dependent and one of them is nonsingular. That is, A is a set of structure matrices, based on \mathfrak{Z} instead of $[\mathfrak{N}, \mathfrak{N}]$, corresponding to an algebra whose center has dimension 2 and commutator has dimension 1; i.e., the algebra has a 1-dimensional Euclidean de Rham factor. We will classify which of these $A +_c B$ are Einstein nilradicals. Note that $A +_c B$ will have no Euclidean de Rham factor since $\{C_1, C_2\}$ are linearly independent, which follows from the fact that $\{B_1, B_2\}$ are linearly independent.

PROPOSITION 8.10. *Let $C = A +_c B$ be the concatenation of A, B given above. Then the two-step nilalgebra associated to C is an Einstein nilradical only for $k \leq 3$.*

Remark. The proof will show that for $k \leq 3$ if $D = g \cdot C$ is a soliton for some $g \in G = GL(2k+3, \mathbb{R}) \times GL(2, \mathbb{R})$, then $D = (D_1, D_2)$ may be chosen to have the following form:

$$D_1 = \begin{pmatrix} 0 & & \\ & F_1 & \\ & & 0 \end{pmatrix}, F_1 = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} E_2 & \\ & F_2 \end{pmatrix}, E_2 = \text{diag}\{\lambda Id_2, \dots, \lambda Id_2\} \text{ (k-many blocks)}, \text{ and } F_2 = \begin{pmatrix} 0 & & \\ & 0 & d \\ & -d & 0 \end{pmatrix}$$

The constants a, d, λ are all positive and related as follows:

$$k = 1 \quad a = d\sqrt{2} \quad \lambda = a$$

$$k = 2 \quad a = 2d \quad \lambda = \frac{a\sqrt{3}}{2}$$

$$k = 3 \quad a = d\sqrt{10} \quad \lambda = d\sqrt{6}$$

Conversely, a routine computation shows that the elements above are solitons. This classification also shows that all concatenated C that admit a soliton lie on a single G orbit, up to scaling by constants.

Before proving the proposition, we need a lemma that makes the above theorem slightly stronger for the particular C chosen.

LEMMA 8.11. *Let C be the concatenation of A, B as above. Let $W \subset \mathfrak{so}(2k+3, \mathbb{R})^2$ be the subspace of block matrices of the type of C . If C admits a soliton $D \in GL(2k+3, \mathbb{R}) \times GL(2, \mathbb{R}) \cdot C \cap W$ then D can be chosen with the following additional property: $D = \begin{pmatrix} E & \\ & F \end{pmatrix}$, where E satisfies $E_1 = 0$.*

PROOF OF THE LEMMA. To construct the desired soliton D with said properties, we will analyze the negative gradient flow of the norm squared of the moment map corresponding to the group $GL(2k+3, \mathbb{R}) \times GL(2, \mathbb{R})$ acting on $\mathfrak{so}(2k+3, \mathbb{R})^2$.

It will be useful to recall some properties of the action of $GL(2, \mathbb{R}) = \{Id\} \times GL(2, \mathbb{R}) \subset GL(2k+3, \mathbb{R}) \times GL(2, \mathbb{R})$ on $\mathfrak{so}(2k+3, \mathbb{R})^2$. If $R = (R_1, R_2) \in \mathfrak{so}(2k+3, \mathbb{R})^2$ and $g \in GL(2, \mathbb{R})$, then $g(R) = S = (S_1, S_2)$, where $\text{span}\{R_1, R_2\} = \text{span}\{S_1, S_2\} \subset \mathfrak{so}(2k+3, \mathbb{R})$. In particular, $GL(2, \mathbb{R})$ leaves W invariant.

As in Theorem 8.6 above, the negative gradient flow starting at $[C] \in \mathbb{P}W$ lies in the orbit $GL(2k, \mathbb{R}) \times GL(3, \mathbb{R}) \times GL(2, \mathbb{R}) \cdot [C] \subset \mathbb{P}W$. Pick a sequence $g_n \in GL(2k, \mathbb{R}) \times GL(3, \mathbb{R}) \times GL(2, \mathbb{R})$ such that $g_n \cdot [C] = \varphi_{t_n}[C]$ for some $t_n \rightarrow \infty$, where φ_t denotes the negative gradient flow. Take $D = \lim g_n \cdot C$. As in Theorem 8.6 above, D is a soliton.

Write $g_n \cdot C = \begin{pmatrix} E^n \\ F^n \end{pmatrix}$. We claim that there exists a sequence $k_n \in SO(2, \mathbb{R})$ such that $k_n g_n \cdot C = \begin{pmatrix} G^n \\ H^n \end{pmatrix}$ with $G_1 = 0$. That is, we can change via the compact group $SO(2, \mathbb{R})$ so that the ‘ E_1 -slot’ = 0. Before showing the existence of such k_n we will use it to finish the proof of the lemma.

As $SO(2, \mathbb{R})$ is compact, we may assume $k_n \rightarrow k \in SO(2, \mathbb{R})$, by passing to a subsequence if necessary. We see that $k_n g_n \cdot [C] \rightarrow k[D] \in \mathbb{P}W$ and since for each n the ‘ E_1 -slot’ = 0, this holds in the limit as well. Moreover, $kD \in W$ is soliton as the set of solitons is K -invariant where $K = O(2k+3, \mathbb{R}) \times O(2, \mathbb{R})$ (cf. Section 7.1). Thus we have constructed a soliton with the desired properties.

To finish the proof of the lemma, we must show the existence of such a $k_n \in SO(2, \mathbb{R})$. Observe that $E^n = (E_1^n, E_2^n)$ is in the $GL(2k, \mathbb{R}) \times GL(2, \mathbb{R})$ -orbit of $A = (A_1, A_2)$. Since A_1 and A_2 are linearly dependent they must be multiples of each other. Thus E_1^n and E_2^n are multiples of each other. To find the desired $k_n = \begin{pmatrix} \cos \theta_n & \sin \theta_n \\ -\sin \theta_n & \cos \theta_n \end{pmatrix}$, one needs to find θ_n such that $\cos \theta_n E_1^n + \sin \theta_n E_2^n = 0$. Clearly such θ_n exists and the lemma is proven. \square

PROOF OF THE PROPOSITION. Suppose that $D = g \cdot C$ is a soliton, we show $k \leq 3$.

The lemma above tells us that our soliton $D \in GL(2k+3, \mathbb{R}) \times GL(2, \mathbb{R}) \cdot C \cap W$ can be chosen with a very special form; that is, we may assume that $D = \begin{pmatrix} E \\ F \end{pmatrix}$ is our soliton where $E \in \mathfrak{so}(2k, \mathbb{R})^2$ with $E_1 = 0$ and E_2 has no kernel (explained below). We will make heavy use of this special form $D = \left\{ \begin{pmatrix} 0 \\ F_1 \end{pmatrix}, \begin{pmatrix} E_2 \\ F_2 \end{pmatrix} \right\}$ to show $k \leq 3$.

To see that E_2 has no kernel, recall that D is a set of structure matrices for an algebra with a 2-dimensional center. If E_2 were to have kernel, then D_1, D_2 would have a common kernel and hence the dimension of the center would be greater than or equal to 3.

Next we show that E_2^2 must be a multiple of the identity and that F_1 and F_2 are orthogonal. Recall that the moment map of $GL(q, \mathbb{R}) \times GL(p, \mathbb{R})$ is $m = m_1 + m_2$. As D is a soliton, $m(D) \cdot D = dD$ for some $d \in \mathbb{R}$. Since the upper left corner of $m_1(D) \cdot D_1$ is zero, one can compute that $m_2(D)$ must be a diagonal matrix since E_2 is nonzero.

Now that $m_2(D)$ is diagonal, $m(D) \cdot D = dD$ implies $m_1(D) \cdot D_2 = d_2 D_2$. That is, $m_1(0, E_2) \cdot E_2 \in \mathbb{R} - \text{span} \langle E_2 \rangle$. From this we see that $m_H(E_2) \cdot E_2 = e E_2$, where m_H is the moment map for the action of $H = GL(2k, \mathbb{R})$ on $\mathfrak{so}(2k, \mathbb{R})$. The action of $GL(2k, \mathbb{R})$ on $\mathfrak{so}(2k, \mathbb{R})$ has open orbits at all the nonsingular

points; moreover, the point $J = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$ is optimal and hence a distinguished point. Hence,

any other distinguished point lies in the orbit $\mathbb{R} \times K \cdot J$ by Theorem 5.10. That is, E_2 is (up to conjugation by $SO(2k, \mathbb{R})$) block diagonal of the form λJ .

Next we refine the lower right block. Consider the group $SO(3, \mathbb{R}) = \{Id_{2k}\} \times SO(3, \mathbb{R}) \times \{Id_2\} \subset O(2k, \mathbb{R}) \times O(3, \mathbb{R}) \times O(2, \mathbb{R}) = K$. Hence $SO(3, \mathbb{R})$ leaves W invariant and carries solitons to solitons. By means of $SO(3, \mathbb{R})$ we can put F_1 in the form $\begin{pmatrix} 0 & a & \\ -a & 0 & \\ & & 0 \end{pmatrix}$. As C , and hence $D = g \cdot C$, corresponds to an

algebra which has no Euclidean de Rham factor, $a \neq 0$. Now write $F_2 = \begin{pmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{pmatrix}$. Since $m_2(D)$ is diagonal and $m_2(D)_{ij} = \langle D_i, D_j \rangle$, it follows that D_1 and D_2 are orthogonal, or equivalently, F_1 and F_2 are orthogonal. Thus $b = 0$.

The stabilizer in $SO(3, \mathbb{R})$ of F_1 is $\begin{pmatrix} \cos \theta & \sin \theta & \\ -\sin \theta & \cos \theta & \\ & & 1 \end{pmatrix}$. Under this group we can further change F_2

so that $c = 0$. Now we have $F_2 = \begin{pmatrix} 0 & & \\ 0 & d & \\ -d & 0 & \end{pmatrix}$. Again, since C corresponds to an algebra which has no Euclidean de Rham factor we know that $d \neq 0$. Now that we have simplified the presentation of our soliton, we can compute $m(D)$

$$m_1(D) = 2 \begin{bmatrix} \lambda^2 & & & & \\ & \ddots & & & \\ & & \lambda^2 & & \\ & & & a^2 & \\ & & & & a^2 + d^2 \\ & & & & & d^2 \end{bmatrix}, \quad m_2(D) = \begin{bmatrix} 2a^2 & 0 \\ 0 & 2k\lambda^2 + 2d^2 \end{bmatrix}$$

The condition $m(D) \cdot D = rD$ produces three numbers which must be equal

$$\begin{cases} 6a^2 + 2d^2 \\ (4 + 2k)\lambda^2 + 2d^2 \\ 2a^2 + 6d^2 + 2k\lambda^2 \end{cases}$$

and this produces two equalities

$$\begin{cases} 4\lambda^2 = 2a^2 + 4d^2 \\ 4a^2 - 4d^2 = 2k\lambda^2 \end{cases}$$

From this we obtain the relation $(4 - k)a^2 = (4 + 2k)d^2$. If $k \geq 4$ then $d^2 \leq 0$, which is a contradiction. We have shown for $k \geq 4$ the two-step nilpotent Lie algebra $C = A +_c B$ cannot admit a soliton metric. \square

Next we construct a new example of a soliton $\mathfrak{N} = \mathfrak{U}_1 \oplus \mathfrak{U}_2 \oplus \mathfrak{Z}$ where \mathfrak{U}_1 does not admit a soliton and \mathfrak{U}_2 does admit a soliton.

EXAMPLE 8.12. Let $\mathfrak{U}_1 \oplus \mathfrak{Z}$ be the algebra with structure matrices $A +_c B$ (from above) with $k = 5$. Here $\dim \mathfrak{U}_1 = 13$ and $\dim \mathfrak{Z} = 2$. Let $\mathfrak{U}_2 \oplus \mathfrak{Z}$ have the usual $(2, 3)$ structure matrices $\begin{pmatrix} 0 & b \\ -b & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & d \\ & -d & 0 \end{pmatrix}$. Then construct an algebra \mathfrak{N} with structure matrices concatenated from the above as follows

$$\left(\begin{array}{ccccccc|ccccccc} 0 & & & & & & & & & & & & & & & & \\ & 0 & & & & & & & & & & & & & & & \\ & & \ddots & & & & & & & & & & & & & & \\ & & & 0 & & & & & & & & & & & & & \\ & & & & 0 & & & & & & & & & & & & \\ & & & & & 0 & a & & & & & & & & & & \\ & & & & & & -a & 0 & & & & & & & & & \\ & & & & & & & & 0 & & & & & & & & \\ \hline & & & & & & & & & 0 & b & & & & & & \\ & & & & & & & & & -b & 0 & & & & & & \\ & & & & & & & & & & & 0 & & & & & \end{array} \right), \quad \left(\begin{array}{ccccccc|ccccccc} 0 & \lambda & & & & & & & & & & & & & & & \\ & -\lambda & 0 & & & & & & & & & & & & & & \\ & & \ddots & & & & & & & & & & & & & & \\ & & & 0 & \lambda & & & & & & & & & & & & \\ & & & & -\lambda & 0 & & & & & & & & & & & \\ & & & & & & 0 & & & & & & & & & & \\ & & & & & & & 0 & c & & & & & & & & \\ & & & & & & & & -c & 0 & & & & & & & \\ \hline & & & & & & & & & & 0 & & & & & & \\ & & & & & & & & & & & 0 & d & & & & \\ & & & & & & & & & & & & -d & 0 & & & \end{array} \right).$$

This is a soliton for $\lambda = 1, a^2 = b^2 = 16/9, c^2 = d^2 = 1/9$.

Remark. Here the algebra $\mathfrak{U}_1 \oplus \mathfrak{Z}$ does not admit a soliton (see Proposition 8.10 with $k = 5$), whereas, the algebra $\mathfrak{U}_2 \oplus \mathfrak{Z}$ does admit a soliton. Similarly, one obtains a soliton above with $k = 4$. I have not checked the cases $k \geq 6$ but these probably do not admit soliton metrics.

This completely answers Question 8.1 in regards to algebras of the type $\mathfrak{N} = \mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \mathfrak{Z}$.

4. Non-Einstein Nilradicals

In this section we describe a procedure for constructing non-Einstein nilradicals for many different (p, q) types; that is, algebras which do not admit a soliton metric. Many of the details have the same flavor as those through out the previous sections and we omit some of the technical work.

Consider two metric two-step nilpotent Lie algebras \mathfrak{N}_1 and \mathfrak{N}_2 . Then one can trivially create the metric two-step nilalgebra $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ where the direct sum is orthogonal and the two subspaces $\mathfrak{N}_1, \mathfrak{N}_2$ commute. Even if the orthogonal direct sum of two algebras is not soliton, in principal it could be possible to endow \mathfrak{N} with a nilsoliton metric such that \mathfrak{N}_1 and \mathfrak{N}_2 are not orthogonal.

Let A^1, A^2 be structure matrices corresponding to the algebras $\mathfrak{N}_1, \mathfrak{N}_2$, respectively. Let $\mathfrak{N}_i = \mathcal{V}_i \oplus \mathfrak{Z}_i$ and $p_i = \dim \mathfrak{Z}_i$ for $i = 1, 2$. Then \mathfrak{N} has structure matrix $C = (C_1, \dots, C_{p_1+p_2}) \in \mathfrak{so}(q_1 + q_2)^{p_1+p_2}$ where

$$\begin{aligned} C_i &= \begin{pmatrix} A_i^1 & \\ & 0 \end{pmatrix} & \text{for } 1 \leq i \leq p_1 \\ C_{j+p_1} &= \begin{pmatrix} 0 & \\ & A_j^2 \end{pmatrix} & \text{for } 1 \leq j \leq p_2 \end{aligned}$$

Denote the construction by $C = A^1 \oplus A^2$.

PROPOSITION 8.13. *Consider $C = A^1 \oplus A^2$ constructed from A^1, A^2 as above. If C is a soliton algebra then so are A^1, A^2 .*

The proof amounts to block matrix multiplication upon writing out $m(C) = m_1(C) + m_2(C)$ and so we leave the details to the reader.

We do not know if the converse is true since admitting a soliton metric corresponds to moving along the group orbit $GL(q_1 + q_2, \mathbb{R}) \times GL(p_1 + p_2, \mathbb{R}) \cdot C$. However, we are able to sidestep this point for a particular case of interest.

Constructing new non-Einstein nilradicals via direct summing. Let \mathfrak{N}_1 , with structure matrix A^1 , be any algebra of type $(2, 3 + 2k)$ as in Proposition 8.10 with $k \geq 4$; note that \mathfrak{N}_1 has no Euclidean de Rham factor. Let \mathfrak{N}_2 , with structure matrix A^2 , be a nilsoliton algebra of type (p, q) . To prove the lack of existence of a soliton metric on the algebra $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ we will study the negative gradient flow corresponding to the group $GL(q_1 + q_2, \mathbb{R}) \times GL(p_1 + p_2, \mathbb{R})$ starting at the point $C = A^1 \oplus A^2$. We will prove that any algebra in the limit cannot be isomorphic to \mathfrak{N} and hence that \mathfrak{N} cannot admit a soliton metric.

Consider the subspace $\mathfrak{so}(q_1, \mathbb{R})^{p_1+p_2} \oplus \mathfrak{so}(q_2, \mathbb{R})^{p_1+p_2}$ of $\mathfrak{so}(q_1 + q_2, \mathbb{R})^{p_1+p_2}$. The vector space $\mathfrak{so}(q_1, \mathbb{R})^{p_1}$ embeds into the aforementioned space via the first p_1 coordinates and similarly the vector space $\mathfrak{so}(q_2, \mathbb{R})^{p_2}$

embeds via the second p_2 coordinates. We are interested in the case $q_1 = 3 + 2k$ and $p_1 = 2$. Now consider the vector space $W_1 \subset \mathfrak{so}(3 + 2k, \mathbb{R})^2$ spanned by $(M, 0)$ and $(0, M')$ where M, M' are of the form

$$\begin{pmatrix} \begin{array}{cc|ccc} 0 & \lambda & & & \\ -\lambda & 0 & & & \\ \hline & & \ddots & & \\ & & & \begin{array}{cc|ccc} 0 & \lambda & & & \\ -\lambda & 0 & & & \\ \hline & & & 0 & a & b \\ & & & -a & 0 & c \\ & & & -b & -c & 0 \end{array} & & \end{array} \end{pmatrix}$$

where $\lambda, a, b, c \in \mathbb{R}$. Embed W_1 into $\mathfrak{so}(q_1 + q_2, \mathbb{R})^{p_1 + p_2}$ above.

Now consider $A \in W_1$ and a soliton $B \in \mathfrak{so}(q_2, \mathbb{R})^{p_2} \subset \mathfrak{so}(q_1 + q_2, \mathbb{R})^{p_1 + p_2}$. If we consider $C = A \oplus B$ then writing out the definitions one obtains

$$m(C) \cdot C = \begin{pmatrix} m(A) \cdot A \\ m(B) \cdot B \end{pmatrix}$$

Moreover, the negative gradient flow starting at C remains tangent to $W_1 \oplus \{\mathbb{R} - \text{span } B\}$ as B is a soliton.

We give a proof by contradiction. Assume that C admits a soliton and let $D \in \omega[C]$ be a limit point of the negative gradient flow starting at $[C]$. By Theorem 5.10, $D \in GL(q_1 + q_2, \mathbb{R}) \times GL(p_1 + p_2, \mathbb{R}) \cdot C$ and D has block decomposition in $W_1 \oplus \{\mathbb{R} - \text{span } B\}$ by the remarks above and the argument of Theorem 8.6. By Proposition 8.13 the ‘A-slot’ of D (component in W_1) must be soliton. But this contradicts Proposition 8.10 since $k \geq 4$.

This provides us with many new examples of non-Einstein nilalgebras in most types (p, q) .

Bibliography

- [BHC62] Armand Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, The Annals of Mathematics **75** (1962), no. 3, 485–535, 2nd ser.
- [Bir71] David Birkes, *Orbits of linear algebraic groups*, Ann. of Math.(2) **93** (1971), no. 3, 459–475.
- [Bor91] Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, 1991.
- [BtD95] Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1995, Translated from the German manuscript, Corrected reprint of the 1985 translation. MR MR1410059 (97i:22005)
- [Che55] Claude Chevalley, *Théorie des groupes de Lie. Tome III. Théorèmes généraux sur les algèbres de Lie*, Actualités Sci. Ind. no. 1226, Hermann & Cie, Paris, 1955.
- [CK04] Bennett Chow and Dan Knopf, *The ricci flow: an introduction*, Mathematical surveys and monographs, vol. 110, AMS, 2004.
- [dC92] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty.
- [Ebe94] Patrick Eberlein, *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sci. cole Norm. Sup.(4) **27** (1994), no. 5, 611–660.
- [Ebe03] ———, *The moduli space of 2-step nilpotent Lie algebras of type (p, q)* , Explorations in complex and Riemannian geometry, Contemporary Mathematics, American Mathematical Society **332** (2003), 37–72.
- [Ebe05] ———, *2-step nilpotent Lie groups arising from semisimple modules*, preprint (2005).
- [Ebe07] ———, *Riemannian 2-step nilmanifolds with prescribed ricci tensor*, to appear in Contemp. Math. (2007).
- [EH96] Patrick Eberlein and Jens Heber, *Quarter pinched homogeneous spaces of negative curvature*, Inter. Jour. Math. **7** (1996), 441–500.
- [EJ] Patrick Eberlein and Michael Jablonski, *Closed orbits and the real m function*, in preparation.
- [Ela72] A.G. Elashvili, *Stationary subalgebras of points of the common state for irreducible linear Lie groups*, Funct. Anal. Appl. **6** (1972), 139–148.
- [Gor85] Carolyn Gordon, *Naturally reductive homogeneous riemannian manifolds*, Can. J. Math. **XXXVII** (1985), no. 3, 467–487.
- [GS82] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513.
- [Heb98] Jens Heber, *Noncompact homogeneous einstein spaces*, Inventiones Mathematicae **133** (1998), no. 2, 279–352.
- [Hei74] Ernst Heintze, *On homogeneous manifolds of negative curvature*, Mathematische Annalen **211** (1974), 23–34.
- [Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, 2001.
- [Hoc65] G. Hochschild, *The structure of Lie groups*, Holden-Day Inc., San Francisco, 1965.
- [Hum81] James E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics 21, Spring-Verlag, New York Heidelberg Berlin, 1981.
- [Kap83] Aroldo Kaplan, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35–42.

- [Kir84] Frances Clare Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes 31, Princeton University Press, Princeton, New Jersey, 1984.
- [KL87] Friedrich Knop and Peter Littleman, *Der grad erzeugender funktionen von invariantenringen. (German) [the degree of generating functions of rings of invariants]*, Math. Z. **196** (1987), no. 2, 211–229.
- [KN78] G. Kempf and L. Ness, *The length of vectors in representation spaces*, Springer Lecture Notes 732 (Copenhagen), Algebraic Geometry, Proceedings, 1978, pp. 233–244.
- [Lau98] Jorge Lauret, *Naturally reductive homogeneous structures on 2-step nilpotent Lie groups*, Rev. Un. Mat. Argentina **41** (1998), no. 2, 15–23.
- [Lau01] ———, *Ricci soliton homogeneous nilmanifolds*, Math. Annalen **319** (2001), 715–733.
- [Lau05] ———, *Minimal metrics on nilmanifolds*, Differential geometry and its applications (2005), 79–97, Matfyzpress, Prague.
- [Lau07] ———, *Einstein solvmanifolds are standard*, preprint (2007).
- [LW] Jorge Lauret and Cynthia Will, *Einstein solvmanifolds: Existence and non-existence questions*, preprint.
- [Mar91] G.A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Springer-Verlag, New York Berlin Heidelberg, 1991.
- [Mar01] Alina Marian, *On the real moment map*, Mathematical Research Letters **8** (2001), 779–788.
- [Mil76] John Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329.
- [Mos50] George Daniel Mostow, *The extensibility of local Lie groups of transformations and groups on surfaces*, The Annals of Mathematics **52** (1950), no. 3, 606–636, 2nd Series.
- [Mos55] G. D. Mostow, *Self-adjoint groups*, Ann. of Math. (2) **62** (1955), 44–55.
- [Mun00] James R. Munkres, *Topology*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2000.
- [New78] P.E. Newstead, *Lectures on introduction to moduli problems and orbit spaces*, Springer-Verlag, published for the Tata Institute of Fundamental Research, Bombay, Berlin New York, 1978.
- [NM84] Linda Ness and David Mumford, *A stratification of the null cone via the moment map*, American Journal of Mathematics **106** (1984), no. 6, 1281–1329.
- [Pop70] V.L. Popov, *Criteria for the stability of the action of a semisimple group on the factorial of a manifold. (russian)*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 523–531.
- [PV94] V.L. Popov and E.B. Vinberg, *Algebraic geometry IV: II. invariant theory*, Springer-Verlag, Berlin Heidelberg, 1994.
- [RS90] R.W. Richardson and P.J. Slodowy, *Minimum vectors for real reductive algebraic groups*, J. London Math. Soc. **42** (1990), 409–429.
- [Ser87] J-P. Serre, *Complex semisimple Lie algebras*, Springer Monographs in Mathematics, Springer, 1987.
- [Sha88] Igor R. Shafarevich, *Basic algebraic geometry I*, second edition ed., Springer-Verlag, Berlin Heidelberg New York, 1988.
- [Whi57] Hassler Whitney, *Elementary structure of real algebraic varieties*, The Annals of Mathematics **66** (1957), no. 3, 545–556, 2nd Ser.
- [Wil82] Edward N Wilson, *Isometry groups on homogeneous nilmanifolds*, Geom. Dedicata **12** (1982), no. 3, 337–346.
- [Wil03] C.E. Will, *Rank-one einstein solvmanifolds of dimension 7*, Diff. Geom. Appl. **19** (2003), 307–318.