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Oscillatory solutions of fourth order conservative systems via the Conley index

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ABSTRACT

In this paper we investigate periodic solutions of second order Lagrangian systems which oscillate around equilibrium points of center type. The main ingredients are the discretization of second order Lagrangian systems that satisfy the twist property and the theory of discrete braid invariants developed by Ghrist et al. (2003) [5]. The problem with applying this topological theory directly is that the braid types in our analysis are so-called *improper*. This implies that the braid invariants do not entirely depend on the topology: the relevant braid classes are *non-isolating* neighborhoods of the flow, so that their Conley index is not universal. In first part of this paper we develop the theory of the braid invariant for improper braid classes and in the second part this theory is applied to second order Lagrangian system and in particular to the Swift–Hohenberg equation.

1. Introduction

The mathematical study of pattern formation phenomena covers a plethora of methods, results and equations. One of the most successful attempts in this area concerns fourth order conservative systems. We refer to [2,12] for comprehensive overviews. These equations exhibit complicated dynamic behavior and the dependence of the dynamics on parameters is intricate. The purpose of this paper is to uncover a topological structure that underlies many of the individual patterns generated by fourth order systems. Throughout, the equation

$$u'''' + \alpha u'' - u + u^3 = 0 \tag{1.1}$$

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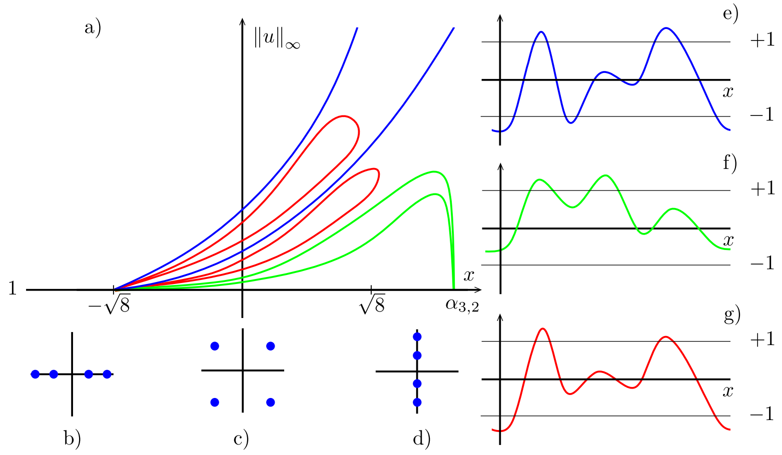

Fig. 1. Bifurcation diagram a) shows three different types of branches, in the plane $(\alpha, \|u\|_\infty)$, which bifurcate at $\alpha = -\sqrt{8}$. Solutions on the branches that extend beyond the boundary of the diagram are of the first type, see e) for an example; branches that form closed loops consist of solutions of the third type, see f) for an example; branches collapsing on $\|u\|_\infty = 1$ consist of solutions of the second type, see g) for an example. Also depicted is the spectrum of the linearization around P_+ and P_- for b) $\alpha \leqslant -\sqrt{8}$; c) $\alpha \in (-\sqrt{8}, \sqrt{8})$; d) $\sqrt{8} \leqslant \alpha$.

is our main example, although the results also apply to a more general class of fourth order conservative systems, which occur as the Euler–Lagrange equations of second order Lagrangians $\int_I L(u, u', u'')\,dt$. For Eq. (1.1) the Lagrangian density is given by

$$L(u, v, w) = \frac{1}{2}w^2 - \frac{\alpha}{2}v^2 + \frac{1}{4}\left(u^2 - 1\right)^2.$$

Related to this variational structure (through Noether's theorem) is a conserved quantity. Solutions of Eq. (1.1) satisfy the energy identity

$$\mathbb{E}[u] = -u'u''' + \frac{1}{2}\left(u''\right)^2 - \frac{\alpha}{2}\left(u'\right)^2 - \frac{1}{4}\left(u^2 - 1\right)^2 = E.$$

In the case $\alpha < 0$ Eq. (1.1) is referred to as the eFK equation (see e.g. [6–8]), while for $\alpha \geqslant 0$ it is the Swift–Hohenberg equation [13]. It appears in physical models for phase transitions, Rayleigh–Bénard convection, non-linear optics, etc., see [2,3,12] for more extensive surveys.

For Eq. (1.1) the energy level $E = 0$ contains the two homogeneous states $u_\pm = \pm 1$ and this energy level acts as an organizing center for the dynamics. Homoclinic solutions to $u_\pm = \pm 1$ and/or a heteroclinic cycle between -1 and $+1$ will, if they exist, lie in this energy level, and it is well known that such connecting orbits may be the source of complicated dynamics. This makes it a natural choice to study the solutions in this *singular* energy level, even though it leads to analytical difficulties. The focus on the singular energy level is not new and we will summarize some of the known results below.

To introduce the central question of this paper, let us summarize the known results on (1.1) most relevant to our problem. The structure of the set of periodic solutions of Eq. (1.1) depends to a large extent on the linearization around the constant solutions $u_\pm = \pm 1$, and hence on the value of the parameter α. In particular, one can identify two critical values of α: $\alpha = +\sqrt{8}$ and $\alpha = -\sqrt{8}$. At these values the linearization around the constant solutions $u_\pm$, i.e. the points $P_\pm = (\pm 1, 0, 0, 0)$ in (u, u', u'', u''') phase space, changes type, as indicated in Fig. 1. In fact, for $\alpha \leqslant -\sqrt{8}$, the equilibria $u_\pm$ are real saddles and there are no periodic solutions on the zero energy level. The set of *all* bounded solutions is very limited, and consists of the three equilibrium points, two monotone antisymmetric heteroclinic loops and (modulo translations) a one parameter family of single bump periodic solutions,

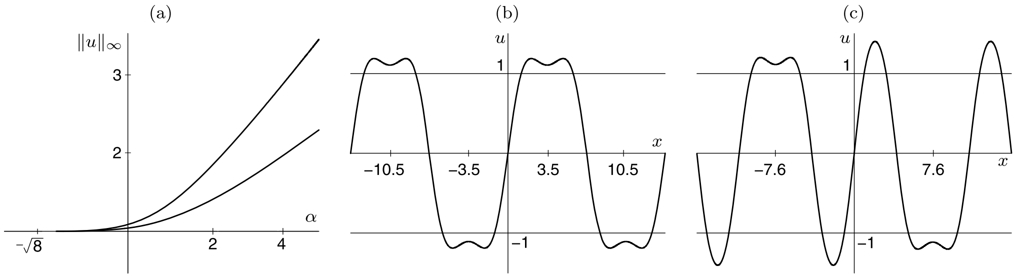

Fig. 2. Bifurcation diagram for the solutions of the first type and corresponding solutions for $\alpha = 1.5$. The number of monotone laps for the solution pictured at (b) is six and it intersects $u_\pm$ two times. For solution (c) the number of monotone laps is ten and the crossing number with $u_\pm$ is six.
Reproduced from [13].

which are even with respect to their extrema and odd with respect to their zeros. These periodic solutions can be parameterized by the energy $E \in (-\frac{1}{4}, 0)$, see [16]. As α increases beyond $-\sqrt{8}$ the equilibria $u_\pm$ become saddle-foci and the set of periodic solutions becomes much richer. There is a plethora of periodic solutions on the energy level zero bifurcating from the heteroclinic loop at $\alpha = -\sqrt{8}$. It has been proved that for $-\sqrt{8} < \alpha \leqslant 0$ the zero energy level contains a great variety of multi-bump periodic solutions. For detailed results we refer to [6–8]. For $0 < \alpha < \sqrt{8}$ the results are more tentative and less complete. For $\alpha > \sqrt{8}$ the equilibria change to centers and small periodic oscillations around the equilibria $u_\pm$ appear. This is the parameter regime of primary interest in the present paper.

It was shown in [15] that at regular energy levels every solution is a concatenation of monotone laps between extrema and the number of the monotone laps is finite and even per period. This also holds for the singular energy level $E = 0$, provided the conventions discussed below are adopted. Fig. 1 shows the bifurcation diagram, where we graph the L^∞-norm of the solutions u^α of (1.1) against the value α. Three branches with very different geometry appear in the bifurcation diagram. Two essential properties are preserved for the solutions lying on the same branch of the bifurcation diagram:

(1) the number of monotone laps;
(2) the number of crossings of the solution with the u_+ and u_-.

The counting of laps and crossings is done with the following conventions. For a regular monotone lap u' does not change sign i.e. $u' < 0$ or $u' > 0$, and a degenerate monotone lap is an inflection point. We have to count both non-degenerate and degenerate monotone laps in order to obtain the invariant along the bifurcation branch, see [12]. The number of crossings of a solution u with $u_\pm$ is the number of zero points of the function $u - u_\pm$ counted over one period *without* multiplicity, i.e. every zero point is counted just once even if it is a multiple zero. The zero points of the function are isolated, hence this number is well defined and finite, and it is preserved along the continuous branches, see [12]. We can make a three way classification of solution branches, making use of the two invariants described above.

1. Solutions of the first type cross the constant solutions u_+ and u_- in such a way that two crossings with u_+ are followed by two crossings with u_- and vice versa.

The existence of an infinite family of periodic solutions of the first type, which extend for all $\alpha > -\sqrt{8}$, was proved in [5]. Two examples are shown in Fig. 2. In the bifurcation diagram we graph the supremum norm $\|u\|_\infty$ against α.

2. The second type consists of solutions which cross the constant solutions u_+ and u_- but crossings do not alternate as for the first type.

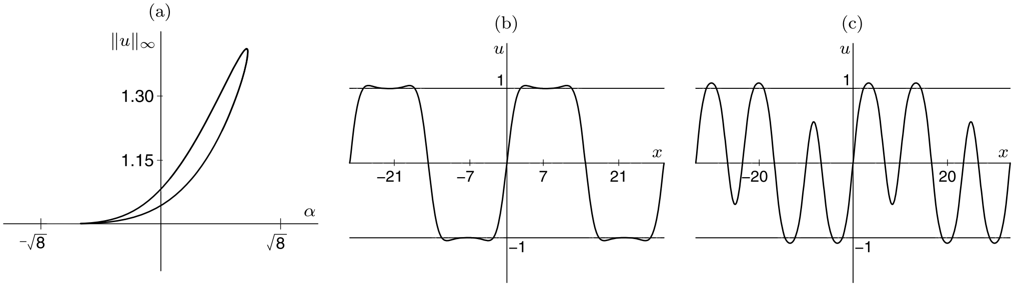

Fig. 3. Bifurcation diagram for the solutions of the second type and corresponding solutions. Both solutions in (b) and (c) lie on the same bifurcation branch and have six monotone laps and cross $u_\pm$ four times. The depicted solutions corresponds to the parameter value $\alpha = -\frac{1}{10}$.
Reproduced from [13].

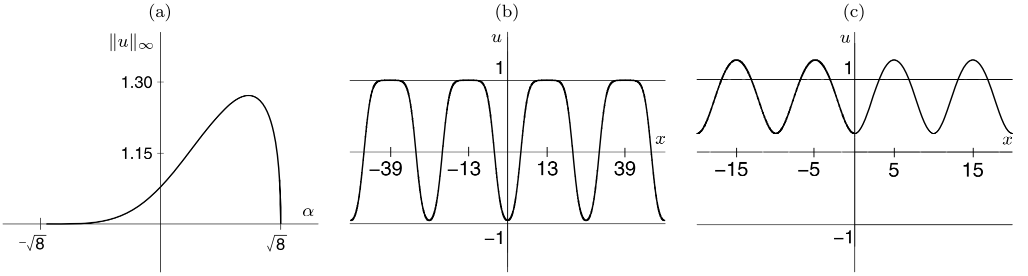

Fig. 4. Bifurcation diagram for the solutions of the third type and corresponding solutions which are in the class $\mathbf{u}_{1,1}$. Solution (b) corresponds to $\alpha = -1$ and (c) to $\alpha = 1$.
Reproduced from [13].

Existence of different solutions of the second type was proven for $\alpha \in (-\sqrt{8}, 0)$. Actually, there is a countable infinity of second type solutions with different numbers of monotone laps. Numerical evidence suggests that these solution continue to exist until some positive α^* (where α^* depends on the solution branch) and two branches of solutions in the bifurcation diagram form a loop (see Fig. 3). Therefore, there are two solutions of the second type with the same crossing number and number of monotone laps for $\alpha \in (-\sqrt{8}, \alpha^*)$ and they coalesce at α^*. For more detailed results we refer to [7,9–11,14].

3. The third kind of periodic solutions crosses only one of the constant solutions u_+ or u_-.

In this paper we are interested in solutions of the third type, see Fig. 4.

1.1. Definition. A periodic solution is of class $\mathbf{u}_{p,q}$ if it is a solution of the third type (i.e., it does not cross $u_- = -1$) with $2p$ monotone laps per period, and it intersects the constant solution $u_+ = 1$ exactly $2q$ times.

Solutions of the third type come as a family of countably many distinct periodic solutions which bifurcate from the heteroclinic loop at $\alpha = -\sqrt{8}$. However, this family does not extend to infinity (as the first type) in parameter space nor do they lie on loops (as the second type). Instead, numerical results indicate that these periodic solutions bifurcate from the constant solution u_+ as α tends to a critical value $\alpha_{p,q}$ of the form

$$\alpha_{p,q} = \sqrt{2}\left(\frac{p}{q} + \frac{q}{p}\right), \quad p, q \in \mathbb{N}\ (p \geqslant q), \tag{1.2}$$

see Fig. 4(a). The parameter values $\alpha_{p,q}$ appear naturally in the linearization around the centers $P_\pm$, see [2,13]. For $q=1$ and $p \in \mathbb{N}$ it was analytically shown in [13] that there exists a family of solutions in the class $\mathbf{u}_{p,1}$ for $\alpha \in (-\sqrt{8}, \alpha_{p,1})$. Moreover, for $p \geqslant 2$ these solutions come in pairs. Numerically computed graphs of two solutions of class $\mathbf{u}_{1,1}$ are shown in Fig. 4.

The shooting technique used in [13] to prove existence of a solution of class $\mathbf{u}_{1,1}$ depends strongly on the particular equation. The method which we develop in this paper generalizes this result in two ways. The application of our method to Eq. (1.1) proves the existence of solutions of the class $\mathbf{u}_{p,q}$ for every pair $p,q \in \mathbb{N}$ that is coprime. The other aspect is that our technique is not limited to this specific equation. It can be applied to conservative equations with a variational formulation as discussed above.

The idea is to use already known solutions of the equation in order to force existence of additional solutions. This idea goes back to [5] where it was shown that a solution of Euler–Lagrange equation of Lagrangian system with a *twist property* corresponds to a fixed point of a flow Ψ^t generated by a parabolic recurrence relation, which is defined on an appropriate space of braids. The space of braids is not connected and its connected components are called braid classes. The braid classes used in [5] are *isolating* neighborhoods for the flow Ψ^t. Therefore, the *Conley index* can be used to show the existence of a fixed point within the class. We will give a more detailed account in the next section. In trying to use these ideas for solutions of the third type the associated braid classes *fail to be isolating*. This is due to the fact that $u_\pm$ are always fixed points on the boundary. This type of braid classes is called improper. Using the ideas from [1] we show that local information near these fixed points allows us to define modified braid classes which are isolating neighborhoods, and for which the invariant set inside the braid class is the same as for the unmodified one. Based on local information about $u_\pm$ we define topological invariants for the modified proper braid classes. We use the non-triviality of this invariant to prove the existence of a solution which corresponds to a fixed point in an improper braid class (i.e. a non-isolating neighborhood).

By applying this result to Eq. (1.1) we show the existence of different solutions of the third type. Namely for any coprime $p > q$ we prove that there is a solution $u \in \mathbf{u}_{p,q}$, for $\alpha \in [\sqrt{8}, \alpha_{p,q})$. We cannot extend the result for $\alpha \geqslant \alpha_{p,q}$ because the local behavior in the fixed point on the boundary of the braid class changes character for this parameter value. Indeed, numerics suggest that the branch of the solutions in the class $\mathbf{u}_{p,q}$ bifurcates from the constant solution $u_+ = 1$ at $\alpha = \alpha_{p,q}$.

1.2. Theorem. *Let $p,q \in \mathbb{N}$ be coprime and $q < p$. Then there exists a solution $u^\alpha \in \mathbf{u}_{p,q}$ of Eq.* (1.1) *with $\mathbb{E}[u^\alpha] = 0$ for every $\alpha \in (\sqrt{8}, \alpha_{p,q})$.*

1.3. Remark. In the above theorem we restrict to the α-parameter range for which $u_\pm$ are centers. For $\alpha \in [0, \sqrt{8}]$ the twist property still holds and $u_\pm$ are saddle-centers, see Fig. 1. We believe that the theorem extends to this parameters range, but while the methods needed are related, they are also expected to involve somewhat different arguments, cf. [5]. Since the current proof already involves considerable effort, we leave the saddle-focus regime for future research.

The outline of the paper is as follows. In Section 2 we explain the connection between solutions of the ODE (1.1) and solutions of parabolic recurrence relations. Braid classes and their associated Conley indices are presented in Section 3, while the up–down braid classes that describe solutions of (1.1) are discussed in Section 4. The twist number, which plays a crucial role in the analysis near $u_\pm$, is introduced in Section 5. With all the ingredients in place, Section 6 is the heart of the paper, where the main results are formulated and proved. Finally, the application of these results to (1.1) in Section 7 leads to the proof of Theorem 1.2.

2. Reduction to a finite dimensional problem

In this section we give a brief survey of the reduction of the problem of finding periodic solutions for Eq. (1.1) to the problem of finding fixed points of a vector field generated by a parabolic recurrence relation. We present this approach in the context of general second order Lagrangians.

If we seek closed characteristics, i.e., a periodic solution of Eq. (1.1) at a given energy level E, we can invoke the following variational principle:

$$\text{Extremize}\left\{J_E[u]\colon u \in \Omega_{\mathrm{per}},\ \tau > 0\right\}, \tag{2.1}$$

where $\Omega_{\mathrm{per}} = \bigcup_{\tau>0} C^2(S^1, \tau)$, the periodic functions with period τ, and

$$J_E[u] = \int_0^\tau \left(L\left(u, u', u''\right) + E\right) dt. \tag{2.2}$$

The function $L \in C^2(\mathbb{R}^3, \mathbb{R})$ is assumed to satisfy $\frac{\partial^2 L}{\partial w^2}(u, v, w) \geqslant \delta > 0$ for all $(u, v, w) \in \mathbb{R}^3$. For the general second order Lagrangian system the (conserved) energy is given by

$$\mathbb{E}[u] = \left(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\right)u' + \frac{\partial L}{\partial u''}u'' - L\left(u, u', u''\right). \tag{2.3}$$

It follows from [15] that the variations in τ guarantee that any critical point u of (2.1) has energy $\mathbb{E}[u] = E$. An energy value E is called regular if $\frac{\partial L}{\partial u}(u, 0, 0) \neq 0$ for all u that satisfy $L(u, 0, 0) + E = 0$. For a regular energy value E the energy manifold $M_E \subset \mathbb{R}^4$ is a smooth non-compact manifold without boundary, and the extrema of a closed characteristic are contained in the closed set $\{u\colon\ L(u, 0, 0) + E \geqslant 0\}$. The connected components I_E of this set are called interval components. Moreover, it follows from [15] that solutions on a regular energy level do not have inflection points. For a singular energy level the interval component I_E contains critical points and the situation is more complicated.

First, we restrict to regular energy levels. It was shown in [15] that for Lagrangian systems $J[u] = \int_I L(u, u', u'')\, dt$, where $L(u, u', u'') = \frac{1}{2}u''^2 + K(u, u')$, at any energy level E which satisfies

$$\frac{\partial K}{\partial v}v - K(u, v) - E \leqslant 0 \quad \text{for all } u \in I_E \text{ and } v \in \mathbb{R}, \tag{2.4a}$$

$$\frac{\partial^2 K}{\partial v^2}v^2 - \frac{5}{2}\left\{\frac{\partial K}{\partial v} - K(u, v) - E\right\} \geqslant 0 \quad \text{for all } u \in I_E \text{ and } v \in \mathbb{R}, \tag{2.4b}$$

there is a unique pair (τ, u_τ) minimizing

$$\inf_{u \in X_\tau, \tau \in \mathbb{R}^+} \int_0^\tau \left(L\left(u, u', u''\right) + E\right) dt,$$

where $X_\tau(u_1, u_2) = \{u \in C^2([0, \tau])\colon\ u(0) = u_1,\ u(\tau) = u_2,\ u'(0) = u'(\tau) = 0,\ u|_{(0,\tau)} > 0 \text{ if } u_1 < u_2$ and $u|_{(0,\tau)} < 0$ if $u_1 > u_2\}$ for $(u_1, u_2) \in I_E \times I_E \setminus \Delta$, and $\Delta = \{(u_1, u_2) \in I_E \times I_E\colon\ u_1 = u_2\}$. Moreover, the function defined by

$$S_E(u_1, u_2) = \inf_{u \in X_\tau, \tau \in \mathbb{R}^+} \int_0^\tau \left(L\left(u, u', u''\right) + E\right) dt, \tag{2.5}$$

for $(u_1, u_2) \in I_E \times I_E \setminus \Delta$ and $S_E|_\Delta = 0$, has the following properties:

(a) $S_E \in C^2(I_E \times I_E \setminus \Delta)$.
(b) $\partial_1\partial_2 S_E(u_1,u_2) > 0$ for all $u_1 \neq u_2 \in I_E$.
(c) $\lim_{u_1 \nearrow u_2} -\partial_1 S_E(u_1,u_2) = \lim_{u_2 \searrow u_1} \partial_2 S_E(u_1,u_2) = \lim_{u_1 \searrow u_2} \partial_1 S_E(u_1,u_2) = \lim_{u_2 \nearrow u_1} -\partial_2 S_E(u_1, u_2) = +\infty$.

The function S_E is a *generating function* and a Lagrangian system possessing such a generating function is called a *twist system*. The second order Lagrangian system associated to Eq. (1.1) is a twist system for $\alpha \geqslant 0$, see [15] for more examples.

The question of finding closed characteristics for a twist system can now be formulated in terms of S_E. Any periodic solution u is a concatenation of monotone laps. Let us take an arbitrary $2p$ periodic sequence $\{u_i\}$ and define u as a concatenation of monotone laps, namely minimizers $u_\tau(u_i, u_{i+1})$, between the consecutive extremal points u_i solving the Euler–Lagrange equation in between any two extrema. The concatenation u does not have to be a solution on $\mathbb{R}$ because the third derivatives of two consecutive monotone laps do not have to match at the common extremal point u_i. It was proved in [15] that the third derivatives match if and only if the extrema sequence $\{u_i\}$ is a critical point of discrete action

$$W_{2p} = \sum_{i=0}^{2p-1} S_E(u_i, u_{i+1}). \tag{2.6}$$

Critical points of W_{2p} satisfy equations

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) \stackrel{\text{def}}{=} \partial_2 S_E(u_{i-1}, u_i) + \partial_1 S_E(u_i, u_{i+1}) = 0, \tag{2.7}$$

where $\mathcal{R}_i(s,t,r)$ is, according to property (a), well defined and C^1 on the domains

$$\Omega_i = \bigl\{(r,s,t) \in I_E^3\colon\ (-1)^{i+1}(s-r) > 0,\ (-1)^{i+1}(s-t) > 0\bigr\}. \tag{2.8}$$

The functions $\mathcal{R}_i$ and domains Ω_i satisfy $\mathcal{R}_i = \mathcal{R}_{i+2}$ and $\Omega_i = \Omega_{i+2}$ for $i \in \mathbb{Z}$. Property (b) implies that $\partial_1\mathcal{R}_i = \partial_1\partial_2 S(u_{i-1}, u_i) > 0$ and $\partial_3\mathcal{R}_i = \partial_1\partial_2 S(u_i, u_{i+1}) > 0$. Property (c) provides information about the behavior of $\mathcal{R}_i$ at the diagonal boundaries of Ω_i:

$$\lim_{s\searrow r} \mathcal{R}_i(r,s,t) = \lim_{s\searrow t} \mathcal{R}_i(r,s,t) = +\infty, \tag{2.9}$$

$$\lim_{s\nearrow r} \mathcal{R}_i(r,s,t) = \lim_{s\nearrow t} \mathcal{R}_i(r,s,t) = -\infty. \tag{2.10}$$

Above-mentioned properties of $\mathcal{R}_i$ give us that $\mathcal{R}_i$ is a parabolic recurrence relation of up–down type as defined below (Definition 2.2). First, we define parabolic recurrence relations.

2.1. Definition. A parabolic recurrence relation $\mathcal{R}$ on $\mathbb{R}^{\mathbb{Z}}$ is a sequence of real-valued functions $\mathcal{R} = (\mathcal{R}_i)_{i\in\mathbb{Z}}$ satisfying

(A1): [monotonicity] $\partial_1\mathcal{R}_i > 0$ and $\partial_3\mathcal{R}_i > 0$ for all $i \in \mathbb{Z}$;
(A2): [periodicity] for some $d \in \mathbb{N}, \mathcal{R}_{i+d} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$.

We see that our $\mathcal{R}$ is not a parabolic recurrence relation in the strict sense because it is not defined on whole space $\mathbb{R}^{\mathbb{Z}}$. In particular, it is not defined for any sequence satisfying $u_i = u_{i+1}$ for some $i \in \mathbb{Z}$. This corresponds to the nature of solutions of Eq. (1.1), namely that minima and maxima alternate.

2.2. Definition. A parabolic recurrence relation $\mathcal{R}$ defined on the domain given by (2.8) is said to be of up–down type if (2.9) and (2.10) are satisfied.

The result of [15] that is pivotal for our current analysis, can be expressed using these definitions as follows.

2.3. Proposition. *Let $J[u] = \int L(u, u', u'')\,dt$ be a second order Lagrangian twist system. Suppose that W_{2p} is the discrete action defined through* (2.5) *and* (2.6) *at the regular energy level E. Then*

(a) *the functions $\mathcal{R}_i = \partial_i W_{2p}$ defined on Ω_i are components of a parabolic recurrence relation $\mathcal{R}$ of up–down type,*
(b) *solutions of $\mathcal{R} = 0$ correspond to periodic solutions on the energy level E.*

The parabolic recurrence relation is thus both exact *and of up–down type.*

In order to find solutions of $\mathcal{R} = 0$ we will employ the Conley index. Conley index theory gives information about the invariant set of a flow inside an isolating neighborhood for this flow. In the case of a gradient vector field, invariant sets have special structure and thus information about critical points can be obtained. There is a natural way to define a flow generated by an up–down parabolic recurrence relation on the set

$$\Omega^{2p} = \big\{\mathbf{u} \in \mathbb{R}^{\mathbb{Z}}\colon\ \mathbf{u} \text{ is } 2p \text{ periodic and } (u_{i-1}, u_i, u_{i+1}) \in \Omega^i,\ \text{for } i \in \mathbb{Z}\big\}. \tag{2.11}$$

Consider the system of differential equations

$$\frac{d}{dt} u_i(t) = \mathcal{R}_i\big(\mathbf{u}(t)\big), \quad \mathbf{u}(t) \in \Omega^{2p},\ t \in \mathbb{R}. \tag{2.12}$$

Eq. (2.12) defines a (local) C^1 flow ψ^t on Ω^{2p}. This flow is not defined on the "diagonal" boundary of Ω^{2p}, but conditions (2.9) and (2.10) give us information about the flow close to this boundary. Finding a periodic solution within the class $\mathbf{u}_{p,q}$ can be reduced to constructing an appropriate isolating neighborhood for the flow ψ^t and calculating its (non-trivial) Conley index.

We will use the concept of up–down discretized braid diagrams to construct this isolating neighborhood. For any $2p$-periodic extrema sequence we can construct a piecewise linear graph by connecting the consecutive points $(i, u_i) \in \mathbb{R}^2$ by straight line segments. The piecewise linear graph, called a strand, is cyclic: one restricts to $0 \leqslant i \leqslant 2p$ and identifies the end points abstractly. A collection of n closed characteristics of period $2p$ then gives rise to a collection of n strands. For multiple strands we can replace the periodicity of a single sequence to a *braid structure* by assigning a crossing type (positive) to every transverse intersection of the graphs: larger slope crosses over smaller slope, see Fig. 5. We thus represent sequences of extrema in the space of closed, positive, piecewise linear braid diagrams. In the next section we briefly recall some basic facts from (discrete) braid theory (for more details see [5]).

3. Braid invariants and the Conley index

We recall now the basic theory of proper braid classes and the Conley type braid invariants, and the implications for parabolic recurrence relations [5]. The parabolic recurrence relations coming from fourth order conservative systems can be put into this framework, as explained in the previous section.

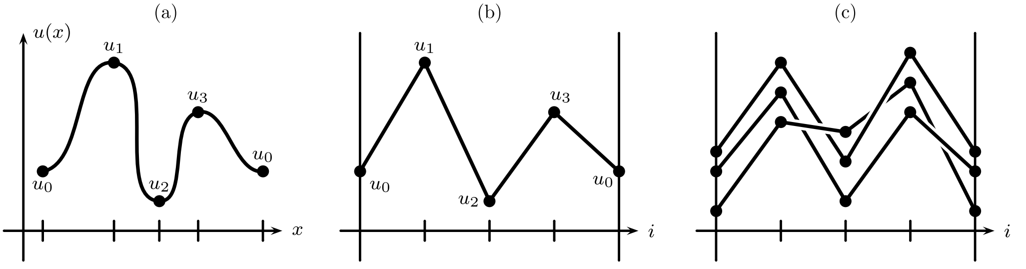

Fig. 5. (a) A periodic function, (b) its piecewise linear graph, and (c) a braid consisting of 3 strands.

3.1. Braid invariants

3.1. Definition. Denote by $\mathcal{D}_d^n$ the space of all closed piecewise linear braid diagrams (PL-braid diagrams) on n strands with period d, i.e., the space of all (unordered) collections $\beta = \{\beta^k\}_{k=1}^n$ of continuous maps $\beta^k : [0,1] \to \mathbb{R}$ such that

(a) β^k is affine linear on $[\frac{i}{d}, \frac{i+1}{d}]$ for all k and for all $i = 0, \dots, d-1$;
(b) $\beta^k(0) = \beta^{\tau(k)}(1)$ for some permutation τ;
(c) for any s such that $\beta^k(s) = \beta^l(s)$ with $k \neq l$, the crossing is transversal: for $\epsilon \neq 0$ sufficiently small

$$\big(\beta^k(s-\epsilon) - \beta^l(s-\epsilon)\big)\big(\beta^k(s+\epsilon) - \beta^l(s+\epsilon)\big) < 0.$$

We note that if $s = 1$ in (c) then the inequality should be interpreted using the permutation from (b).

Any PL-braid diagram corresponds to some n-collection $\mathbf{u} = \{\mathbf{u}^k\}_{k=0}^{n-1}$ of anchor points $\mathbf{u}^k = \{u_i^k\}$, where

$$u_i^k = \beta^k(i/d \bmod 1). \tag{3.1}$$

The converse to this statement is not true, because condition (c) of Definition 3.1 is not satisfied for arbitrary collection of sequences. A collection $\mathbf{u}$ for which this condition is violated corresponds to a singular PL-braid diagram. We switch between the notation u_i^k of the anchor points and β^k of the piecewise linear braid diagrams throughout this section, using β only if necessary. Discretized braid diagrams will primarily be denoted by $\mathbf{u}$. Given anchor points $\mathbf{u}$, the associated piecewise linear braid diagram is given by $\beta(\mathbf{u})$.

Two representatives $\mathbf{u}, \mathbf{u}' \in \mathcal{D}_d^n$ are of the same discretized braid class $[\mathbf{u}] = [\mathbf{u}']$, if and only if they are in the same connected component of $\mathcal{D}_d^n$. Note that if $[\mathbf{u}] = [\mathbf{u}']$, then $\beta(\mathbf{u})$ and $\beta(\mathbf{u}')$ are isotopic as closed positive topological braid diagrams (and braids), see [5]. However, two discretizations of a topological braid are not necessarily equivalent in $\mathcal{D}_d^n$, i.e. connected in $\mathcal{D}_d^n$. The connected component $[\mathbf{u}]$ of $\mathcal{D}_d^n$ are called *braid classes* of period d. The singular braids $\mathbf{u}$ are defined by $\Sigma_d^n \stackrel{\text{def}}{=} \overline{\mathcal{D}}_d^n \backslash \mathcal{D}_d^n$ and consist of braids $\mathbf{u}$ failing (c) in Definition 3.1. We suppress the indices and denote the semi-algebraic sub-variety of singular braids by Σ. The set $\Sigma_- \subset \Sigma$ denotes the collapsed singularities, where entire strands have collapsed onto each other or onto strands of the skeleton, see [5] for details.

For pairs of braids we can define the space of braid pairs using the fact that the union of two braid diagrams is again a braid diagram satisfying (a) and (b) of Definition 3.1. Consider

$$\mathcal{D}_d^{n,m} \stackrel{\text{def}}{=} \big\{(\mathbf{u},\mathbf{v}) \in \mathcal{D}_d^n \times \mathcal{D}_d^m \;\big|\; \mathbf{u} \cup \mathbf{v} \in \mathcal{D}_d^{n+m}\big\}. \tag{3.2}$$

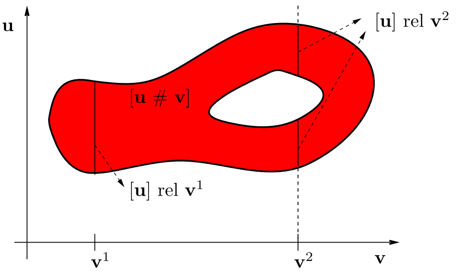

Fig. 6. A relative braid class $[\mathbf{u}\,\#\,\mathbf{v}]$ depicted in gray (red in the web version). Two fibers $[\mathbf{u}]\,\mathrm{rel}\,\mathbf{v}^1$ and $[\mathbf{u}]\,\mathrm{rel}\,\mathbf{v}^2$ are shown as well.

If a pair $(\mathbf{u},\mathbf{v})$ is in $\mathcal{D}_d^{n,m}$ we write $\mathbf{u}\,\#\,\mathbf{v}$. Note that for $\mathbf{u}\,\#\,\mathbf{v}\in\mathcal{D}_d^{n,m}$ it holds $\mathbf{u}\in\mathcal{D}_d^n$ and $\mathbf{v}\in\mathcal{D}_d^m$. As before, the connected components of $\mathcal{D}_d^{n,m}$ are denoted by $[\mathbf{u}\,\#\,\mathbf{v}]$ and are called *relative braid classes* (of period d). (See Fig. 6.) Associated with $[\mathbf{u}\,\#\,\mathbf{v}]$ we have the projection

$$\pi:\mathcal{D}_d^{n,m}\to\mathcal{D}_d^m,\quad \mathbf{u}\,\#\,\mathbf{v}\mapsto\mathbf{v}.$$

For each $\mathbf{v}'\in\pi([\mathbf{u}\,\#\,\mathbf{v}])$ we can define the fiber $[\mathbf{u}']\,\mathrm{rel}\,\mathbf{v}'\stackrel{\mathrm{def}}{=}\{\mathbf{u}'\in\mathcal{D}_d^n\mid\mathbf{u}'\,\#\,\mathbf{v}'\in[\mathbf{u}\,\#\,\mathbf{v}]\}$. The fiber $[\mathbf{u}']\,\mathrm{rel}\,\mathbf{v}'$ is called a *relative braid class with fixed skeleton* $\mathbf{v}'$. Depending on the period d a fiber $[\mathbf{u}']\,\mathrm{rel}\,\mathbf{v}'$ may consist of more than one connected component. The set of connected components relative to a fixed braid $\mathbf{v}\in\mathcal{D}_d^m$ is denoted by $\mathcal{D}_d^n\,\mathrm{rel}\,\mathbf{v}$.

When we interpret d as a parameter, it is natural to consider the generalization to continuous positive braid diagrams $\mathbb{D}^n$, which satisfy Definition 3.1 *without* condition (a). We denote equivalence classes of continuous positive braid diagrams by $[\cdot]_{C^0}$. The discretized braid classes in $\mathcal{D}_d^n$ can be interpreted as subsets of continuous braid classes in $\mathbb{D}^n$ through the piecewise linear interpolations β. The concept of relative braid classes, fibers, singular braids, etc. all have natural counterparts in the continuous category.

3.2. Definition. A relative braid class $[\mathbf{u}\,\#\,\mathbf{v}]\subset\mathcal{D}_d^{n,m}$ is called *bounded* if every fiber $[\mathbf{u}']_{C^0}\,\mathrm{rel}\,\mathbf{v}'$, with $\mathbf{v}'\in\pi([\beta(\mathbf{u}\,\#\,\mathbf{v})]_{C^0})$, is a bounded set.

As before we can define the singular relative braids as $\Sigma_d^n\,\mathrm{rel}\,\mathbf{v}\stackrel{\mathrm{def}}{=}\overline{\mathcal{D}}_d^n\,\mathrm{rel}\,\mathbf{v}\backslash\mathcal{D}_d^n\,\mathrm{rel}\,\mathbf{v}$ and $\Sigma_-\,\mathrm{rel}\,\mathbf{v}\stackrel{\mathrm{def}}{=}\Sigma_-^{n+m}\cap(\mathcal{D}^n\,\mathrm{rel}\,\mathbf{v})$.

3.3. Definition. A relative braid class $[\mathbf{u}\,\#\,\mathbf{v}]\subset\mathcal{D}_d^{n,m}$ is called *proper* if for every fiber $[\mathbf{u}']_{C^0}\,\mathrm{rel}\,\mathbf{v}'$, with $\mathbf{v}'\in\pi([\beta(\mathbf{u}\,\#\,\mathbf{v})]_{C^0})$, it holds that $\mathrm{cl}([\mathbf{u}']_{C^0}\,\mathrm{rel}\,\mathbf{v}')\cap(\Sigma_-\,\mathrm{rel}\,\mathbf{v}')=\emptyset$. If $[\mathbf{u}\,\#\,\mathbf{v}]$ is not proper it is called improper.

For each fiber of a bounded proper relative braid class $[\mathbf{u}\,\#\,\mathbf{v}]$ we define a topological invariant. Fix a fiber $[\mathbf{u}']\,\mathrm{rel}\,\mathbf{v}'$, with $\mathbf{v}'\in\pi([\mathbf{u}\,\#\,\mathbf{v}])$ and let $N=\mathrm{cl}([\mathbf{u}']\,\mathrm{rel}\,\mathbf{v}')$. By assumption N is compact and $\partial N\cap(\Sigma_-\,\mathrm{rel}\,\mathbf{v}')=\emptyset$. The "exit set" $N^-\subset\partial N$ is defined as follows: for each $\mathbf{u}'\in\partial N$ there exists a *small enough* neighborhood W in $\overline{\mathcal{D}}_d^n$ such that $W-\Sigma\,\mathrm{rel}\,\mathbf{v}'$ consists of finitely many components W_j. Set $W_0=W\cap N$, then

$$N^-\stackrel{\mathrm{def}}{=}\mathrm{cl}\big\{\mathbf{u}'\in\partial N\;\big|\;|W_0|_{\mathrm{word}}\geqslant|W_j|_{\mathrm{word}},\ \forall j>0\big\},$$

where $|W_j|_{\mathrm{word}}$ is a word metric defined by the number of pairwise crossings in the diagram $\beta(\mathbf{u})$ representing the class W_j, see [5] for more details.

For a fiber $[\mathbf{u}']\,\mathrm{rel}\,\mathbf{v}'$ there are finitely many components (N_i,N_i^-). Now define the index $\mathbf{h}(\mathbf{u}'\,\mathrm{rel}\,\mathbf{v}')=\bigvee_i[N_i,N_i^-]$, where $[N_i,N_i^-]$ denotes the homotopy type of the pointed space

$(N_i/N_i^-, [N_i^-])$. We note that $\mathbf{h}(\mathbf{u}' \operatorname{rel} \mathbf{v})$ is also equal to the homotopy type of the pair $(\bigcup N_i, [\bigcup N_i^-])$ and, as was proved in [5], is an invariant.

3.4. Proposition. *The homotopy type* $\mathbf{h}(\mathbf{u}' \operatorname{rel} \mathbf{v}') = \bigvee_i [N_i, N_i^-]$ *is independent of the fiber* $[\mathbf{u}'] \operatorname{rel} \mathbf{v}'$ *in* $[\mathbf{u} \# \mathbf{v}]$.

Due to Proposition 3.4 we can define

$$\mathbf{H}(\mathbf{u} \# \mathbf{v}; d) = \bigvee_i \left[N_i, N_i^-\right]. \tag{3.3}$$

The homological analogue is defined as $\mathbf{CH}([\mathbf{u} \# \mathbf{v}], d) = \bigoplus_k H_k(N, N^-; \mathbb{Z})$. Again it was proved in [5] that this is indeed an invariant. Proposition 3.4 was proved in [5] by associating discrete relative braids to parabolic recurrence relations.

3.2. Parabolic recurrence relations

Let $\mathcal{R}$ be a parabolic recurrence relation (see Definition 2.1). For any braid $\mathbf{v} \in \mathcal{D}_d^m$ one can choose a parabolic recurrence relation such that all strands $\mathbf{v}^k$ in $\mathbf{v}$ satisfy $\mathcal{R}_i(v_{i-1}^k, v_i^k, v_{i+1}^k) = 0$, or $\mathcal{R}(\mathbf{v}) = 0$ for short. Denote by Ψ^t the local flow generated by the vector field $\mathcal{R}$. As such Ψ^t becomes a flow in $\overline{\mathcal{D}}_d^n$. Given a proper bounded relative braid class $[\mathbf{u} \# \mathbf{v}]$, fix a fiber $[\mathbf{u}'] \operatorname{rel} \mathbf{v}'$. Choose a parabolic recurrence relation such that $\mathcal{R}(\mathbf{v}') = 0$. Then, by the structure of parabolic flows, the set $N = \operatorname{cl}([\mathbf{u}'] \operatorname{rel} \mathbf{v}')$ is an isolating neighborhood of Ψ^t in the sense of Conley, see [4] and Proposition 4.2 below. It holds that the Conley index is given by

$$h\big(N; \Psi^t\big) = \mathbf{h}\big(\mathbf{u}' \operatorname{rel} \mathbf{v}'\big) = H(\mathbf{u} \# \mathbf{v}; d) = \bigvee_i \left[N_i, N_i^-\right].$$

Continuation properties of the Conley index yield Proposition 3.4.

A fundamental result is that the invariant $\mathbf{H}$ is independent of the period d in the following sense. Define the extension operator $\mathbf{E} : \overline{\mathcal{D}}_d^n \to \overline{\mathcal{D}}_{d+1}^n$ as follows:

$$\big(\mathbf{E}(\mathbf{u})\big)_i^k \stackrel{\text{def}}{=} \begin{cases} u_i^k & \text{for } i = 0, \ldots, d, \\ u_d^k & \text{for } i = d+1. \end{cases}$$

Given a bounded proper relative braid class $[\mathbf{u} \# \mathbf{v}]$ in $\mathcal{D}_d^{n+m}$, then $[\mathbf{E}(\mathbf{u}) \# \mathbf{E}(\mathbf{v})]$ is a bounded proper relative braid class in $\mathcal{D}_{d+1}^{n+m}$. The main result in [5] is:

3.5. Proposition. *It holds that* $\mathbf{H}(\mathbf{u} \# \mathbf{v}; d) = \mathbf{H}(\mathbf{E}(\mathbf{u}) \# \mathbf{E}(\mathbf{v}); d+1)$.

One conclusion from Proposition 3.5 is that given an equivalence class of continuous positive relative braid diagrams of $[\beta(\mathbf{u}) \# \beta(\mathbf{v})]_{C^0}$, determined by the representative $\beta(\mathbf{u}) \operatorname{rel} \beta(\mathbf{v})$, the index $\mathbf{H}$ is independent of the chosen discretization d, see [5]. Therefore we may define the topological invariant

$$\mathbf{H}\big(\beta(\mathbf{u}) \# \beta(\mathbf{v})\big) \stackrel{\text{def}}{=} \mathbf{H}(\mathbf{u} \# \mathbf{v}; d), \tag{3.4}$$

for any discretization d as described above. The index $\mathbf{H}(\beta(\mathbf{u}) \# \beta(\mathbf{v}))$ is an invariant for topological bounded proper relative braid classes $[\beta(\mathbf{u}) \# \beta(\mathbf{v})]_{C^0}$.

3.6. Remark. For more details we refer to [5] where definitions of properness, boundedness, etc. for topological classes are given.

The braid invariant $\mathbf{H}$ has Morse theoretical implications for parabolic recurrence relations, see [5] again.

3.7. Lemma. *Let Ψ^t be a parabolic flow on $\mathcal{D}_d^n$ which fixes a skeleton $\mathbf{v} \in \mathcal{D}_d^m$ and let $[\mathbf{u} \,\#\, \mathbf{v}]$ be a bounded and proper relative braid class. If $\mathbf{H}(\beta(\mathbf{u}) \,\#\, \beta(\mathbf{v})) \neq 0$ (homotopically non-trivial), then the relative braid class $[\mathbf{u}]$ rel $\mathbf{v}$ has at least one fixed point for the parabolic flow, and thus contains a zero for the associated parabolic recurrence relation.*

4. Parabolic recurrence relations for conservative systems

4.1. Braid classes of up–down type

By Proposition 2.3 closed characteristics correspond to sequences of local minima and maxima satisfying a parabolic recurrence relation of up–down type. The extrema alternate in the sense that $(-1)^i(u_{i\pm1} - u_i) > 0$ — the (natural) up–down restriction — and therefore an n-collection of extrema sequences $\{\mathbf{u}^k\}_{k=0}^{n-1}$ can be seen as a point in the space of up–down piecewise linear braid diagrams.

4.1. Definition. The space $\mathcal{E}_{2p}^n$ of up–down PL-braid diagrams on n strands with period $2p$ is the subset of $\mathcal{D}_{2p}^n$ determined by the relation $(-1)^i(u_{i+1}^k - u_i^k) > 0$ for $k = 1, \dots, n$ and $i = 0, \dots, 2p-1$. Let $\overline{\mathcal{E}}_{2p}^n$ be the subset of all braid diagrams in $\overline{\mathcal{D}}_{2p}^n$ satisfying $(-1)^i(u_{i+1}^k - u_i^k) > 0$ and as before the singular braid diagrams are defined as $\Sigma^{\mathcal{E}} = \overline{\mathcal{E}}_{2p}^n \setminus \mathcal{E}_{2p}^n$ while collapsed singular diagrams are denoted by $\Sigma_-^{\mathcal{E}}$.

The set $\overline{\mathcal{E}}_{2p}^n$ has a boundary in $\overline{\mathcal{D}}_{2p}^n$ which can be characterized as follows:

$$\partial \overline{\mathcal{E}}_{2p}^n = \left\{ \mathbf{u} \in \overline{\mathcal{E}}_{2p}^n \colon u_i^k = u_{i+1}^k \text{ for at least one } i \text{ and } k \right\}. \tag{4.1}$$

Such braids, called horizontal singularities, are not included in the definition of $\overline{\mathcal{E}}_{2p}^n$, because the recurrence relation (2.7) does not induce a well-defined flow on the boundary $\partial \overline{\mathcal{E}}_{2p}^n$. Up–down parabolic recurrence relations hence define a well-defined parabolic (semi-)flow Ψ^t on $\overline{\mathcal{E}}_{2p}^n$. This has the important property that $\overline{\mathcal{E}}_{2p}^n$ is forward invariant with respect to Ψ^t, i.e. $\Psi^t(\overline{\mathcal{E}}_{2p}^n) \subset \overline{\mathcal{E}}_{2p}^n$ for all $t \geqslant 0$. The main properties of the flow can be summarized as follows, see [5].

4.2. Proposition. *Let Ψ^t be a parabolic flow of up–down type on $\overline{\mathcal{E}}_{2p}^n$.*

(a) *For each point $\mathbf{u} \in \Sigma^{\mathcal{E}} - \Sigma_-^{\mathcal{E}}$, the local orbit $\{\Psi^t(\mathbf{u}) \colon t \in [-\epsilon, \epsilon]\}$ intersects $\Sigma^{\mathcal{E}} - \Sigma_-^{\mathcal{E}}$ uniquely at $\mathbf{u}$ for all ϵ sufficiently small.*
(b) *For any such $\mathbf{u}$, the word metric of the braid diagram $\Psi^t(\mathbf{u})$ for $t > 0$ is strictly less than that of the diagram $\Psi^t(\mathbf{u})$, $t < 0$.*
(c) *The flow blows up in a neighborhood of $\partial \overline{\mathcal{E}}_{2p}^n$ in such a manner that the vector field points into $\overline{\mathcal{E}}_{2p}^n$.*
(d) *The flow is forward invariant: $\Psi^t(\overline{\mathcal{E}}_{2p}^n) \subset \overline{\mathcal{E}}_{2p}^n$ for all $t \geqslant 0$.*

The boundary $\partial \overline{\mathcal{E}}_{2p}^n$ can be regarded as a repelling set.

4.3. Remark. An important interpretation of Proposition 4.2 is that parabolic flows never increase the intersection number of a (piecewise linearly interpolated) braid diagram. Furthermore, the intersection number strictly decreases at any point on the boundary of a (relative) braid class, and all singular braids (except possibly collapsed ones) are non-stationary.

If v is a closed characteristic of a second order Lagrangian system, then its sequence of extrema $\mathbf{v} = \{v_i\}$ is a zero of the associated parabolic recurrence relation (up–down) $\mathcal{R}$ and thus a fixed point for parabolic flow Ψ^t generated by $\mathcal{R}$. In the case of braids with the up–down restriction we can again define braid classes and relative braid classes, see [5]. Define the space of relative braids of up–down type

$$\mathcal{E}_{2p}^{n,m} = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{E}_{2p}^{n} \times \mathcal{E}_{2p}^{m} \mid \mathbf{u} \cup \mathbf{v} \in \mathcal{E}_{2p}^{n+m}\}.$$

Elements in this space are again denoted by $\mathbf{u} \# \mathbf{v}$ and the connected components, or relative braid classes, by $[\mathbf{u} \# \mathbf{v}]_{\mathcal{E}}$. The space of relative braids with a fixed skeleton $\mathbf{v} \in \mathcal{E}_{2p}^{m}$ is denoted by $\mathcal{E}_{2p}^{n} \operatorname{rel} \mathbf{v}$. The fibers in $[\mathbf{u} \# \mathbf{v}]_{\mathcal{E}}$ for a fixed skeleton $\mathbf{v}' \in \pi([\mathbf{u} \# \mathbf{v}]_{\mathcal{E}})$ are denoted by $[\mathbf{u}']_{\mathcal{E}} \operatorname{rel} \mathbf{v}' \subset \mathcal{E}_{2p}^{n} \operatorname{rel} \mathbf{v}'$. The notions of boundedness and properness are defined in the same way as before, see also [5].

Parabolic recurrence relations of up–down type and the associated braid classes satisfy an important universality principle. Let Ψ^t fix a skeleton $\mathbf{v} \in \mathcal{E}_{d}^{m}$ and let $[\mathbf{u} \# \mathbf{v}]_{\mathcal{E}}$ be a bounded and proper relative braid class. Then $N_{\mathcal{E}} \stackrel{\text{def}}{=} \operatorname{cl}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$ is an isolating neighborhood in the sense of Conley and therefore its Conley index $h(N_{\mathcal{E}}; \Psi^t)$ is well defined. We now relate any up–down braid class to an associated unrestricted braid class. Define the *extended skeleton* $\mathbf{v}^* = \mathbf{v} \cup \mathbf{v}^+ \cup \mathbf{v}^-$, where

$$v_i^+ = \max_{k,i} v_i^k + 1 + (-1)^{i+1}, \qquad v_i^- = \min_{k,i} v_i^k - 1 + (-1)^{i+1}. \tag{4.2}$$

The following crucial property was proved in [5].

4.4. Proposition. *It holds that* $h(N_{\mathcal{E}}; \Psi^t) = \mathbf{H}(\mathbf{u} \# \mathbf{v}^*, d) = \mathbf{H}(\beta(\mathbf{u}) \# \beta(\mathbf{v}^*))$.

4.5. Remark. If $\mathbf{H}(\beta(\mathbf{u}) \# \beta(\mathbf{v}^*)) \neq 0$ (i.e. homotopically non-trivial), then by [5, Lemma 35] the relative braid class $[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$ contains at least one fixed point for the parabolic flow, and thus a zero for the associated parabolic recurrence relation of up–down type.

In [5] it was also proved that Proposition 4.4 can be used in the setting of braid invariants for up–down type relative braid classes. In the up–down case we can also define $\mathbf{H}(\mathbf{u} \# \mathbf{v}, \mathcal{E}; 2p)$, i.e., the invariant is independent of the fiber chosen, and $\mathbf{H}(\mathbf{u} \# \mathbf{v}, \mathcal{E}; 2p) = \mathbf{H}(\mathbf{u} \# \mathbf{v}^*; 2p)$. This principle gives us a powerful tool to compute the Conley index of isolating neighborhood given by bounded proper relative braid classes of up–down type via universal braid class invariants.

4.2. Fourth order equations

Let us go back now to the classification of solutions of Eq. (1.1), relate the three types of solutions in Fig. 1 to braid classes, and put them in the context of the definitions presented in this section. The three types of solutions are distinguished according to their intersections with the constant solutions $u_\pm = \pm 1$. The most straightforward way of relating a solution to a relative braid class is to take the two constant strands $\pm\mathbf{1}$ as a skeleton and define the relative braid class by the free strand $\mathbf{u}$ which intersects the constant strands $\pm\mathbf{1}$ in the same manner as the solution u intersects $u_\pm$. However, the flow Ψ^t is well defined only for the braids with up–down restriction. Hence instead of taking the constant strands we have to use the skeleton $\mathbf{v} = \mathbf{u}_+ \cup \mathbf{u}_-$, where the strands $\mathbf{u}_\pm$ correspond to solutions of Eq. (1.1) which oscillate around $u_\pm$ with a small amplitude (on a slightly positive energy level) and the free strand $\mathbf{u}$ intersects the skeleton strands in the same manner as u intersects $u_\pm$. Fig. 7 shows the three different braid classes which correspond to the three different types of solutions. The first two braid classes are proper and the third one is not. All these braid classes are obviously unbounded. It was shown in [5] how to use properties of Eq. (1.1) to find extra skeletal strands which make the class bounded. We will give more details in Section 7.

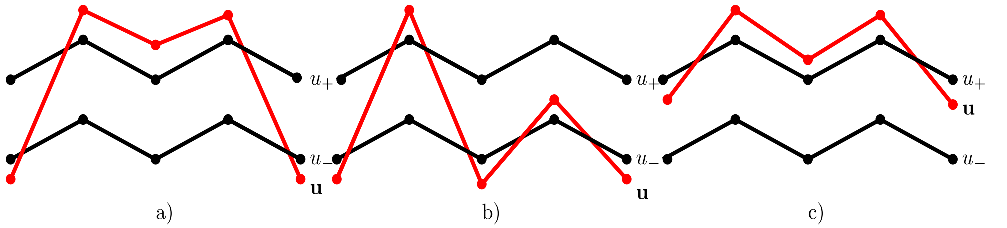

Fig. 7. Representatives of the three different relative braid classes. A fixed point in the relative braid class defined by the depicted representative, with the free strand shown in gray (red in the web version), corresponds to the solution of a) type I, b) type II and c) type III. Braid classes a) and b) are proper, but c) is not.

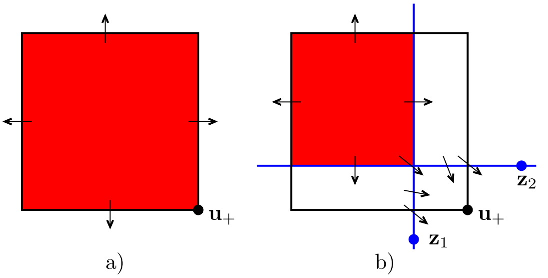

Fig. 8. Figure a) schematically shows the behavior of the vector field $\mathcal{R}$ on the boundary of the improper braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ corresponding to the third type of solution. The strand $\mathbf{u}_+$ is a fixed point for the flow Ψ^t and some trajectories approach this point as $t \to \infty$. Figure b) shows the behavior of the perturbed vector field on the boundary of the proper braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\bar{\mathbf{v}}$, where $\bar{\mathbf{v}} = \mathbf{v} \cup \mathbf{z}_1 \cup \mathbf{z}_2$ and $\mathbf{z}_1, \mathbf{z}_2$ are fixed strands for this perturbed vector field.

According to [5] the braid invariant $\mathbf{H}$ for any braid class corresponding to a solution of the first type is non-trivial. Conley index theory then guarantees the existence of a fixed point for Ψ^t in this class. A fixed point in this braid class corresponds to the solution of Eq. (1.1) of the first type. Thus there are many different solutions of the first type and their bifurcation branches exist for all $\alpha \geqslant 0$, e.g. see Fig. 2.

For the second braid class in Fig. 7 the braid invariant $\mathbf{H}$ is trivial and thus does not provide information about fixed points. However, if we know that there exists a non-degenerate (hyperbolic) solution of the second type then it corresponds to a fixed point in a braid class with a trivial Conley index. Hence there must be another fixed point in this class which corresponds to a different solution of the same type. This explains that the bifurcation curves form loops in Figs. 1 and 3. We should point out that the existence of a local minimum of the second type was shown in [7,6] for $\alpha \in (-\sqrt{8}, 0)$, which would be enough to find a second fixed point, except that the twist property is not known to hold for $\alpha < 0$.

In the third case, Fig. 7 c), the braid class is not proper (not an isolating neighborhood), since the free strand can collapse on a skeletal strand $\mathbf{u}_+$. Using the information about the flow Ψ^t near the strand $\mathbf{u}_+$, we will perturb the parabolic recurrence relation on a neighborhood of the boundary of the improper braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ and construct some new fixed strands which will make the class proper without changing the invariant set inside the class. In Fig. 8 we schematically depict the behavior of the vector field $\mathcal{R}$ on the boundary of the improper braid class, as well as on the boundary of a new proper braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\bar{\mathbf{v}}$ created by adding extra strands which are fixed points of this perturbed vector field. We will show via the invariant $\mathbf{H}(\mathbf{u} \,\#\, \bar{\mathbf{v}}^*)$ that the Conley index $h(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}); \Psi^t)$ is non-trivial.

4.6. Remark. In this paper we restrict ourselves to improper braid classes with one free strand, i.e., $[\mathbf{u} \,\#\, \mathbf{v}] \subset \mathcal{D}^{1,m}_{2p}$. In this case the free strand $\mathbf{u}$ of an improper braid class can collapse only onto single skeletal strands. Hence the set Σ_- consists of isolated points which are fixed points for the flow Ψ^t.

5. Linearization of the discrete action

In this section we study the linearization of W_{2p} at the fixed points and introduce the notion of a rotation number. The following discussion is based on [1], and proofs of the results listed in this section can be found in [1] as well.

Let $\mathbf{u} \in \mathcal{E}^1_{2p}$ be a critical point of $W_{2p} = \sum_{i=0}^{2p-1} S_E(u_i, u_{i+1})$ and define $P_i = (u_i, w_i)$, where $w_i = \partial_1 S_E(u_i, u_{i+1})$. It was shown in [1] that we can define the differentiable functions F_i on some neighborhood of P_i by the relation

$$\bigl(u', w'\bigr) = F_i(u, w) \quad \Leftrightarrow \quad w = \partial_1 S_E\bigl(u, u'\bigr) \quad \text{and} \quad w' = -\partial_2 S_E\bigl(u, u'\bigr).$$

It holds that $P_{i+1} = F_i(P_i)$ because $\mathbf{u} \in \mathcal{E}^1_{2p}$ is a critical point of W_{2p}.

We define the rotation number as follows. Take a vector $Q_0 \in T_{P_0}\mathbb{R}^2$ such that $Q_0 \neq 0$, and define $Q_i \in T_{P_i}\mathbb{R}^2$ by

$$Q_i = dF_i(P_{i-1})Q_{i-1}, \quad \text{for all } i.$$

We use the natural identification of the tangent spaces $T_{P_i}\mathbb{R}^2$ with $\mathbb{R}^2$. Let the vector Q_i have components (ξ_i, η_i), and let θ_i be the angle between Q_{i-1} and Q_i, oriented in the clockwise sense. This angle is only defined up to a multiple of 2π, which we specify using the following rule:

$$\text{if } \xi_{i-1}\xi_i \geqslant 0, \quad \text{then } -\pi < \theta_i \leqslant \pi, \tag{5.1a}$$

$$\text{if } \xi_{i-1}\xi_i < 0, \quad \text{then } 0 < \theta_i < 2\pi. \tag{5.1b}$$

Then we define the twist number $\tau(\mathbf{u})$ of the orbit $\mathbf{u}$ to be

$$\tau(\mathbf{u}) = \lim_{n\to\infty} \frac{1}{2n} \sum_{i=-2pn+1}^{2pn} \theta_i/2\pi. \tag{5.2}$$

Roughly speaking, $2\pi\tau(\mathbf{u})$ is the average angle about which $dF(P_0)$ rotates the vector u_0, where $F = F_{2p-1} \circ \cdots \circ F_0$. Or, more importantly in our setting, $2\tau(\mathbf{u})$ is the average number of times the sequence ξ_i changes sign in an interval of the length $2p$. This holds due to the choice made in (5.1a) and (5.1b). In particular, $0 \leqslant \tau(\mathbf{u}) \leqslant p$.

If we differentiate ∇W_{2p} at the point $\mathbf{u}$ we get the following expression for the i-th component of the linearization L:

$$(L\boldsymbol{\xi})_i = \alpha_i\xi_{i-1} + \beta_i\xi_i + \alpha_{i+1}\xi_{i+1}, \tag{5.3}$$

where $\boldsymbol{\xi} = (\xi_0, \ldots, \xi_{2p-1})$ and

$$\alpha_i = \partial_1\partial_2 S_E(u_{i-1}, u_i) > 0, \tag{5.4a}$$

$$\beta_i = \partial_2^2 S_E(u_{i-1}, u_i) + \partial_1^2 S_E(u_i, u_{i+1}). \tag{5.4b}$$

The fact that $\alpha_i > 0$ follows from the monotonicity property $\partial_1\partial_2 S_E > 0$ of the generating function, see Section 2. Thus L is a symmetric (periodic tridiagonal) Jacobi matrix, and the following is known (see [17]).

5.1. Proposition. *The spectrum of L is given by*

$$\mathrm{spec}(L) = \{\lambda_0 > \lambda_1 \geqslant \lambda_2 > \lambda_3 \geqslant \cdots \geqslant \lambda_{2p-1}\}.$$

In particular, for all i we have $\lambda_{2i} > \lambda_{2i+1}$.

Let us summarize the results obtained for the linearization L in [1]. We use the symbols $\lfloor a \rfloor$ and $\lceil a \rceil$ to denote the lower integer part and upper integer part of a, respectively.

5.2. Lemma. *Let $\boldsymbol{\xi}^j$ be an eigenvector of L corresponding to the eigenvalue λ_j, and let $0 \leqslant k \leqslant l \leqslant 2p-1$. Then any nonzero linear combination of $\boldsymbol{\xi}^k, \boldsymbol{\xi}^{k+1}, \ldots, \boldsymbol{\xi}^l$ has at least $2\lfloor (k+1)/2 \rfloor$ and at most $2\lfloor (l+1)/2 \rfloor$ sign changes.*

5.3. Lemma. *If $\tau \notin \mathbb{N}$, then the linearization L has $2\lceil \tau(\mathbf{u}) \rceil - 1$ positive eigenvalues. If $\tau \in \mathbb{N}$, then the linearization L has either $2\tau - 1$ or 2τ positive eigenvalues.*

When $L\xi = 0$ then all ξ_i can be computed from (ξ_0, ξ_1) via

$$\begin{pmatrix} \xi_{2p} \\ \xi_{2p+1} \end{pmatrix} = M(\mathbf{u}) \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}, \tag{5.5}$$

with

$$M(\mathbf{u}) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_{2p-1}}{\alpha_{2p}} & \frac{-\beta_{2p-1}}{\alpha_{2p}} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_0}{\alpha_1} & \frac{-\beta_0}{\alpha_1} \end{pmatrix}, \tag{5.6}$$

where α_i, β_i are given by (5.4a), (5.4b). The matrix $M(\mathbf{u})$ is conjugate to the matrix $dF(P_0) = dF_{2p-1}(P_{2p-1}) \circ \cdots \circ dF_0(P_0)$, see [1, Lemma 3.1], hence the rotation numbers of $M(\mathbf{u})$ and $dF(P_0)$ are the same.

5.4. Remark. Lemmas 5.2 and 5.3 are valid for any symmetric (periodic tridiagonal) Jacobi matrix L with positive off-diagonal entries. Exactness of the parabolic recurrence relation guarantees that the Jacobi matrix is symmetric, but the results hold for any symmetric linearization of a (possibly non-exact) parabolic recurrence relation $\mathcal{R}$. In that case the rotation number is defined as the rotation number of the matrix M given by (5.6).

5.5. Remark. For the simplest case, $\alpha_i = \overline{\alpha} > 0$ and $\beta_i = \overline{\beta}$, the rotation number can be determined explicitly. For $|\overline{\beta}| \leqslant 2\overline{\alpha}$, we can write $\beta = -2\overline{\alpha}\cos(\pi\omega)$ for a unique $\omega \in [0,1]$, and we find $\tau = p\omega$. For $\overline{\beta} < -2\overline{\alpha}$ we have $\tau = 0$, while for $\overline{\beta} > 2\alpha$ it holds that $\tau = p$.

6. The invariant set of an improper braid class

The closure of a proper braid class is an isolating neighborhood and Conley index theory may provide information about qualitative properties of the maximal invariant set within the braid class. Therefore, if we show that a certain invariant set inside an *improper* (non-isolating) braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ is identical to the maximal invariant set in the closure of some *proper* braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\bar{\mathbf{v}}$, then we can use the Conley index, and thus the global braid invariant $\mathbf{H}$, to study qualitative properties of this invariant set in the improper class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$. We will apply these ideas to parabolic recurrence relations of up–down type which are *exact*:

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = \partial_2 S_{i-1}(u_{i-1}, u_i) + \partial_1 S_i(u_i, u_{i+1}).$$

Hence, the results apply to fourth order equations, see Section 2. These ideas extend to general braid classes.

6.1. Main results

As outlined in Section 4.2 the basic idea behind creating a corresponding proper braid class from an improper braid class is to add skeletal strands which will prevent the free strand from collapsing onto the skeleton, cf. Fig. 8. In order to compare the invariant sets of parabolic flows on both braid classes we perturb the parabolic recurrence $\mathcal{R}$ in such a way that we can find the new skeletal strands, and the invariant sets are the same. We start with a bounded *improper* braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v} \subset \mathcal{E}^{1,m}_{2p}$ with skeleton $\mathbf{v} \in \mathcal{E}^m_{2p}$. In order to simplify the exposition and since for our purposes it suffices, we assume throughout that $\mathbf{u}$ consists of one strand and $\mathbf{u}$ can collapse only on the skeletal strand $\mathbf{v}^1$. This implies the boundary conditions $u_0 = u_{2p}$ and $v^1_0 = v^1_{2p}$. We denote by $I(\mathbf{u}, \mathbf{v}^1)$ the number of intersections of the piecewise linear interpolations $\beta(\mathbf{u})$ and $\beta(\mathbf{v}^1)$ on a single period, i.e., $I(\mathbf{u}, \mathbf{v}^1)$ is the braid word length of $\mathbf{u} \cup \mathbf{v}^1$. Hence $0 \leqslant I(\mathbf{u}, \mathbf{v}^1) \leqslant 2p$ and $I(\mathbf{u}, \mathbf{v}^1)$ is even. More generally, for braids $\{\mathbf{z}^k\}_{k=1}^n$ and $\{\mathbf{w}^k\}_{k=1}^m$ such that $\mathbf{z} \cup \mathbf{w}$ is also a braid, we denote by $I(\mathbf{z}, \mathbf{w})$ the number of intersections between (piecewise linear) strands in $\mathbf{z}$ and strands in $\mathbf{w}$, i.e.,

$$I(\mathbf{z}, \mathbf{w}) = |\mathbf{z} \cup \mathbf{w}|_{\text{word}} - |\mathbf{z}|_{\text{word}} - |\mathbf{w}|_{\text{word}}.$$

For improper braid classes we use the following notion of maximal invariant set:

$$\overline{\mathrm{INV}}_{\Psi^t}\big([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}\big) \stackrel{\text{def}}{=} \big\{\mathbf{u}' \operatorname{rel} \mathbf{v} \in [\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v} \;\big|\; \mathrm{cl}\big(\Psi^t(\mathbf{u}')\big) \subset [\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}\big\}. \tag{6.1}$$

In particular, orbits of Ψ^t that limit to $\mathbf{v}^1$ are *not* in $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$. We want to understand the structure of this invariant set via its Conley index. Since the braid class is not an isolating neighborhood, we need to adopt a careful approach.

Throughout, let $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v} \subset \mathcal{E}^1_{2p}$ rel $\mathbf{v}$ be a bounded improper braid class and let $\mathcal{R}$ be a parabolic recurrence relation of up–down type fixing the skeleton $\mathbf{v}$, and let $\mathbf{u}$ be a single strand that can collapse on the skeletal strand $\mathbf{v}^1$ only. The rotation number $\tau(\mathbf{v}^1)$ of $\mathbf{v}^1$, as introduced in Section 5, plays a crucial role. The rotation number of any fixed point of an *exact* parabolic recurrence relation is well defined, as explained in Section 5. Remark 5.4 implies that the definition of the rotation number, as well as the properties described in Lemmas 5.2 and 5.3, extend to fixed points of non-exact (up–down) parabolic recurrence relations provided the Jacobi matrix of the fixed point is symmetric.

6.1. Lemma. *Let $\mathcal{R}$ be a parabolic recurrence relation of up–down type fixing the skeleton $\mathbf{v}$, such that $\mathbf{v}^1$ has symmetric Jacobi matrices. If $2\tau(\mathbf{v}^1) \neq I(\mathbf{u}, \mathbf{v}^1)$, then the invariant set $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$ is an isolated invariant set.*

If $N \subset [\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$ is any isolating neighborhood of $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$, then $h(N; \Psi^t)$ denotes its Conley index. The next theorem states that the Conley index of the invariant set depends on the sign of $2\tau(\mathbf{v}^1) - I(\mathbf{u}, \mathbf{v}^1)$ only.

6.2. Theorem. *Let Ψ^t_1 and Ψ^t_2 be two parabolic flows associated to exact parabolic recurrence relations $\mathcal{R}^1$ and $\mathcal{R}^2$ that both fix the skeleton $\mathbf{v}$. Let $\mathbf{v}^1$ have symmetric Jacobi matrices for both recurrence relations, and let τ_1 and τ_2 be the twist numbers of $\mathbf{v}^1$ with respect to Ψ^t_1 and Ψ^t_2, respectively. Assume that $2\tau_i \neq I(\mathbf{u}, \mathbf{v}^1)$, $i = 1, 2$. Let N_i, $i = 1, 2$, be isolating neighborhoods for $\overline{\mathrm{INV}}_{\Psi^t_i}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$:*

(i) *If $2\tau_1 < I(\mathbf{u}, \mathbf{v}^1)$ and $2\tau_2 < I(\mathbf{u}, \mathbf{v}^1)$, then $h(N_1; \Psi^t_1) \cong h(N_2; \Psi^t_2)$.*
(ii) *If $2\tau_1 > I(\mathbf{u}, \mathbf{v}^1)$ and $2\tau_2 > I(\mathbf{u}, \mathbf{v}^1)$, then $h(N_1; \Psi^t_1) \cong h(N_2; \Psi^t_2)$.*

Lemma 6.1 and Theorem 6.2 will be proved in Section 6.2.

To convert the improper braid class into a proper (i.e. isolating) one, we will augment the skeleton with suitably chosen strands. First, define the distance between braids $\mathbf{u}', \mathbf{u}''$ (with n and m strands respectively) by

$$\sigma\big(\mathbf{u}', \mathbf{u}''\big) := \min\big\{\big|u_i'^k - u_i''^l\big| > 0\colon\ 0 \leqslant i \leqslant 2p-1,\ 1 \leqslant k \leqslant n,\ 1 \leqslant l \leqslant m\big\}. \tag{6.2}$$

Note that the minimum is taken over all anchor points of $\mathbf{u}'$ and $\mathbf{u}''$ that do *not* coincide.

6.3. Definition. Given a relative braid $\mathbf{u} \# \mathbf{v}$ in $\mathcal{D}_{2p}^{1,m}$, let (p', q') be coprime, with $0 < q' < pp'$, or $(p', q') = (1, 0)$. Then we define $\mathbf{z} = \{\mathbf{z}^k\}_{k=1}^{p'} = \mathbf{z}(p', q', \delta)$ through

$$z_i^k = z_i^k\big(p', q', \delta\big) \stackrel{\text{def}}{=} v_i^1 + \delta \cos 2\pi \frac{q'}{p'}\left(k - 1 + \frac{i}{2p}\right), \tag{6.3}$$

for $k = 1, \ldots, p'$, $i = 0, \ldots, 2p$, and

$$0 < |\delta| < \min\big\{\sigma\big(\mathbf{u}, \mathbf{v}^1\big), \sigma\big(\mathbf{v}, \mathbf{v}^1\big)\big\}. \tag{6.4}$$

The associated *augmentation* of $\mathbf{v}$ is defined by $\bar{\mathbf{v}} \stackrel{\text{def}}{=} \mathbf{v} \cup \mathbf{z}$.

Due to the restriction $q' < pp'$ and (6.4), all the $[\mathbf{u} \# \bar{\mathbf{v}}]$ are relative braid classes. For $|\delta|$ sufficiently small, $\mathbf{z}$ is an up–down braid. Furthermore, if

$$I\big(\mathbf{u}, \mathbf{v}^1\big) \neq 2\frac{q'}{p'}, \tag{6.5}$$

then the relative braid class $[\mathbf{u} \# \mathbf{v} \cup \mathbf{z}]_{\mathcal{E}}$ is *proper*. In particular, since p' and q' are coprime and $I(\mathbf{u}, \mathbf{v}^1)$ is even, inequality (6.5) is always satisfied if $p' \geqslant 2$. Under condition (6.5) properness of $[\mathbf{u} \# \mathbf{v} \cup \mathbf{z}]_{\mathcal{E}}$ follows by noting that $I(\mathbf{u}, \mathbf{z}) = p' I(\mathbf{u}, \mathbf{v}^1)$, while $I(\mathbf{v}^1, \mathbf{z}) = 2q'$.

Under condition (6.5) the braid class $[\mathbf{u} \# \bar{\mathbf{v}}^*]$, where $\bar{\mathbf{v}}^*$ is the extended skeleton of the augmentation $\bar{\mathbf{v}}$, see Section 4, is a bounded *proper* relative braid class. We want to derive a relation between the invariant set $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$ defined in (6.1) and the braid invariant $\mathbf{H}(\mathbf{u} \# \bar{\mathbf{v}}^*)$ for augmentations $\bar{\mathbf{v}} = \mathbf{v} \cup \mathbf{z}(p', q', \delta)$. To understand the crucial interaction between the intersection numbers $I(\mathbf{z}, \mathbf{v}^1)$ and $I(\mathbf{u}, \mathbf{v}^1)$ and the flow Ψ^t near $\mathbf{v}^1$, we need to take into account also the rotation number $\tau = \tau(\mathbf{v}^1)$. The next theorem describes the relation between the invariant set defined in (6.1) and the dynamics of a (perturbed) parabolic flow on $[\mathbf{u} \# \bar{\mathbf{v}}]_{\mathcal{E}}$.

6.4. Theorem. *Assume that $2\tau(\mathbf{v}^1) \neq I(\mathbf{u}, \mathbf{v}^1)$. Let N be an isolating neighborhood for $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$. Let $\bar{\mathbf{v}} = \mathbf{v} \cup \mathbf{z}$, where $\mathbf{z} = \mathbf{z}(p', q', \delta)$ is defined in* (6.3) *with (p', q') coprime and δ as in* (6.4)*.*

(i) *If $2\tau(\mathbf{v}^1) > I(\mathbf{u}, \mathbf{v}^1)$ and $2\frac{q'}{p'} > I(\mathbf{u}, \mathbf{v}^1)$, then*

$$h\big(N; \Psi^t\big) \cong \mathbf{H}\big(\mathbf{u} \# \bar{\mathbf{v}}^*; 2p\big) \cong \mathbf{H}\big(\beta(\mathbf{u}) \# \beta\big(\bar{\mathbf{v}}^*\big)\big).$$

(ii) *If $2\tau(\mathbf{v}^1) < I(\mathbf{u}, \mathbf{v}^1)$ and $2\frac{q'}{p'} < I(\mathbf{u}, \mathbf{v}^1)$, then*

$$h\big(N; \Psi^t\big) \cong \mathbf{H}\big(\mathbf{u} \# \bar{\mathbf{v}}^*; 2p\big) \cong \mathbf{H}\big(\beta(\mathbf{u}) \# \beta\big(\bar{\mathbf{v}}^*\big)\big).$$

When $\mathbf{H} \neq 0$, then $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}) \neq \emptyset$.

In particular one may make the "simplest" choices for the augmentations.

6.5. Corollary. *Assume that* $2\tau(\mathbf{v}^1) \neq I(\mathbf{u}, \mathbf{v}^1)$. *Let* N *be an isolating neighborhood for* $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$. *Let* δ *satisfy* (6.4).

(i) *If* $2\tau(\mathbf{v}^1) > I(\mathbf{u}, \mathbf{v}^1)$, *then*

$$h\big(N; \Psi^t\big) \cong \mathbf{H}\big(\mathbf{u} \# \big(\mathbf{v} \cup \mathbf{z}(1, p, \delta)\big); 2p\big).$$

(ii) *If* $2\tau(\mathbf{v}^1) < I(\mathbf{u}, \mathbf{v}^1)$, *then*

$$h\big(N; \Psi^t\big) \cong \mathbf{H}\big(\mathbf{u} \# \big(\mathbf{v} \cup \mathbf{z}(1, 0, \delta)\big); 2p\big).$$

When $\mathbf{H} \neq 0$, *then* $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}) \neq \emptyset$.

The proof of Theorem 6.4 proceeds in several steps. In Section 6.3 we study perturbations of parabolic recurrence relations; these perturbations make the recurrence relations locally linear near $\mathbf{v}^1$. Section 6.4 deals with the construction and properties of the braid classes $[\mathbf{u}]_{\mathcal{E}} \# (\mathbf{v} \cup \mathbf{z}(p', q', \delta))$, and the connection is made between the invariant sets of the improper braid class $[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$ for the original flow and of the proper braid class $[\mathbf{u}]_{\mathcal{E}} \# (\mathbf{v} \cup \mathbf{z}(p', q', \delta))$ for the perturbed flow. Finally, in Section 6.5 all ingredients are put together to prove Theorem 6.4.

6.6. Remark. One can think of extensions of Theorem 6.4 in several directions. The arguments can be fairly easily extended to the case when the free strand can collapse on several skeletal strands. Furthermore, a result as in Theorem 6.4 can be proved for arbitrary (i.e. not up–down restricted) exact improper braid classes in $\mathcal{D}_d^{n,m}$ with some minor modifications. Finally, an extension to general non-exact parabolic flows requires a more substantial modification of the construction, since the arguments involving the twist number (Section 5) need to be adapted. We leave this for future research.

6.2. Invariance and continuation

In this subsection we prove Lemma 6.1 and Theorem 6.2, starting with the former. We follow the same line of arguing as in [1, Section 7].

Denoting $\widetilde{N} \stackrel{\text{def}}{=} [\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$, it suffices to show that for all $\mathbf{w}$ in the compact boundary $\partial\widetilde{N}$ there is a neighborhood $U_{\mathbf{w}}$ of $\mathbf{w}$ such that $U_{\mathbf{w}}$ and $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$ are disjoint.

Let $\mathbf{w} \in \partial\widetilde{N}$. If $\mathbf{w} \neq \mathbf{v}^1$, then Proposition 4.2, combined with continuity of the flow, implies that there is an open neighborhood $U_{\mathbf{w}}$ of $\mathbf{w}$ such that $U_{\mathbf{w}} \cap \overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}) = \emptyset$. We are left with the case $\mathbf{w} = \mathbf{v}^1$.

Identify $\mathcal{E}^1_{2p}$ with a subset of $\mathbb{R}^{2p}$ via

$$\mathbf{u} \leftrightarrow \mathbf{x} = \big(u_0 - v^1_0, \ldots, u_{2p-1} - v^1_{2p-1}\big) \in \mathbb{R}^{2p},$$

so that $\mathbf{v}^1$ becomes the origin. By following the ideas in the proof of (the second occurrence of) Lemma 7.2 in [1], which we repeat here because we shall need a generalization later in Section 6.4, we write the linear part $\mathcal{L}$ of $\mathcal{R}$ at $\mathbf{v}^1$ as $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$, where $\mathcal{L}_\pm$ are self-adjoint, $\mathcal{L}_+\mathcal{L}_- = \mathcal{L}_-\mathcal{L}_+ = 0$ and

$$(\mathbf{x}, \mathcal{L}_+\mathbf{x}) > 0, \qquad (\mathbf{x}, \mathcal{L}_-\mathbf{x}) \leqslant 0, \quad \text{for all } 0 \neq \mathbf{x} \in \mathbb{R}^{2p}.$$

Let $\{\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_{2p-1}\}$ and $\{\lambda_0 > \lambda_1 \geqslant \lambda_2, \ldots, \lambda_{2p-1}\}$ be the eigenvectors and eigenvalues of $\mathcal{L}$.

We now proceed with the case that $2\tau(\mathbf{v}^1) > I(\mathbf{u},\mathbf{v}^1)$. The case $2\tau(\mathbf{v}^1) < I(\mathbf{u},\mathbf{v}^1)$ is analogous, see below. In the former case, since $I(\mathbf{u},\mathbf{v}^1)$ is even, we must have $2\lceil\tau(\mathbf{v}^1)\rceil \geqslant I(\mathbf{u},\mathbf{v}^1)+2$. It then follows from Lemma 5.3 that $\mathcal{L}$ has at least $I(\mathbf{u},\mathbf{v}^1)+1$ positive eigenvalues. The null space of $\mathcal{L}_+$ is thus spanned by $\{\boldsymbol{\xi}_{m+1},\dots,\boldsymbol{\xi}_{2p-1}\}$, where $m \geqslant I(\mathbf{u},\mathbf{v}^1)$. Hence Lemma 5.2 implies that if $\mathbf{x}\neq 0$ and $\mathcal{L}_+\mathbf{x}=0$, then $\mathbf{x}$ has at least $I(\mathbf{u},\mathbf{v}^1)+2$ sign changes and therefore $\mathbf{x}$ does not lie in $\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$. We infer the existence of a constant $K>0$ such that

$$G(\mathbf{x}) \stackrel{\text{def}}{=} (\mathbf{x},\mathcal{L}_+\mathbf{x}) \geqslant K\|\mathbf{x}\|^2 > 0 \quad \text{for all } 0\neq\mathbf{x}\in\mathrm{cl}\big([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}\big), \tag{6.6}$$

and hence also $\|\mathcal{L}_+\mathbf{x}\| \geqslant K\|\mathbf{x}\|$ on $\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$. Close to $\mathbf{x}=0$ the flow Φ^t is given by $\mathbf{x}'(t)=\mathcal{L}\mathbf{x}(t)+o(\mathbf{x}(t))$, hence

$$\frac{d}{dt}G(\mathbf{x}) = 2(\mathcal{L}_+\mathbf{x},\mathcal{L}_+\mathbf{x}) + o\big(\|\mathbf{x}\|^2\big) \geqslant \big(2K^2+o(1)\big)\|\mathbf{x}\|^2 > 0,$$

for all $\mathbf{x}\in[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$. Next, let $\eta>0$ be so small that $\frac{dG(\mathbf{x}(t))}{dt}>0$ whenever $G(\mathbf{x}(t))\leqslant\eta$ and $\mathbf{x}(t)\in[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$. Define $U_{\mathbf{v}^1}=\{\mathbf{x}\colon\ G(\mathbf{x})<\eta\}$. If the orbit $\mathbf{x}(t)$, $t\in\mathbb{R}$, lies in $[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$ and intersects $U_{\mathbf{v}^1}$ for some t_0, then $\mathbf{x}(t)\in U_{\mathbf{v}^1}$ and $\frac{d}{dt}G(\mathbf{x}(t))>0$ for all $t\leqslant t_0$. It follows that $\mathbf{x}(t)\to 0$ as $t\to-\infty$. Hence the closure of the orbit does not lie in $[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$, which proves that $U_{\mathbf{v}^1}$ and $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$ are disjoint.

The case that $2\tau(\mathbf{v}^1) < I(\mathbf{u},\mathbf{v}^1)$ is similar, but one needs to exchange the roles of L_+ and L_-, as well as consider $t\to\infty$ rather than $t\to-\infty$. This finishes the proof of Lemma 6.1.

6.7. Remark. The above arguments show that the set $S=\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$ is an isolated invariant set. Recalling that $\widetilde{N}=[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$, the closed set

$$N = \widetilde{N} \setminus \bigcup_{\mathbf{w}\in\partial N} U_{\mathbf{w}} \tag{6.7}$$

is an isolating neighborhood of S. Moreover, assume that $\mathcal{R}(s)$, $s\in[0,1]$, is a continuous path in the space of C^1 up–down parabolic recurrence relations fixing $\mathbf{v}$, and such that $\mathbf{v}^1$ has symmetric Jacobi matrices along the path. Denote the associated flows by $\Psi^t(s)$, and $S(s)=\overline{\mathrm{INV}}_{\Psi^t(s)}([\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v})$. If $\tau(s;\mathbf{v}^1)\neq I(\mathbf{u},\mathbf{v}^1)$ for any $s\in[0,1]$, then using the arguments given above it is not hard to prove that for $\delta>0$ sufficiently small the set $N(0)$, defined by (6.7) using the flow $\Psi^t(0)$, is a uniform isolating neighborhood of $S(s)$ for all $s\in[0,\delta]$.

The crucial property of isolated invariant sets (and their Conley indices) is stability under continuation. In particular, we will employ continuation to specially constructed systems of the following form. Suppose for now that none of the skeletal strands have common anchor points. Let

$$S_i^{a,b}(u_i,u_{i+1}) = 2|u_{i+1}-u_i|^{1/2} + a_i u_i u_{i+1} + b_i {u_i}^2 + V_i(u_i),$$

with $a_i\geqslant 0$, $b_i\in\mathbb{R}$, and $V_i\in C^2(\mathbb{R},\mathbb{R})$. Let

$$\mathcal{R}_i^{a,b}(u_{i-1},u_i,u_{i+1}) = a_{i-1}u_{i-1} + 2b_i u_i + a_i u_{i+1} + V_i'(u_i) + W(u_{i-1},u_i) - W(u_i,u_{i+1}), \tag{6.8}$$

where

$$W(u_i,u_{i+1}) = \frac{u_{i+1}-u_i}{|u_{i+1}-u_i|^{3/2}},$$

which has the monotonicity properties $\partial_1 W > 0$ and $\partial_2 W < 0$, so that $\mathcal{R}_i^{a,b}$ is a parabolic recurrence relation of up–down type for any $a_i \geqslant 0$, $b_i \in \mathbb{R}$. We choose a function V_i satisfying the equalities

$$V_i'\big(v_i^k\big) = -a_{i-1} v_{i-1}^k - 2b_i v_i^k - a_i v_{i+1}^k - W\big(v_{i-1}^k, v_i^k\big) + W\big(v_i^k, v_{i+1}^k\big), \tag{6.9}$$

for all $i = 0, \ldots, 2p$, $k = 1, \ldots, m$. This construction fails if skeletal anchor points coincide, since at such points where $v_i^k = v_i^{k'}$ there are two (or more) conflicting equalities (6.9). To overcome this, one needs adapt the construction of $\mathcal{R}^{a,b}$ (or $S^{a,b}$) in the same way as in [5, Appendix A]; we leave the details to the reader.

We denote the parabolic flow associated to the parabolic recurrence relation $\mathcal{R}_i^{a,b}$ constructed in (6.8) by $\Psi_{a,b}^t$. For convenience, we additionally require that $V_i''(v_i^k) = 0$ for all i, k. Then the components of the Jacobi matrix in Section 5 can be written as

$$\alpha_i = a_{i-1} + \partial_1 W(v_{i-1}, v_i) > 0,$$
$$\beta_i = 2b_i + \partial_2 W(v_{i-1}, v_i) - \partial_1 W(v_i, v_{i+1}).$$

We see that we may use the parameters $a_i \geqslant 0$ and b_i to construct families of parabolic recurrence relations with varying Jacobi matrices.

In particular, for $\bar{a} \geqslant \max_i W(v_{i-1}^1, v_i^1)$ and $\bar{b} \in \mathbb{R}$ we may take

$$a_i = \bar{a} - \partial_1 W\big(v_i^1, v_{i+1}^1\big),$$
$$b_i = \frac{1}{2}\bar{b} - \partial_2 W\big(v_{i-1}^1, v_i^1\big) + \partial_1 W\big(v_i^1, v_{i+1}^1\big).$$

For the associated parabolic recurrence relation, denoted by $\mathcal{R}^{\bar{a},\bar{b}}$, the linearization around $\mathbf{v}^1$ is given by

$$L(\boldsymbol{\xi})_i = \bar{a}\xi_{i-1} + \bar{b}\xi_i + \bar{a}\xi_{i+1}.$$

We are now sufficiently prepared to finish the proof of Theorem 6.2 using the continuation arguments from [1, Section 8]. We sketch the main arguments for part (i) of the theorem, and refer to [1, Section 8] for more details. For $n = 1, 2$ the linear interpolation $\mathcal{R}(s) = (1-s)\mathcal{R}^n + s\mathcal{R}^{\bar{a},\bar{b}}$, $s \in [0,1]$, consists of exact up–down parabolic recurrence relations fixing $\mathbf{v}$. We denote by $\Psi^t(s)$ the parabolic flow associated to $\mathcal{R}(s)$, and $\tau(s)$ is the rotation number of $\mathbf{v}^1$ with respect to $\Psi^t(s)$. By choosing $\bar{b} \gg \bar{a}$ one can guarantee that the Jacobi matrix $J_{ji} = \partial_j(\mathcal{R}_i^{\bar{a},\bar{b}} - \mathcal{R}_i^n)$ is positive definite. Then Lemma 5.3 implies that if $\tau(0) > I(\mathbf{u}, \mathbf{v}^1)$ then $\tau(s) > I(\mathbf{u}, \mathbf{v}^1)$ for all $s \in [0,1]$. Hence by Lemma 6.1 the sets $\overline{\mathrm{INV}}_{\Psi^t(s)}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$, $s \in [0,1]$, are isolated invariant sets. By using Remark 6.7 we infer that the Conley index does not change along the path. We conclude that both $\mathcal{R}^1$ and $\mathcal{R}^2$ can be continued to the same $\mathcal{R}^{\bar{a},\bar{b}}$ for some $\bar{b} \gg \bar{a}$ while preserving isolation, and part (i) of Theorem 6.2 follows. For part (ii) of Theorem 6.2 one employs the same construction, but with $\bar{b} \ll \bar{a}$.

6.3. *Perturbations of the parabolic recurrence relations*

The parabolic recurrence relation $\mathcal{R}$, generated by Eq. (1.1) is of up–down type and 2-periodic. However, we will deal with a more general setting, namely that $\mathcal{R}$ is $2p$-periodic i.e. $\mathcal{R}_{i+2p} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$. Every component $\mathcal{R}_i$ depends only on (u_{i-1}, u_i, u_{i+1}) and we use the notation $\mathbf{u}_i = (u_{i-1}, u_i, u_{i+1})$. Throughout this section we use a smooth bump function $\omega^\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ which satisfies

$$\omega^\varepsilon(x_1, x_2, x_3) = \begin{cases} 1 & \text{for } \|x\| \leqslant \frac{\varepsilon}{2}, \\ 0 & \text{for } \|x\| \geqslant \varepsilon, \end{cases}$$

where $\|\mathbf{x}\| = \|(x_1, x_2, x_3)\| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. Moreover we may assume that $|\frac{\partial \omega^\varepsilon}{\partial x_i}| < A\varepsilon^{-1}$ and $|\frac{\partial^2 \omega^\varepsilon}{\partial u_i \partial u_j}| < B\varepsilon^{-2}$, for $1 \leqslant i, j \leqslant 3$, for some $A, B > 0$ (independent of ε).

We construct a perturbation of the vector field $\mathcal{R}$ which is linear near $\mathbf{v}^1$.

6.8. Definition. Let $\varepsilon > 0$ and $\alpha, \beta \in \mathbb{R}^{2p}$ be such that

$$\alpha = \partial_1 \mathcal{R}(\mathbf{v}^1) \quad \text{and} \quad \beta = \partial_2 \mathcal{R}(\mathbf{v}^1), \tag{6.10}$$

where $\partial_j \mathcal{R}(\mathbf{v}^1) = (\partial_j \mathcal{R}_0(\mathbf{v}^1), \ldots, \partial_j \mathcal{R}_{2p-1}(\mathbf{v}^1))$. Then

$$\mathcal{N}_i^\varepsilon(\mathbf{u}_i) \stackrel{\text{def}}{=} \omega^\varepsilon(\mathbf{u}_i - \mathbf{v}_i^1)\mathcal{L}_i(\mathbf{u}_i) + \big(1 - \omega^\varepsilon(\mathbf{u}_i - \mathbf{v}_i^1)\big)\mathcal{R}_i(\mathbf{u}_i), \tag{6.11}$$

where $\mathcal{L}_i = \mathcal{L}_i^{\alpha\beta}(\mathbf{u}_i) \stackrel{\text{def}}{=} \alpha_i(u_{i-1} - v_{i-1}^1) + \beta_i(u_i - v_i^1) + \alpha_{i+1}(u_{i+1} - v_{i+1}^1)$.

The following two lemmas summarize the properties of $\mathcal{N}^\varepsilon$.

6.9. Lemma. *There exists an $\varepsilon_0 > 0$ such that $\mathcal{N}^\varepsilon$ is a parabolic recurrence relation of up–down type for any $0 < \varepsilon < \varepsilon_0$.*

Proof. Every $\mathcal{N}_i^\varepsilon$ is well defined on the set Ω_i and $\mathcal{N}_i^\varepsilon(\mathbf{u}_i) = \mathcal{R}(\mathbf{u}_i)$ if $\mathbf{u}_i \notin B_i(\varepsilon)$, where

$$B_i(\varepsilon) := \{\mathbf{x} \in \mathbb{R}^3\colon\ \|\mathbf{x} - \mathbf{v}_i^1\| \leqslant \varepsilon\}. \tag{6.12}$$

Thus $\mathcal{N}_i$ has all required properties on the complement of the set $B_i(\varepsilon)$. The up–down restriction for the braid $\mathbf{v}^1$ implies that

$$\rho_1 \stackrel{\text{def}}{=} \min\big(|v_i^1 - v_{i-1}^1|,\ i \in \{0, \ldots, 2p-1\}\big) \tag{6.13}$$

is positive. If we choose $\varepsilon < \frac{\rho_1}{2}$ then a sufficiently small neighborhood of $\partial\Omega_i$ is in the complement of $B_i(\varepsilon)$, and $\mathcal{N}^\varepsilon$ is of up–down type since the limits (2.9) and (2.10) for $\mathcal{N}_i^\varepsilon$ are the same as for $\mathcal{R}_i$.

In order to prove the monotonicity conditions for $\mathcal{N}$ we need to show that both $\partial_1 \mathcal{N}_i^\varepsilon(\mathbf{u}_i) > 0$ and $\partial_3 \mathcal{N}_i^\varepsilon(\mathbf{u}_i) > 0$ on Ω_i. We carry out the proof for the first inequality; the second follows in an analogous way. For the first inequality we show that there exists a universal constant $C_i > 0$ such that for

$$\partial_1 \mathcal{N}_i^\varepsilon(\mathbf{u}_i) \geqslant \partial_1 \mathcal{R}_i(\mathbf{v}_i^1) - \varepsilon C_i, \quad \mathbf{u}_i \in B_i(\varepsilon) \text{ for all } \varepsilon < \frac{\rho_1}{2}. \tag{6.14}$$

From monotonicity of $\mathcal{R}$ (i.e. $\partial_1 \mathcal{R}_i > 0$) combined with inequality (6.14) we infer that $\partial_1 \mathcal{N}_i^\varepsilon(\mathbf{u}_i) > 0$ for $\mathbf{u}_i \in B_i(\varepsilon)$, for $0 < \varepsilon < \min\{\frac{\rho_1}{2}, \frac{\partial_1 \mathcal{R}_i(\mathbf{v}_i^1)}{C_i}\}$. In order to prove inequality (6.14) we use that $\mathcal{R}_i(\mathbf{v}_i^1) = 0$ and (6.10) to estimate

$$|\mathcal{L}_i(\mathbf{u}_i) - \mathcal{R}_i(\mathbf{u}_i)| \leqslant \frac{1}{2}\|d^2\mathcal{R}_i(\mathbf{x}_i)(\mathbf{u}_i - \mathbf{v}_i^1, \mathbf{u}_i - \mathbf{v}_i^1)\|$$

where $\mathbf{x}_i = (1-s)\mathbf{u}_i + s\mathbf{v}_i^1$, for some $s \in (0, 1)$. Therefore

$$|\mathcal{L}_i(\mathbf{u}_i) - \mathcal{R}_i(\mathbf{u}_i)| \leqslant \frac{D_i}{2}\|\mathbf{u}_i - \mathbf{v}_i^1\|^2, \tag{6.15}$$

for $\mathbf{u}_i \in B_i(\varepsilon)$, where $D_i = \max_{\mathbf{x}\in B_i(\rho_1/2)} \|d^2\mathcal{R}_i(\mathbf{x})\|$, and $\varepsilon < \rho_1/2$. By the same token we show that

$$\left|\partial_j \mathcal{L}_i(\mathbf{u}_i) - \partial_j \mathcal{R}_i(\mathbf{u}_i)\right| \leqslant D_i \left\|\mathbf{u}_i - \mathbf{v}_i^1\right\|, \tag{6.16}$$

for $j = 1, 2, 3$ and every $\mathbf{u}_i \in B_i(\varepsilon)$. We write $\mathcal{N}_i^\varepsilon = \mathcal{L}_i + (1-\omega^\varepsilon)(\mathcal{R}_i - \mathcal{L}_i)$. Using the estimate $|\partial_1\omega^\varepsilon| < A\varepsilon^{-1}$, we obtain

$$\begin{aligned}
\partial_1 \mathcal{N}_i^\varepsilon(\mathbf{u}_i) &= \partial_1 \mathcal{L}_i(\mathbf{u}_i) - \partial_1\omega^\varepsilon(\mathbf{u}_i)\big(\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i(\mathbf{u}_i)\big) + \big(1-\omega^\varepsilon(\mathbf{u}_i)\big)\partial_1\big(\mathcal{R}(\mathbf{u}_i) - \mathcal{L}_i(\mathbf{u}_i)\big)\\
&\geqslant \alpha_i - \frac{A}{\varepsilon}\left|\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i(\mathbf{u}_i)\right| - \left|\partial_1\big(\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i(\mathbf{u}_i)\big)\right|\\
&\geqslant \partial_1\mathcal{R}_i\big(\mathbf{v}_i^1\big) - \frac{A}{\varepsilon}\left|\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i(\mathbf{u}_i)\right| - \left|\partial_1\big(\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i(\mathbf{u}_i)\big)\right|\\
&\geqslant \partial_1\mathcal{R}_i\big(\mathbf{v}_i^1\big) - \varepsilon - \frac{A\varepsilon D_i}{2} - \varepsilon D_i,
\end{aligned}$$

for any $\mathbf{u}_i \in B_i(\varepsilon)$ and $\varepsilon < \rho_1/2$. The last inequality guarantees the existence of the universal constant C_i in (6.14). Since positivity of $\partial_3\mathcal{N}_i^\varepsilon$ can be shown in an analogous way, the monotonicity condition for parabolic recurrence relation is satisfied. □

6.10. Remark. Inequality (6.15) implies that for ε_0 small enough there exist a constant C^{ε_0} such that $\|\mathcal{N}^\varepsilon(\mathbf{u}) - \mathcal{R}(\mathbf{u})\| < \varepsilon^2 C^{\varepsilon_0}$ and $\|\partial_j\mathcal{N}^\varepsilon(\mathbf{u}) - \partial_j\mathcal{R}(\mathbf{u})\| < \varepsilon C^{\varepsilon_0}$, $j = 1, 2, 3$, for all $\mathbf{u} \in \mathcal{E}_{2p}^1$ and $0 < \varepsilon < \varepsilon_0$.

6.11. Lemma. *There exists an $\varepsilon_0 > 0$ and a positive constants K^{ε_0} such that $\mathcal{N}^\varepsilon$ can be written in the form*

$$\mathcal{N}^\varepsilon(\mathbf{u}) = \mathcal{L}(\mathbf{u}) + P^\varepsilon(\mathbf{u}), \tag{6.17}$$

where

$$\left\|P^\varepsilon(\mathbf{u})\right\| \leqslant K^{\varepsilon_0}\left\|\mathbf{u} - \mathbf{v}^1\right\|^2, \tag{6.18}$$

for all $\mathbf{u} \in \mathcal{E}_{2p}^1$ such that $\|\mathbf{u} - \mathbf{v}^1\| < \varepsilon_0$ and $0 < \varepsilon < \varepsilon_0$.

Proof. We show that there exist P_i^ε and $K_i^{\varepsilon_0}$ such that (6.17) holds for every component $\mathcal{N}_i^\varepsilon$ and (6.18) holds for every P_i^ε. Then the lemma holds for $P^\varepsilon = (P_1^\varepsilon, \dots, P_{2p}^\varepsilon)^T$ and $K^{\varepsilon_0} = \sqrt{2p}\max_{i\in\{0,\dots,2p-1\}} K_i^{\varepsilon_0}$.

Due to the usual estimate on the remainder of the Taylor series it is enough to show that for every $i, k, l \in \{0, \dots, 2p-1\}$ there exists a constant $K_{i,k,l}^{\varepsilon_0}$ with the property

$$\left|\frac{\partial^2\mathcal{N}_i^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i)\right| \leqslant K_{i,k,l}^{\varepsilon_0}, \tag{6.19}$$

for $\mathbf{u}_i \in B_i(\varepsilon_0)$. Let us compute

$$\begin{aligned}
\frac{\partial^2\mathcal{N}_i^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i) &= \omega^\varepsilon(\mathbf{u}_i)\frac{\partial^2\mathcal{L}_i^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i) + \big(1-\omega^\varepsilon(\mathbf{u}_i)\big)\frac{\partial^2\mathcal{R}_i}{\partial u_k \partial u_l}(\mathbf{u}_i)\\
&\quad + \frac{\partial\omega^\varepsilon}{\partial u_l}(\mathbf{u}_i)\left(\frac{\partial\mathcal{L}_i}{\partial u_k}(\mathbf{u}_i) - \frac{\partial\mathcal{R}_i}{\partial u_k}(\mathbf{u}_i)\right) + \frac{\partial\omega^\varepsilon}{\partial u_k}(\mathbf{u}_i)\left(\frac{\partial\mathcal{L}_i}{\partial u_l}(\mathbf{u}_i) - \frac{\partial\mathcal{R}_i}{\partial u_l}(\mathbf{u}_i)\right)\\
&\quad + \frac{\partial^2\omega^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i)\big(\mathcal{L}_i(\mathbf{u}_i) - \mathcal{R}_i(\mathbf{u}_i)\big).
\end{aligned} \tag{6.20}$$

Using the estimates $|\frac{\partial\omega^\varepsilon}{\partial u_l}| < A\varepsilon^{-1}$ and $|\frac{\partial^2\omega^\varepsilon}{\partial u_l \partial u_l}| < B\varepsilon^{-2}$, the bounds (6.15) and (6.16) imply

$$\left|\frac{\partial^2 \mathcal{N}_i^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i)\right| \leqslant D_i + 2\frac{A}{\varepsilon}\varepsilon D_i + \frac{B}{\varepsilon^2}\varepsilon^2 \frac{D_i}{2},$$

for $\mathbf{u}_i \in B_i(\varepsilon_0)$, where $D_i = \max_{\mathbf{x}\in B_i(\varepsilon_0)} \|d^2\mathcal{R}_i(\mathbf{x})\|$. □

6.4. The construction of proper braid classes

In the previous subsection we defined the perturbation $\mathcal{N}^\varepsilon$ of the parabolic recurrence relation $\mathcal{R}$. Now we show that for every $\varepsilon > 0$ the improper braid class $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}$ can be associated with a proper braid class $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}$ where $\mathcal{N}^\varepsilon(\bar{\mathbf{v}}) = 0$ for any $0 < \varepsilon < \varepsilon_0$, with $\bar{\mathbf{v}}$ depending (trivially) on ε.

In view of Theorem 6.2, see also Section 6.5, we need to consider only the particular parabolic recurrence relations $\mathcal{R}^{\bar{a},\bar{b}}$, fixing $\mathbf{v}$, which where constructed in Section 6.2. Their linearization around $\mathbf{v}^1$ is given by

$$\mathcal{L}\big(\mathbf{v}^1 + \boldsymbol{\xi}\big)_i = \bar{a}\xi_{i-1} + \bar{b}\xi_i + \bar{a}\xi_{i+1}.$$

Let τ be any number in $[0, p]$. We fix $\bar{a} = 1$ for definiteness. Then we choose

$$\bar{b} = \bar{b}_\tau \stackrel{\text{def}}{=} -2\cos\frac{\pi\tau}{p}.$$

It follows from Remark 5.5 that for this choice the rotation number of $\mathbf{v}^1$ is equal to τ. In this subsection we restrict our attention to twist numbers $\tau \in [0, p] \cap \mathbb{Q}$. We write $\tau = \frac{q'}{p'}$ with (p', q') coprime and $0 < q' < pp'$, or $(p', q') = (1, 0)$. With the above choices the equations $\mathcal{L}(\mathbf{v}^1 + \boldsymbol{\xi})_i = 0$ have explicit $2pp'$-periodic solutions

$$\xi_i = \delta\cos\frac{\pi\tau i}{p}, \quad i \in \mathbb{Z},$$

for any $\delta \in \mathbb{R}$. We recall from (6.3) that

$$z_i^k\big(p', q', \delta\big) = v_i^1 + \delta\cos 2\pi\frac{q'}{p'}\left(k - 1 + \frac{i}{2p}\right).$$

It follows that $\{\mathbf{z}^k\}_{k=1}^{p'} = \mathbf{z}(p', q', \delta)$ is a stationary braid for the perturbed parabolic recurrence relation $\mathcal{N}^\varepsilon$ for sufficiently small δ. Here $\mathcal{N}^\varepsilon$ is the perturbation as constructed in Section 6.3 of $\mathcal{R}^{1,\bar{b}_\tau}$ defined above. In particular, choosing a $\delta \in (0, \varepsilon/2)$ the parabolic flow associated to $\mathcal{N}^\varepsilon$ fixes the augmented skeleton $\bar{\mathbf{v}} = \mathbf{v} \cup \mathbf{z}$.

6.12. Lemma. *The fiber $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}$ is a well-defined relative braid class in $\mathcal{E}_{2p}^1 \operatorname{rel} \bar{\mathbf{v}}$. Moreover, if the braid class $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}$ is bounded and $I(\mathbf{u}, \mathbf{v}^1) \neq 2\tau(\mathbf{v}^1)$, then $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}$ is bounded and proper.*

Proof. Every up–down braid $\mathbf{u} \in [\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}$ which satisfies $|u_i - v_i^1| \geqslant \frac{\varepsilon}{2}$, for all i, does not have a common anchor point with the strands $\mathbf{z}$. Thus the fiber $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}$ is indeed a well-defined relative braid class in $\mathcal{E}_{2p}^1 \operatorname{rel} \bar{\mathbf{v}}$.

First we show that the braid class $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}$ does not depend on the choice of $\mathbf{u}$. Let $\mathbf{u}^1$ and $\mathbf{u}^2$ be arbitrary braids in $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}$ which satisfy $|u_i^{1,2} - v_i^1| \geqslant \frac{\varepsilon}{2}$. We will show that $\mathbf{u}^1$ and $\mathbf{u}^2$ are in the same braid class relative to $\bar{\mathbf{v}}$.

Let $\mathbf{u}(t)\operatorname{rel}\mathbf{v}\in\mathcal{E}^1_{2p}\operatorname{rel}\mathbf{v}$ be a path between them, which, without loss of generality, evolves just one anchor point at the time. This means that there is a partition of the interval $[0,1]$, given by $0=t_0<t_1<\cdots<t_{m-1}<t_m=1$, such that only anchor point u_{i_j} evolves for $t\in(t_j,t_{j+1}]$ where $i_j\in\{0,\dots,2p-1\}$. The path $\mathbf{u}(t)\operatorname{rel}\mathbf{v}$ does not have to be in $\mathcal{E}^1_{2p}\operatorname{rel}\bar{\mathbf{v}}^\varepsilon$, because non-transverse crossings with some strands in $\mathbf{z}$ may occur. We will modify the path $\mathbf{u}(t)\operatorname{rel}\mathbf{v}$ in order to avoid this. If $|u_{i_j}(t_{j+1})-v^1_{i_j}|<\frac{\varepsilon}{2}$, then we perturb the function $\widetilde{u}_{i_j}(t):(t_j,t_{j+1}]\to\mathbb{R}$ as follows

$$\widetilde{u}_{i_j}(t)=\begin{cases}u_{i_j}(t_j)(1-\bar{t})+(v_{i_j}+\varepsilon/2)\bar{t} & \text{if } u_{i_j}(t_{j+1})\geqslant v_{i_j},\\ u_{i_j}(t_j)(1-\bar{t})+(v_{i_j}-\varepsilon/2)\bar{t} & \text{otherwise,}\end{cases}$$

where $\bar{t}=\frac{t-t_j}{t_{j+1}-t_j}$. We set $\widetilde{u}_{i_j}(t)=\widetilde{u}_{i_j}(t_{j+1})$, for all $t>t_{j+1}$, until the original path moves u_{i_j} again. The fact that $u_{i_j}(1)\notin(v^1_i-\varepsilon/2,v^1_i+\varepsilon/2)$ implies that there is a $j'>j$ such that $\mathbf{u}(t)\operatorname{rel}\mathbf{v}$ evolves the point u_{i_j} for $t\in(t_{j'},t_{j'+1}]$. We then define $\widetilde{u}_{i_j}(t):(t_{j'},t_{j'+1}]\to\mathbb{R}$ as a linear function connecting $\widetilde{u}_{i_j}(t_{j+1})$ with $u_{i_j}(t_{j'+1})$. We repeat this procedure for any anchor point ending within $\frac{\varepsilon}{2}$ from $\mathbf{v}^1$. This perturbation does not create non-transverse intersections with $\mathbf{v}^1,\mathbf{v}^2,\dots,\mathbf{v}^n$. Namely, along the perturbed path only one anchor point u_i can be in the interval $(v^1_i-\varepsilon/2,v^1_i+\varepsilon/2)$ at a time. If u_i passes through this interval then $u_{i-1}\leqslant v^1_i-\varepsilon/2<v^1_i+\varepsilon/2\leqslant u_{i+1}$ or $u_{i+1}\leqslant v^1_i-\varepsilon/2<v^1_i+\varepsilon/2\leqslant u_{i-1}$, since the original path was in $\mathcal{E}^1_{2p}\operatorname{rel}\mathbf{v}$. Hence a non-transverse crossing with the strands in $\mathbf{z}$ is not possible because all the anchor points of $\mathbf{z}$ are within distance $\delta<\frac{\varepsilon}{2}$ of $\mathbf{v}^1$. Furthermore, since $\varepsilon<\rho_1$ the perturbed path still satisfies the up–down restriction. This shows that $\mathbf{u}^1\operatorname{rel}\mathbf{v}$ and $\mathbf{u}^2\operatorname{rel}\mathbf{v}$ define the same braid class $[\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\bar{\mathbf{v}}$. Finally, properness when $I(\mathbf{u},\mathbf{v}^1)\neq 2\tau(\mathbf{v}^1)$ follows from the considerations in Section 6.1. □

Let Φ^t_ε be the parabolic flows associated to $\mathcal{N}^\varepsilon$. The next lemma makes the connection between the maximal invariant sets $\operatorname{INV}_{\Phi^t_\varepsilon}(\operatorname{cl}([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\bar{\mathbf{v}}))$ and $\overline{\operatorname{INV}}_{\Phi^t_\varepsilon}([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\mathbf{v})$, with the latter one defined in (6.1).

6.13. Theorem. *If $I(\mathbf{u},\mathbf{v}^1)\neq 2\tau(\mathbf{v}^1)$, then there exists an $\varepsilon_0>0$ such that*

$$\overline{\operatorname{INV}}_{\Phi^t_\varepsilon}\big([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\mathbf{v}\big)=\operatorname{INV}_{\Phi^t_\varepsilon}\big(\operatorname{cl}\big([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\bar{\mathbf{v}}\big)\big)$$

for $0<\varepsilon<\varepsilon_0$.

Proof. We have already seen that by choosing ε_0 sufficiently small we guarantee that $[\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\bar{\mathbf{v}}$ is a well-defined relative up–down braid class. We start by proving the inclusion

$$\overline{\operatorname{INV}}_{\Phi^t_\varepsilon}\big([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\mathbf{v}\big)\subset\operatorname{INV}_{\Phi^t_\varepsilon}\big(\operatorname{cl}\big([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\bar{\mathbf{v}}\big)\big).\tag{6.21}$$

The sets $\overline{\operatorname{INV}}_{\Phi^t_\varepsilon}(\operatorname{cl}([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\mathbf{v}))$ and $\partial\operatorname{cl}([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\mathbf{v})$ are compact and disjoint. Thus there exists an $\varepsilon_1<\rho_1$ such that their distance (in the supremum norm) satisfies

$$\operatorname{dist}\big(\overline{\operatorname{INV}}_{\Phi^t_\varepsilon}\big([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\mathbf{v}\big),\partial\operatorname{cl}\big([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\mathbf{v}\big)\big)>\varepsilon_1.\tag{6.22}$$

For any point $\mathbf{w}\in\overline{\operatorname{INV}}_{\Phi^t_\varepsilon}(\operatorname{cl}([\mathbf{u}]_{\mathcal{E}}\operatorname{rel}\mathbf{v}))$ with $|w_i-v^1_i|<\epsilon_1$ the following two inequalities hold:

$$\big|w_{i\pm1}-v^1_{i\pm1}\big|>\varepsilon_1,\tag{6.23}$$

$$\big(w_{i-1}-v^1_{i-1}\big)\big(w_{i+1}-v^1_{i+1}\big)<0.\tag{6.24}$$

Indeed, let

$$\mathbf{s} = \big(w_0, \dots, w_{i-2}, v_{i-1}^1, v_i^1, w_{i+1}, \dots, w_{2p-1}\big) \in \partial\,\mathrm{cl}\big([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}\big),$$

and if $|w_{i-1} - v_{i-1}^1| \leqslant \epsilon_1$, then since $\mathbf{w}$ is in the invariant set we conclude that

$$\mathrm{dist}\big(\overline{\mathrm{INV}}_{\Phi_\varepsilon^t}\big([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}\big), \partial\,\mathrm{cl}\big([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}\big)\big) \leqslant \mathrm{dist}(\mathbf{w}, \mathbf{s}) \leqslant \varepsilon_1.$$

This contradicts (6.22) and thus (6.23) holds. When we assume, for contradiction, that

$$\big(w_{i-1} - v_{i-1}^1\big)\big(w_{i+1} - v_{i+1}^1\big) \geqslant 0,$$

then we obtain a similar contradiction for

$$\mathbf{s} = \big(w_0, \dots, w_{i-1}, v_i^1, w_{i+1}, \dots, w_{2p-1}\big) \in \partial\,\mathrm{cl}\big([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}\big),$$

and therefore (6.24) holds as well.

Let $\mathbf{w} \in \overline{\mathrm{INV}}_{\Phi_\varepsilon^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}))$, then we show that $\mathbf{w} \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}$ for $\varepsilon < \varepsilon_1$. If $|w_i - v_i^1| \geqslant \frac{\varepsilon}{2}$ for all i, then $\mathbf{w} \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}$, since $\bar{\mathbf{v}} = \mathbf{v} \cup \mathbf{z}$ and $|z_i^k - v_i^1| < \varepsilon/2$ by construction. If $|w_i - v_i^1| < \frac{\varepsilon}{2}$ for some i, then it follows from Eqs. (6.23) and (6.24) that w_i can be moved out of interval $(v_i^1 - \frac{\varepsilon}{2}, v_i^1 + \frac{\varepsilon}{2})$ without changing the intersection number with the skeletal strands $\bar{\mathbf{v}}$. Therefore $\mathbf{w} \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}$.

Furthermore, if $\mathbf{w} \in \overline{\mathrm{INV}}_{\Phi_\varepsilon^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}))$, then it follows from (6.22) that $\Phi_\varepsilon^t(\mathbf{w})$ stays away from the boundary $\partial\,\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$. Hence for $\varepsilon < \varepsilon_1$ we have $\Phi_\varepsilon^t(\mathbf{w}) \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}$ for all t. Therefore $\mathbf{w} \in \mathrm{INV}_{\Phi_\varepsilon^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}))$, which proves the inclusion (6.21) for $\varepsilon < \varepsilon_1$.

We are left with proving the opposite inclusion. Let $\mathbf{w} \in \mathrm{INV}_{\Phi_\varepsilon^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}))$ and suppose that $\mathrm{dist}(\Phi_\varepsilon^t(\mathbf{w}), \partial\,\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}))$ is uniformly bounded away from 0 for all t, then $\mathbf{w} \in \overline{\mathrm{INV}}_{\Phi_\varepsilon^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}))$. It therefore suffices to prove that there exists $\varepsilon_2 > 0$, $\varepsilon_2 \leqslant \varepsilon_1$, such that

$$\mathrm{dist}\big(\mathrm{INV}_{\Phi_\varepsilon^t}\big(\mathrm{cl}\big([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}\big)\big), \partial\,\mathrm{cl}\big([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}\big)\big) > \varepsilon \quad \text{for all } 0 < \varepsilon < \varepsilon_2. \tag{6.25}$$

Let $\mathbf{y} \in \partial\,\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$. We claim there exists an $\varepsilon_{\mathbf{y}} > 0$, $\varepsilon_{\mathbf{y}} \leqslant \varepsilon_1$, such that the ball $B_{\varepsilon_{\mathbf{y}}}(\mathbf{y}) = \{\mathbf{x} \in \mathcal{E}_{2p}^1\colon \|\mathbf{x} - \mathbf{y}\| < \varepsilon_{\mathbf{y}}\}$ has empty intersection with $\mathrm{INV}_{\Phi_\varepsilon^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}))$ for all $\varepsilon < \varepsilon_1$. The compact set $\partial\,\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ can then be covered by a finite covering $U = \{B_{\varepsilon_{\mathbf{y}_i}}(\mathbf{y}_i)\}_{i=1}^N$. Hence (6.25) holds for $\varepsilon_2 = \min_{1 \leqslant i \leqslant N} \varepsilon_{\mathbf{y}_i}$.

We start with the boundary point $\mathbf{v}^1$, for which the argument is a variation on the one in Section 6.2. Hence, identify $\mathcal{E}_{2p}^1$ with a subset of $\mathbb{R}^{2p}$ via

$$\mathbf{u} \leftrightarrow \mathbf{x} = \big(u_0 - v_0^1, \dots, u_{2p-1} - v_{2p-1}^1\big) \in \mathbb{R}^{2p}.$$

We consider the case $I(\mathbf{u}, \mathbf{v}^1) > 2\tau(\mathbf{v}^1)$ only (the other case is analogous). The linearization of $\mathcal{N}^\varepsilon$ at $\mathbf{v}^1$ is given by $\mathcal{L}$ for all ε, hence $\tau(\mathbf{v}^1)$ does not depend on ε.

Let $\mathbf{x}_\varepsilon(t)$ be a flow line of Φ_ε^t. By the arguments in Section 6.2, there exist $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$ such that if for some $0 < \varepsilon < \varepsilon_4$ the orbit $\mathbf{x}_\varepsilon(t)$ lies in $[\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}$ and intersects the small neighborhood of the origin B_{ε_3} for some t_0, then $\mathbf{x}_\varepsilon(t) \to 0$ as $t \to -\infty$. Hence, $I(\mathbf{x}_\varepsilon(t), \mathbf{z}) \to 2q'$ as $t \to -\infty$. On the other hand, if $\mathbf{x}_\varepsilon(t) \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}$ then $I(\mathbf{x}_\varepsilon(t), \mathbf{z}) = p' I(\mathbf{u}, \mathbf{v}^1)$. Since $p' I(\mathbf{u}, \mathbf{v}^1) < 2q'$, this implies that $\mathbf{x}_\varepsilon(t)$ leaves the class $[\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}$, for some $t_0 < 0$ and $\mathbf{x}_\varepsilon \notin \mathrm{INV}_{\Phi_\varepsilon^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}))$. Hence, $B_{\varepsilon_3}(\mathbf{v}^1) \cap \mathrm{INV}_{\Phi_\varepsilon^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}})) = \emptyset$ for all $0 < \varepsilon < \varepsilon_4$, and we may choose $\varepsilon_{\mathbf{v}^1} = \min\{\varepsilon_3, \varepsilon_4\}$.

We are now left with the case $\mathbf{y} \in \partial\,\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ and $\mathbf{y} \neq \mathbf{v}^1$. The flow Φ_ε^t is transverse to the set $\partial\,\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}) \setminus \{\mathbf{v}^1\}$ by Proposition 4.2. We may assume that it points out of the set $\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$

at $\mathbf{y}$ (otherwise we go through the same argument in the reversed time direction). According to Proposition 4.2 the flow Φ_ε^t cannot enter the class $\mathrm{cl}([\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v})$ after leaving it. This combined with the continuity of the flow implies that there exists an $\varepsilon_\mathbf{y} > 0$ such that for all $\varepsilon < \varepsilon_\mathbf{y}$

$$B_{\varepsilon_\mathbf{y}}(\mathbf{y}) \cap \mathrm{INV}_{\Phi_\varepsilon^t}\big(\mathrm{cl}\big([\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}\big)\big) = \emptyset.$$

Finally, since $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}} \subset [\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}$, we also have

$$B_{\varepsilon_\mathbf{y}}(\mathbf{y}) \cap \mathrm{INV}_{\Phi_\varepsilon^t}\big(\mathrm{cl}\big([\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}\big)\big) = \emptyset.$$

This proves (6.25) for $\mathbf{y} \in \partial\,\mathrm{cl}([\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v})$, $\mathbf{y} \neq \mathbf{v}^1$, and we conclude that indeed

$$\mathrm{INV}_{\Phi_\varepsilon^t}\big(\mathrm{cl}\big([\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}\big)\big) \subset \overline{\mathrm{INV}}_{\Phi_\varepsilon^t}\big([\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}\big).$$

This finishes the proof of the theorem. □

6.5. *Proof of Theorem 6.4*

We deal with part (i) of the theorem only, as the proof of part (ii) is entirely analogous. Let Ψ^t be any (exact) parabolic flow of up–down type such that $\Psi^t(\mathbf{v}) = \mathbf{v}$, and $2\tau(\mathbf{v}^1) > I(\mathbf{u}, \mathbf{v}^1)$. Let $\tau_0 = \frac{q'}{p'}$, and let Ψ_0^t be the flow associated to a parabolic recurrence $\mathcal{R}^{1,\bar{b}_{\tau_0}}$. Let Φ_ε^t be the flow associated to $\mathcal{N}^\varepsilon$, the perturbation of $\mathcal{R}^{1,\bar{b}_{\tau_0}}$ discussed in Section 6.4, for some sufficiently small $\varepsilon > 0$. Let N and N_ε be isolating neighborhoods of $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v})$ and $\overline{\mathrm{INV}}_{\Phi_\varepsilon^t}([\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v})$, respectively. By Theorem 6.2,

$$h\big(N; \Psi^t\big) \cong h\big(N_\varepsilon; \Phi_\varepsilon^t\big). \tag{6.26}$$

Moreover, by Theorem 6.13,

$$\overline{\mathrm{INV}}_{\Phi_\varepsilon^t}\big([\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}\big) = \mathrm{INV}_{\Phi_\varepsilon^t}\big(\mathrm{cl}\big([\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}\big)\big),$$

which implies that

$$h\big(N_\varepsilon; \Phi_\varepsilon^t\big) \cong h\big(\mathrm{cl}\big([\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}\big); \Phi_\varepsilon^t\big), \tag{6.27}$$

since $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}}$ is a proper relative braid class and thus $\mathrm{cl}([\mathbf{u}]_\mathcal{E} \operatorname{rel} \bar{\mathbf{v}})$ is an isolating neighborhood. Recalling Proposition 4.4, combining (6.26) with (6.27) proves Theorem 6.4.

7. The application to fourth order differential equations

In this section we study solutions of Eq. (1.1) on the zero energy level. We focus on solutions of the third type, i.e., functions which intersect the constant solution $u_+ = +1$, but not the constant solution $u_- = -1$. As we mentioned in Section 1 these functions can be classified by the number of monotone loops $2p$ and number of intersections $2q$ with u_+. To prove Theorem 1.2 we will show existence of a solution $u^\alpha \in \mathbf{u}_{p,q}$, for $\alpha \in (\sqrt{8}, \alpha_{p,q})$, where $\alpha_{p,q}$ is given by (1.2).

Our strategy is based on taking a free strand u which intersects $2q$ times u_+ and does not intersect u_-. One obstacle to applying the machinery developed in the previous sections is that the strands $\mathbf{u}_\pm$ corresponding to the discretization of the constant solutions $u_\pm = \pm 1$ do not obey the up–down restriction. Hence, we cannot include them in the skeleton $\mathbf{v}$ in order to define the braid class $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}$. To overcome this problem we have to use a more elaborate approach. First we will show that for small positive energy values E there exist two solutions of (1.1) which oscillates around

u_+ and u_-, respectively Then we define the braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$. The strands associated to the small oscillations around $u_\pm$ are included in $\mathbf{v}$ and the free strand is braided with them in the way mentioned above. We use the results from the previous section to prove existence of a fixed point within the braid class. This provides a solution u^E of (1.1), for small positive E, such that $\mathbb{E}[u^E] = E$. Finally we will use a limit process $E \to 0$ for solutions u^E to find a solution $u \in \mathbf{u}_{p,q}$ at the zero energy level.

7.1. Small oscillations

We show in this section that for small positive energy levels there exist solutions that oscillate around $u_\pm$. The rotation number of these solutions is also computed.

7.1. Lemma. *For every $\alpha > \sqrt{8}$ and sufficiently small $E > 0$ there exists a periodic solution u_+^E of Eq. (1.1) with two extrema per period such that $\min u_+^E < 1 < \max u_+^E$ and $\mathbb{E}[u_+^E] = E$. Moreover $u_+^E \to +1$ as $E \to 0$.*

Proof. The transformation $u(t) = 1 + \epsilon w(t)$ transforms Eq. (1.1) into

$$w'''' + \alpha w'' + 2w + 3\epsilon w^2 + \epsilon^2 w^3 = 0. \tag{7.1}$$

The rescaled energy functional is given by

$$\mathbb{E}_\epsilon[w] = -w'w''' + \frac{1}{2}\left(w''\right)^2 - \frac{\alpha}{2}\left(w'\right)^2 - F_\epsilon(w), \tag{7.2}$$

where $F_\epsilon(w) = w^2 + \epsilon w^3 + \frac{1}{4}\epsilon^2 w^4$. If $\epsilon = 0$, then (7.1) reduces to the linear equation

$$w'''' + \alpha w'' + 2w = 0. \tag{7.3}$$

The eigenvalues of the latter are given by

$$\lambda_i^2 = \frac{1}{2}\left[\alpha - (-1)^i \sqrt{\alpha^2 - 8}\right].$$

Thus, for $\alpha > \sqrt{8}$, $w_0(t) = -\cos(\lambda_1 t)$ is its solution with two extrema per period and energy $\mathbb{E}_0[w_0] = \frac{\lambda_1^4}{2} - 1 > 0$. Eq. (7.1) contains only even derivatives, which implies that every solution satisfying

$$w'(0) = w'(T) = w'''(0) = w'''(T) = 0, \quad T \in \mathbb{R}^+,$$

is $2T$-periodic. Define $G : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$G(A, T, \epsilon) = \begin{pmatrix} w'_{\epsilon,A}(T) \\ w'''_{\epsilon,A}(T) \end{pmatrix},$$

where $w_{\epsilon,A}$ is the solution of (7.1) with initial data

$$w_{\epsilon,A}(0) = A, \qquad w'_{\epsilon,A}(0) = 0,$$
$$w''_{\epsilon,A}(0) = \sqrt{2\left(F_\epsilon(A) + \mathbb{E}_0[w_0]\right)}, \qquad w'''_{\epsilon,A}(0) = 0.$$

If $G(w_{\epsilon,A}, T, \epsilon) = (0,0)^T$, then $w_{\epsilon,A}$ is a $2T$-periodic solution of (7.1). The condition $w''_{\epsilon,A}(0) = \sqrt{2(F_\epsilon(A) + \mathbb{E}_0[w_0])}$ implies that $\mathbb{E}_\epsilon[w_{\epsilon,A}] = \mathbb{E}_0[w_0]$.

To prove the existence of periodic solutions of (7.1) for $\epsilon > 0$ we will employ the implicit function theorem for the function G. For $\epsilon = 0$ we have $w_{0,-1} = w_0$ and

$$G\left(-1, \frac{\pi}{\lambda_1}, 0\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The solutions $w_{\epsilon,A}$ can be expanded as

$$w_{\epsilon,A}(t) = C(A)\cos(\lambda_1 t) + D(A)\cos(\lambda_2 t) + g(\epsilon, A, t), \tag{7.4}$$

where $g = O(\epsilon)$ and $C(A)$, $D(A)$ satisfy

$$\begin{gathered} C(A) + D(A) = A, \quad \text{and} \\ -\lambda_1^2 C(A) - \lambda_2^2 D(A) = \sqrt{2\big(F_\epsilon(A) + \mathbb{E}_0[w_0]\big)}. \end{gathered}$$

Using (7.4) we obtain that

$$\begin{aligned} \det\left(\frac{\partial G}{\partial A}\,\frac{\partial G}{\partial T}\right)_{(-1,\frac{\pi}{\lambda_1},0)} &= \det\begin{pmatrix} \partial_A w'_{\epsilon,A}(T) & w''_{\epsilon,A}(T) \\ \partial_A w'''_{\epsilon,A}(T) & w''''_{\epsilon,A}(T) \end{pmatrix}_{(-1,\frac{\pi}{\lambda_1},0)} \\ &= \frac{\lambda_1^4}{\lambda_1^2 - \lambda_2^2}\sin\frac{\lambda_2}{\lambda_1}\pi \neq 0. \end{aligned}$$

From the implicit function theorem we conclude the existence of two continuous functions $A : (-\delta, \delta) \to \mathbb{R}$ and $T : (-\delta, \delta) \to \mathbb{R}$, $\delta > 0$, such that $A(0) = -1$, $T(0) = \frac{\pi}{\lambda_1}$ and

$$G\big(A(\epsilon), T(\epsilon), \epsilon\big) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

for $\epsilon \in [0, \delta)$. The periodic solutions $w_\epsilon(t) := w_{A(\epsilon),\epsilon}(t)$ converge to $w_0 = -\cos\lambda_1 t$ as $\epsilon \to 0$ in C^3 norm. Therefore, w_ϵ has two extrema per period (one negative, one positive) for ϵ small enough. Let $\epsilon(E) = \sqrt{\frac{E}{E_0[w_0]}}$. Then the solution $u_+^E(t) = 1 + \epsilon(E) w_{\epsilon(E)}(t)$ of Eq. (1.1) satisfies the energy identity

$$\mathbb{E}\big[u_+^E(t)\big] = \epsilon(E)^2 \mathbb{E}_{\epsilon(E)}\big[w_{\epsilon(E)}(t)\big] = E,$$

which proves the lemma. $\square$

7.2. Remark. An analogous construction can be carried out to construct u_-^E. The solutions u_-^E have similar properties as u_+^E and $u_-^E \to -1$ as $E \to 0$.

One should keep in mind that every solution u_+^E of Eq. (1.1) is a solution for some value of the parameter α, although we do not indicate this in the notation. We can associate the solution u_+^E with a braid $\mathbf{u}_+^E \in \mathcal{E}_2^1$ via its sequence of extrema. The following lemma estimates the rotation number $\tau(\mathbf{u}_+^E)$.

7.3. Lemma. *Let $\alpha > \sqrt{8}$, and let $\mathbf{u}_+^E \in \mathcal{E}_2^1$ be the braid corresponding to the solution u_+^E. Then for every $\varepsilon > 0$ there exists an $E_0 > 0$ such that*

$$\left|\tau\left(\mathbf{u}_+^E\right) - \frac{\lambda_2}{\lambda_1}\right| < \epsilon \quad \textit{for all } 0 < E < E_0, \tag{7.5}$$

where $\lambda_i^2 = \frac{1}{2}[\alpha - (-1)^i\sqrt{\alpha^2 - 8}]$.

Proof. As we mentioned in Section 5, the twist maps $F_i^E(x, y)$ corresponding to the generating function S_E for a Lagrangian system with Euler–Lagrange equation given by (1.1), can be defined as follows. Let u_i be a solution of Eq. (1.1) with the initial values

$$u_i(0) = x, \qquad u_i'(0) = 0, \qquad u_i''(0) = (-1)^i\sqrt{2E + \left(x^2 - 1\right)^2}, \qquad u_i'''(0) = y.$$

Let $t_0 > 0$ be the first nonzero time for which $u'(t_0) = 0$. Then

$$F_i^E(x, y) = \left(u_i(t_0), u_i'''(t_0)\right).$$

We want to compute the rotation number of

$$d\left(F_1^E \circ F_0^E\right)\left(u_+^E(0), \left(u_+^E\right)'''(0)\right) = dF_1^E\left(F_0^E\left(u_+^E(0), \left(u_+^E\right)'''(0)\right)\right)dF_0^E\left(u_+^E(0), \left(u_+^E\right)'''(0)\right).$$

Let us first compute $dF_0^E(u_+^E(0), (u_+^E)'''(0))$. To do so we will use, as before, the transformation $u(t) = 1 + \epsilon(E)w$, where $\epsilon(E) = \sqrt{\frac{2E}{\lambda_1^4 - 2}}$. One observes that

$$dF_0^E\left(u_+^E(0), \left(u_+^E\right)'''(0)\right) = d\widetilde{F}^E\left(w_E(0), w_E'''(0)\right),$$

where $\widetilde{F}^E$ is defined in the same manner as F_0^E, but with w a solution of Eq. (7.1) with initial data

$$w(0) = \tilde{x}, \qquad w'(0) = 0,$$
$$w''(0) = \sqrt{2\left(\frac{\lambda_1^4}{2} - 1 + \tilde{x}^2 + \epsilon(E)\tilde{x}^3 + \frac{1}{4}\epsilon(E)^2\tilde{x}^4\right)}, \qquad w'''(0) = \tilde{y}.$$

Continuous dependence upon E implies that for every $\varepsilon_1 > 0$ there exists an E_1 such that

$$\left\| D\widetilde{F}^E\left(w_E(0), w_E'''(0)\right) - D\widetilde{F}^0\left(w_0(0), w_0'''(0)\right)\right\| < \varepsilon_1 \quad \text{for all } 0 < E < E_1, \tag{7.6}$$

where $w_0 = -\cos(\lambda_1 t)$. The value of $D\widetilde{F}^E(w_0(0), w_0'''(0))$ in the general direction $(\cos\theta, \sin\theta)^T$, for $0 \leqslant \theta < 2\pi$, is computed as follows:

$$\begin{aligned} d\widetilde{F}^0\left(w_0(0), w_0'''(0)\right)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} &= \frac{d}{d\mu}\widetilde{F}^0\left(w_0(0) + \mu\cos\theta, w_0'''(0) + \mu\sin\theta\right)_{\mu=0} \\ &= \begin{pmatrix} \partial_\mu y_{\mu,\theta}(P_\theta(0)) \\ -\partial_\mu y_{\mu,\theta}'''(P_\theta(0)) - y_{\mu,\theta}''''(P_\theta(0))\frac{d}{d\mu}P_\theta(0)|_{\mu=0}\end{pmatrix}, \end{aligned}$$

where $P_\theta(\mu)$ is the first positive time for which $y'_{\mu,\theta}(P_\theta(\mu)) = 0$ (a maximum). The function $y_{\mu,\theta}$ is a solution of Eq. (7.3) with initial conditions

$$
\begin{aligned}
y_{\mu,\theta}(0) &= w_0(0) + \mu\cos\theta,\\
y'_{\mu,\theta}(0) &= 0,\\
y''_{\mu,\theta}(0) &= \sqrt{2\left(\frac{\lambda_1^4}{2} - 1 + \big(w_0(0) + \mu\cos\theta\big)^2\right)},\\
y'''_{\mu,\theta}(0) &= w_0'''(0) + \mu\sin\theta.
\end{aligned}
\tag{7.7}
$$

We evaluate $\frac{d}{d\mu}P_\theta(0)|_{\mu=0}$ by differentiating the equation $y'_{\mu,\theta}(P_\theta(\mu)) = 0$, with respect to the parameter μ:

$$
\frac{d}{d\mu}P_\theta(\mu)\bigg|_{\mu=0} = -\frac{\partial_\mu y'_{\mu,\theta}(P_\theta(0))|_{\mu=0}}{y''_{\mu,\theta}(P_\theta(0))|_{\mu=0}}.
$$

Linearity of Eq. (7.3) enables us to compute all components of the expression $d\widetilde{F}^E(w_0(0), w_0'''(0))(\cos\theta,\sin\theta)^T$, for any θ. By doing so for $\theta = 0$ and $\theta = \frac{\pi}{2}$, we obtain

$$
d\widetilde{F}^E\big(w_0(0), w_0'''(0)\big) = \begin{pmatrix} \cos(\frac{\lambda_2}{\lambda_1}\pi) & -\frac{\lambda_1}{\lambda_2}\sin(\frac{\lambda_2}{\lambda_1}\pi) \\ \frac{\lambda_2}{\lambda_1}\sin(\frac{\lambda_2}{\lambda_1}\pi) & \cos(\frac{\lambda_2}{\lambda_1}\pi) \end{pmatrix},
$$

which is conjugate to

$$
\begin{pmatrix} \cos(\frac{\lambda_2}{\lambda_1}\pi) & -\sin(\frac{\lambda_2}{\lambda_1}\pi) \\ \sin(\frac{\lambda_2}{\lambda_1}\pi) & \cos(\frac{\lambda_2}{\lambda_1}\pi) \end{pmatrix}.
$$

From continuous dependence of the rotation number and Eq. (7.6) we infer that we can choose E_0 in such a way that for all $0 < E < E_0$ the matrix dF_0^E is conjugate to the rotation matrix

$$
\begin{pmatrix} \cos(2\tau_E\pi) & -\sin(2\tau_E\pi) \\ \sin(2\tau_E\pi) & \cos(2\tau_E\pi) \end{pmatrix},
\tag{7.8}
$$

where $|\tau_E - \frac{\lambda_2}{2\lambda_1}| < \frac{\varepsilon}{2}$.

By the same token we get a similar result for dF_1^E and $d(F_1^E \circ F_0^E)$. In particular, by composing dF_0^E and dF_1^E one gets that $d(F_1^E \circ F_0^E)$ is also conjugate to a rotation matrix of the form (7.8) for some $\tilde{\tau}_E$ which satisfies $|\tilde{\tau}_E - \frac{\lambda_2}{\lambda_1}| < \epsilon$. It follows from Eq. (5.2) that the rotation number is given by $\tau(\mathbf{u}_+^E) = \tilde{\tau}_E + k$, for some $k \in \mathbb{N}$. The fact that $\frac{\lambda_2}{2\lambda_1} < \frac{1}{2}$ for $\alpha > \sqrt{8}$, implies that $k = 0$, which concludes the proof. $\square$

7.4. Remark. From now on, if there is no ambiguity, we will indicate a p-fold cover/repetition of $\mathbf{u}_+^E$ by the same symbol. The rotation number $\tau(\mathbf{u}_+^E)$ of the p-fold $\mathbf{u}_+^E \in \mathcal{E}_{2p}^1$ is p times the rotation number of $\mathbf{u}_+^E$.

7.2. The solution u^E with positive energy

We will construct a solution u of Eq. (1.1) on the zero energy level as a limit of solutions u^E on positive energy levels, of which the existence is established in the following theorem.

7.5. Theorem. *Let $p, q \in \mathbb{N}$ be coprime such that $0 < q < p$ and $\alpha \in (\sqrt{8}, \alpha_{p,q})$. Then for sufficiently small E there exists a solution u^E of (1.1) with $\mathbb{E}[u^E] = E$ and its sequence of extrema $\mathbf{u}^E$ is $2p$-periodic. Moreover, $I(\mathbf{u}^E, \mathbf{u}_+^E) = 2q$ and $I(\mathbf{u}^E, \mathbf{u}_-^E) = 0$, where $\mathbf{u}^E$, $\mathbf{u}_+^E$ and $\mathbf{u}_-^E$ are the sequences of extrema in $\mathcal{E}_{2p}^1$ corresponding to the solutions u^E, u_+^E and u_-^E respectively.*

Proof. To prove this theorem we employ the relative braid class $[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v} \subset \mathcal{E}_{2p}^1 \operatorname{rel} \mathbf{v}$. This braid class will turn out to contain a fixed point $\mathbf{u}^E$ which is a sequence of extrema for a solution u^E. Let us start by identifying the skeleton

$$\mathbf{v} = \mathbf{v}^1 \cup \mathbf{v}^2 \cup \mathbf{v}^3 \in \mathcal{E}_{2p}^3.$$

We define $\mathbf{v}^1 = \mathbf{u}_+^E$ and $\mathbf{v}^2 = \mathbf{u}_-^E$. To construct the strand $\mathbf{v}^3$, which acts as a kind of outer bound, we use the dissipativity of the Lagrangian system generated by Eq. (1.1). Dissipativity implies the existence of $u_1^*, u_2^* \in \mathbb{R}$ such that $u_1^* < v_i^1$, $v_i^2 < u_2^*$ for all i and $\mathcal{R}_{2i}(u_{2i-1}, u_1^*, u_{2i+1}) < 0$ for $u_1^* < u_{2i\pm1} < u_2^*$ while $\mathcal{R}_{2i+1}(u_{2i}, u_2^*, u_{2i+1}) > 0$, for $u_1^* < u_{2i}, u_{2i+2} < u_2^*$. For more details see [15]. Let

$$\Omega_i = \begin{cases} \{(u_{i-1}, u_i, u_{i+1}) \in \mathbb{R}^3\colon\ u_1^* < u_{i\pm1} < u_i < u_2^*\}, & i \text{ odd}, \\ \{(u_{i-1}, u_i, u_{i+1}) \in \mathbb{R}^3\colon\ u_1^* < u_i < u_{i\pm1} < u_2^*\}, & i \text{ even}. \end{cases}$$

Denote by Ω^{2p} the set of $2p$-periodic sequences $\{u_i\}$ for which $(u_{i-1}, u_i, u_{i+1}) \in \Omega_i$. Furthermore define the set

$$C = \bigl\{\mathbf{u} \in \Omega^{2p}\colon\ I\bigl(\mathbf{u}, \mathbf{v}^1\bigr) = I\bigl(\mathbf{u}, \mathbf{v}^2\bigr) = 2p\bigr\}.$$

Since $I(\mathbf{v}^1, \mathbf{v}^2) = 0$ the vector field $\mathcal{R}$ is transverse to ∂C. Moreover, the set C is contractible, compact and $\mathcal{R}$ is pointing outward at the boundary ∂C due to the dissipativity. The set C is therefore negatively invariant for the induced flow Ψ^t. Consequently, there exists a fixed point $\mathbf{v}^3$ of Ψ^t in the interior of C, see Remark 4.5. We define $[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v} \in \mathcal{E}_{2p}^1 \operatorname{rel} \mathbf{v}$ by its representative $\mathbf{u}$ satisfying

(i) $(-1)^i u_i > (-1)^i v_i^3$,
(ii) $u_i > v_i^2$,
(iii) $I(\mathbf{u}, \mathbf{v}^1) = 2q$,

where $0 < 2q < 2p$, see Fig. 9. For $p \geqslant 2$, $[\mathbf{u}]_{\mathcal{E}} \operatorname{rel} \mathbf{v}$ is a bounded improper, free[1] up–down braid class where $\mathbf{u}$ can collapse only onto $\mathbf{v}^1$. It follows from Lemma 7.3 and Remark 7.4 that for every $\varepsilon_1 > 0$ we can choose $E > 0$ so small that the rotation number of $\mathbf{v}^1 = \mathbf{u}_+^E$ satisfies the inequality

$$\left|\tau\bigl(\mathbf{v}^1\bigr) - p\frac{\lambda_2}{\lambda_1}\right| < \varepsilon_1,$$

[1] A braid class is free if it consists of one connected component, see [5].

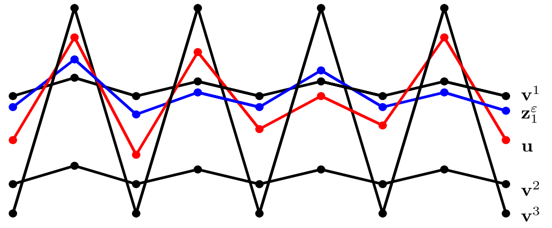

Fig. 9. A representative of the braid class $[\mathbf{u} \operatorname{rel} \mathbf{v}^\varepsilon]$ for $p = 4$, $q = 3$, $p' = 1$ and $q' = 2$. The free strand is shown in red and the skeleton strand $\mathbf{z}_1^\varepsilon$ is blue. If we leave out the strand $\mathbf{z}_1^\varepsilon$ from the skeleton we get a representative of the braid class $[\mathbf{u} \operatorname{rel} \mathbf{v}]$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

where $\lambda_i^2 = \frac{1}{2}[\alpha - (-1)^i\sqrt{\alpha^2 - 8}]$. If $\alpha \in (\sqrt{8}, \alpha_{p,q})$, then $0 < \frac{q}{p} < \frac{\lambda_2}{\lambda_1}$ and therefore, for any $\alpha \in (\sqrt{8}, \alpha_{p,q})$, we can choose ε_1 in such a way that $\tau(\mathbf{v}^1) > q$. Hence according to Lemma 6.1, $S = \overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v})$ is an isolated invariant set and by Theorem 6.4 it follows

$$h\bigl(S; \Psi^t\bigr) \cong \mathbf{H}\bigl(\mathbf{u} \,\#\, \bar{\mathbf{v}}^*; 2p\bigr) \cong \mathbf{H}\bigl(\beta(\mathbf{u}) \,\#\, \beta\bigl(\bar{\mathbf{v}}^*\bigr)\bigr),$$

where the extended skeleton $\bar{\mathbf{v}}^*$ is described by (4.2). It follows from Proposition 7.6 below that $\mathbf{H}(\beta(\mathbf{u}) \,\#\, \beta(\bar{\mathbf{v}}^*)) \cong S^{2q-1} \vee S^{2q} \neq 0$, which implies that the braid class $[\mathbf{u}]_\mathcal{E} \operatorname{rel} \mathbf{v}$ contains a fixed point of the flow Ψ^t, see again Remark 4.5. □

7.6. Proposition. *For the proper bounded braid class* $[\mathbf{u}] \operatorname{rel} \bar{\mathbf{v}}^*$ *defined in Theorem* 7.5 *it holds that* $\mathbf{H}(\beta(\mathbf{u}) \,\#\, \beta(\bar{\mathbf{v}}^*)) \cong S^{2q-1} \vee S^{2q}$.

Proof. For a detailed proof (among other integrable braid classes) see [5, Section 10]. □

7.3. The limiting process $E \to 0$

We have proved existence of a solution u^E of (1.1) in the parameter range $\alpha \in (\sqrt{8}, \alpha_{p,q})$ on small positive energy levels E. Its sequence of extrema $\mathbf{u}^E$ is $2p$ periodic and $I(\mathbf{u}^E, \mathbf{u}_+^E) = 2q$, while $I(\mathbf{u}^E, \mathbf{u}_-^E) = 0$. We will construct a sequence $\{u_n\}_{n=0}^\infty$ given by $u_n = u^{E_n}$, with $E_n \downarrow 0$ as $n \to \infty$, such that $u = \lim_{n\to\infty} u_n$ is a solution of (1.1) in the periodic class $\mathbf{u}_{p,q}$ and $\mathbb{E}[u] = 0$. First we show that there is a convergent sequence $\{u_n\}_{n=0}^\infty$. Let $u_n = u^{E_n}$ denote the aforementioned solutions in the energy levels $E_n > 0$, with $E_n \downarrow 0$ as $n \to \infty$.

7.7. Lemma. *There exists a convergent subsequence, again denoted by u_n, such that $u_n \to u$ for $n \to \infty$ in the C^4 norm on bounded intervals. Moreover, u is a solution of* (1.1) *on the zero energy level.*

Proof. Recall the definition of u_1^* and u_2^* in the proof of Theorem 7.5. We will show that the sequence $\{\frac{d^i}{dt^i}u_n(0)\}_{n=0}^\infty$ is bounded for $i \in \{0, 1, 2, 3\}$. It follows from the construction of the solutions u^E that $u_1^* < u_n(t) < u_2^*$ for all t and $u_n'(0) = 0$. The energy equation implies that $u_n''(0) = \sqrt{2E_n + \frac{(u_n^2(0)-1)^2}{2}}$ and therefore $\{u_n''(0)\}_{n=0}^\infty$ is bounded. By standard estimates on the third derivative one can get that the sequence $\{u_n'''(0)\}_{n=0}^\infty$ is bounded as well. Now choose a subsequence $\{u_n\}_{n=0}^\infty$ such that $\frac{d^i}{dt^i}u_n(0) \to u^i$ for $n \to \infty$. The sequence $\{u_n\}_{n=0}^\infty$ converges in the C^4-norm to a solution u of Eq. (1.1) which satisfies the initial conditions $\frac{d^i}{dt^i}u(0) = u^i$. For the energy it holds that $\mathbb{E}[u] = \lim_{n\to\infty} \mathbb{E}[u_n] = 0$. □

The following lemma shows that if the limit solution u is not constant then it is in the periodic class $\mathbf{u}_{p,q}$.

7.8. Lemma. *Let u be the limit of the sequence $\{u_n\}_{n=0}^\infty$ given by the previous lemma. If $u \not\equiv 1$ then $u \in \mathbf{u}_{p,q}$.*

Proof. Let T_n be the period of u_n. Every solution u_n has $2p$ extrema per period, denoted by $\mathbf{u}_n = (u_0^n, \dots, u_{2p-1}^n)$ and $I(\mathbf{u}_n, \mathbf{u}_+^{E_n}) = 2q$ for $n \in \mathbb{N}$. Let $t_n^i \in [0, T_n)$ be the time at which the solution u_n attains the minimum (maximum) u_i^n. The energy $\mathbb{E}[u_n] = E$ is positive and hence any two extremal points u_i^n and u_{i+1}^n are connected by a non-degenerate monotone lap.

According to Lemma 7.7 the limit $u = \lim_{n\to\infty} u_n$ lies in the zero energy level. We show now that $u \in \mathbf{u}_{p,q}$. First, we claim that u is not a constant solution of (1.1), i.e., $u \not\equiv \pm 1$. Namely, we excluded the case $u \equiv 1$ in the assumption of the lemma, while it follows from $I(\mathbf{u}_n, \mathbf{u}_+^{E_n}) = 2q$ that for every $n \in \mathbb{N}$ there is t_n such that $u_n(\bar{t}_n) > 1$, hence u_n cannot converge to $u_- \equiv -1$.

We will prove periodicity of u by contradiction. If u is not periodic then it must be that $T_n \to \infty$ as $n \to \infty$. Then u is non-constant and bounded, and it consists of finitely many monotone laps, hence u converges monotonically to an equilibrium, i.e., $u(t) \to u_-$ or $u(t) \to u_+$ monotonically as $t \to \infty$, which is impossible since the equilibrium points ± 1 are centers ($\alpha > \sqrt{8}$). Consequently, T_n is uniformly bounded for all $n \in \mathbb{N}$, and T_n (at least along a subsequence) converges to some $T \in \mathbb{R}$, which implies that u is T periodic.

Next we show that $u \in \mathbf{u}_{p,q}$. We start by showing that u has $2p$ monotone laps per period. Degenerate monotone laps (inflection points) can occur on the singular energy level $E = 0$. According to the definition of the solution class we have to count also these degenerate laps. To show that u has $2p$ monotone laps per period it is enough to prove that no sets of more than two extremal points can collapse onto each other and if two extremal points collapse then a degenerate monotone lap is created.

Suppose there are three different extremal points collapsing onto each other. Then the sequences $\{t_n^{i-1}\}$, $\{t_n^i\}$, $\{t_n^{i+1}\}$ converge to the same t_0. The equalities $u_n'(t_n^{i-1}) = u_n'(t_n^i) = u_n'(t_n^{i+1}) = 0$ imply that there exist $\widetilde{t}_n \in (t_n^{i-1}, t_n^i)$ and $\widehat{t}_n \in (t_n^i, t_n^{i+1})$ such that $u_n''(\widetilde{t}_n) = u_n''(\widehat{t}_n) = 0$. In turn, we infer that there are $\bar{t}_n \in (\widetilde{t}_n, \widehat{t}_n)$ such that $u_n'''(\bar{t}_n) = 0$. By continuity, the limit function satisfies $u'(t_0) = u''(t_0) = u'''(t_0) = 0$. Since $\mathbb{E}[u] = 0$, it holds that $u(t_0) = \pm 1$ and u is a constant solution. However, we assume that u cannot be constant. If there is a collapsing monotone lap (two extremal points collapse on one) then the same argumentation as above implies that it collapses on an inflection point of the limit u, and the number of monotone laps is thus preserved.

Next, we will show that the solution u intersects the constant solution $u_+ \equiv 1$ exactly $2q$ times per period. Let the i-th intersection between the linear interpolations $\beta(u_n)$ and $\beta(u_+^{E_n})$ occur at $\sigma_n^i \in [0, 1]$. Since $\beta(u_n)$ and $\beta(u_+^{E_n})$ are piecewise linear, for every σ_n^i there is an anchor point k_n^i such that $\sigma_n^i < k_n^i/d < \sigma_n^{i+1}$. For the solutions $u_n(t)$ this means that there are corresponding times s_n^i and t_n^i such that $s_n^i < t_n^i < s_n^{i+1}$ and $u_n(s_n^i) \to 1$ as $n \to \infty$ (since they correspond to intersection with $u_+^{E_n} \to 1$) and $u_n'(t_n^i) = 0$. If two crossing points s_n^i and s_n^j with $i < j$ collapse i.e., $s_n^i - s_n^j \to 0$, then $u_n(t_n^k) \to 1$ as $n \to \infty$ for all k such that $s_n^i \leqslant t_n^k \leqslant s_n^j$. We showed that more than two extremal points cannot collapse, and thus more than three crossings cannot collapse.

Now suppose that three crossings collapse (the case of just two collapsing crossings is dealt with later), i.e., $s_n^i, s_n^{i+1}, s_n^{i+2} \to E$ as $n \to \infty$. If there are three (or more) extrema of u_n in $[s_n^i, s_n^{i+2}]$ for all large n, then the argument above (about the number of monotone laps) shows that $u \equiv 1$, a contradiction. Hence we may restrict our attention to the case that the crossings are between *consecutive* extremal points $u_i^n, u_{i+1}^n, u_{i+2}^n, u_{i+3}^n$, i.e., $s_n^i < t_n^{i+1} < s_n^{i+1} < t_n^{i+2} < s_n^{i+2}$. In particular, the crossings and extrema do not coincide, hence $u_n(t_n^{i+1}) \neq (u_+^{E_n})_{i+1}$. Furthermore, $s_n^i, s_n^{i+2} \to \bar{t}$ as $n \to \infty$. As before one can show that $u(\bar{t}) = 1$ and $u'(\bar{t}) = u''(\bar{t}) = 0$. We assert that $u(t_n^i) \to A \neq 1$ and $u(t_n^{i+3}) \to B \neq 1$ for $n \to \infty$, otherwise at least three extremal points would collapse. If i is odd then t_{i+1} corresponds to a minimum, hence $B < 1 < A$ and $u'''(\bar{t}) < 0$, whereas if i is even then t_{i+1} corresponds to a maximum, hence $A < 1 < B$ and $u'''(\bar{t}) > 0$. We now restrict attention to the latter case (the proof in the former case is analogous).

To arrive at a contradiction we argue as follows. We recall that by construction

$$u_+^{E_n} = 1 + \sqrt{\frac{E_n}{E_0}} \cos \lambda_1 t + o(\sqrt{E_n}), \quad \text{as } n \to \infty \ (E_n \to 0),$$

where $E_0 = \frac{\lambda_1^4}{2} - 1$. Hence $|(u_+^{E_n})_i - 1| > \frac{1}{2}\sqrt{\frac{E_n}{E_0}}$ for all i and n sufficiently large. For the maximum of u_n at t_n^{i+1} we distinguish two cases: $u_n(t_n^{i+1}) > (u_+^{E_n})_{i+1}$ or $u_n(t_n^{i+1}) < (u_+^{E_n})_{i+1}$, since equality was excluded above. In the latter case, since there is a crossing between t_n^i and t_n^{i+1}, we also have $u_n(t_n^i) > (u_+^{E_n})_i$ for the minimum at t_n^i. Moreover, using the up–down restriction and the fact that we have three consecutive crossings, we infer that

$$\left(u_+^{E_n}\right)_i < u_n\left(t_n^j\right) < \left(u_+^{E_n}\right)_{i+1} \quad \text{for } j = i-1, i, i+1, i+2.$$

Since $u_+^{E_n} \to 1$ as $n \to \infty$, this implies that four consecutive extrema of u_n converge to 1, and thus $u \equiv 1$ by the arguments used above, a contradiction.

We are thus left with the case $u_n(t_n^{i+1}) > (u_+^{E_n})_{i+1} > 1$, for which the proof is somewhat more involved. For the minimum at t_n^{i+2} we get the inequality $u_n(t_n^{i+2}) < (u_+^{E_n})_{i+2} < 1$. We then estimate

$$u_n\left(t_n^{i+1}\right) - u_n\left(t_n^{i+2}\right) \geqslant \left(u_+^{E_n}\right)_{i+1} - \left(u_+^{E_n}\right)_{i+2} > \sqrt{\frac{E_n}{E_0}}.$$

Let $\delta_n = \sqrt{\frac{E_n}{E_0}}$, then it follows form the mean value theorem that for every n there exists a $c_n \in (t_n^{i+1}, t_n^{i+2})$ such that

$$-u_n'(c_n) = \frac{u_n(t_n^{i+1}) - u_n(t_n^{i+2})}{t_n^{i+2} - t_n^{i+1}}.$$

We note that $u_n(c_n) \to 1$ and $u_n'(c_n) \to 0$ as $n \to \infty$, since $c_n \in (s_n^i, s_n^{i+2})$, see above. Due to $t_{i+2}^n - t_{i+1}^n \to 0$, we can estimate

$$-u_n'(c_n) > \sqrt{\frac{E_n}{E_0}} > 0, \tag{7.9}$$

for n large enough. If we divide the energy equation

$$E_n = -u_n'(c_n)u_n'''(c_n) + \frac{1}{2}\left(u_n''(c_n)\right)^2 - \frac{\alpha}{2}\left(u_n'(c_n)\right)^2 - \frac{1}{4}\left(u_n^2(c_n) - 1\right)^2$$

by the positive number $-u_n'(c_n)$ and use inequality (7.9), we get

$$u_n'''(c_n) \leqslant \sqrt{E_0 E_n} + \frac{\alpha}{2}\left|u_n'(c_n)\right| - \frac{(u_n^2(c_n) - 1)^2}{4u_n'(c_n)}. \tag{7.10}$$

We estimate the last term in the right-hand side of (7.10) using

$$\left|u_n(c_n) - 1\right| < u_n\left(t_n^{i+1}\right) - u_n\left(t_n^{i+2}\right) = -u_n'(c_n)\left(t_n^{i+2} - t_n^{i+1}\right) \leqslant -u_n'(c_n),$$

and hence

$$\left|\frac{(u_n^2(c_n) - 1)^2}{u_n'(c_n)}\right| \leqslant \left|u_n'(c_n)\right|\left(u(c_n) + 1\right)^2.$$

We conclude that the right-hand side of (7.10) tends to 0 as $n \to \infty$, implying that $u'''(\bar{t}) \leqslant 0$, which is a contradiction with $u'''(\bar{t}) > 0$. Thus three crossings cannot collapse.

Now we show, by contradiction, that two crossings cannot collapse. If two crossings collapse then there exist $s_n^{i-1} \leqslant t_n^i \leqslant s_n^i$ such that $s_n^{i-1}, s_n^i \to \bar{t}$. We can assume that $u_n(t_n^{i\pm 1}) \not\to 1$ otherwise the proof is analogous to the case of three collapsing intersections. Then, for $t = \bar{t}$ the solution u has an extremum and $u(\bar{t}) = 1$. As before, this contradicts that $\mathbb{E}[u] = 0$ and $u \not\equiv 1$.

Finally, $u(t) > -1$ for all t because otherwise there would be an extremum (minimum) of u at the point $\bar{t}$, with $u(\bar{t}) = -1$ and $\mathbb{E}[u] = 0$, again a contradiction. □

The final step is to show that $\{u_n\}_{n=0}^{\infty}$ does not converge to the constant solution $u_+ = 1$. Let $\mathbb{E}[u_n] = E_n$ and define the sequences $\{w^n\}_{n=0}^{\infty}$ and $\{w_+^n\}_{n=0}^{\infty}$ as follows

$$u_n = 1 + \varepsilon_n w^n,$$
$$u_+^{E_n} = 1 + \varepsilon_n w_+^n, \tag{7.11}$$

where $\varepsilon_n = \|u_n - 1\|_{L^\infty}$. Then w^n, w_+^n are solutions of equation

$$w'''' + \alpha w'' + 2w + 3\varepsilon_n w^2 + \varepsilon_n^2 w^3 = 0.$$

Let $\mathbb{E}_\varepsilon$ be the rescaled energy functional (7.2) associated to this equation. Then $\mathbb{E}_{\varepsilon_n}[w^n] = \mathbb{E}_{\varepsilon_n}[w_+^n] > 0$. If $u_n \to 1$ then $\varepsilon(n) \to 0$, $w^n \to w$ and $w_+^n \to w_+$, where w and w_+ are solutions of the linear equation

$$w'''' + \alpha w'' + 2w = 0, \tag{7.12}$$

and $\mathbb{E}_0[w] = \mathbb{E}_0[w_+] \stackrel{\text{def}}{=} E \geqslant 0$. By construction $w_+ = \sqrt{\frac{E}{E_0}}\cos(\lambda_1 t)$, where $E_0 = \frac{\lambda_1^4}{2} - 1$ (note that $w_+ = 0$ if $E = 0$). The following two lemmas summarize the properties of solutions to the linear equation (7.12).

7.9. Lemma. *Let $\alpha > \sqrt{8}$ be such that $\frac{\lambda_2}{\lambda_1}$ is irrational. Then there is no periodic solution of (7.12) on the energy level zero. The only periodic solution on a positive energy level is w_+.*

Proof. Every solution of (7.12) can be written as

$$x(t) = A\cos(\lambda_1 t + \varphi_1) + B\cos(\lambda_2 t + \varphi_2), \tag{7.13}$$

where $A, B, \varphi_1, \varphi_2 \in \mathbb{R}$. The ratio of the frequencies $\frac{\lambda_2}{\lambda_1}$ is irrational. Thus if x is periodic then either $A = 0$ or $B = 0$. Plugging (7.13) into the energy equation proves the lemma. □

7.10. Lemma. *Let $\alpha > \sqrt{8}$ be such that $\frac{\lambda_2}{\lambda_1}$ is rational, i.e., there are $p', q' \in \mathbb{N}$ coprime and $\frac{\lambda_2}{\lambda_1} = \frac{q'}{p'}$. Assume that $E > 0$ and $w_+ = \sqrt{\frac{E}{E_0}}\cos(\lambda_1 t)$ where $E_0 = \frac{\lambda_1^4}{2} - 1$. Then every solution w of (7.12) with $\mathbb{E}[x] = E$, which is not equal to w_+, has the property that its sequence of extrema $\mathbf{w}$ is $2p'$-periodic and intersects $\mathbf{w}_+$ exactly $2q'$ times per period.*

Proof. Since $\frac{\lambda_2}{\lambda_1}$ is rational, it follows from (7.13) that all solutions on the positive energy level E are periodic with the period $\frac{2\pi}{\lambda_1}p'$. Without loss of generality we may assume that w attains its minimum at $t = 0$.

First we show that the number of extremal points per period $\frac{2\pi}{\lambda_1}p'$ is $2p'$ for all solutions of (7.12). Let w_1 and w_2 be two different solutions. We interpolate between them and let $y(s,t)$ be a solution of (7.12) for every fixed $s \in [0,1]$ with initial conditions

$$\begin{aligned}
y(s,0) &= sw_1(0) + (1-s)w_2(0),\\
y'(s,0) &= 0,\\
y''(s,0) &= \sqrt{2\big(E + \big(sw_1(0) + (1-s)w_2(0)\big)^2\big)},\\
y'''(s,0) &= sw_1'''(0) + (1-s)w_2'''(0).
\end{aligned}$$

For every fixed $s \in [0,1]$ it holds that $\mathbb{E}_0[y(s,t)] = E$. The fact that the energy level $E > 0$ is regular implies that $y(s,t)$ is a concatenation of regular monotone laps (degenerate monotone lap cannot occur) for any fixed s. If two extremal points would collapse or a new one would be created along the path $y(s,t)$ then a degenerate monotone lap occurs, which is impossible. Therefore the number of extremal points per period $\frac{2\pi}{\lambda_1}p'$ is constant along the path $y(s,t)$. This implies that w_1 and w_2 have the same number of extremal points per period $\frac{2\pi}{\lambda_1}p'$. By counting the number of extremal points of the solution w_+ on the interval $[0, \frac{2\pi}{\lambda_1}p')$ one infers that this number is $2p'$. Hence the extremal sequence of any solution is $2p'$-periodic.

It follows from the proof of Lemma 7.3 that the rotation number $\tau(\mathbf{w}_+) = \frac{\lambda_2}{\lambda_1} = \frac{q'}{p'}$. This combined with the fact that the extremal sequence $\mathbf{w}$ of an arbitrary solution is $2p'$ periodic implies that $I(\mathbf{w}, \mathbf{w}_+) = 2q'$ for all solutions whose initial data are sufficiently close to the initial data of the solution w_+ but $w \not\equiv w_+$.

Again, by interpolating between the solutions we will prove that $I(\mathbf{v}, \mathbf{v}_+) = 2q'$ for an arbitrary solution not equal to w_+. Let w_1 and w_2 be two solutions such that $w_1, w_2 \not\equiv w_+$ and let $y(s,t)$ be the connecting path between them defined as above. It may happen that $y(s,t) = w_+$ for some s_0, but by a small perturbation of the path of initial conditions, say varying $y'''(s,0)$ slightly, we can avoid that. Therefore, we suppose that $y(s,t)$ is not equal to w_+ for any s. Let $\mathbf{y}(s)$ be an extremal sequence of $y(s,t)$. We show that $I(\mathbf{v}, \mathbf{y}(s))$ is constant by contradiction. If it is non-constant, then there exists an $s_0 \in [0,1]$, for which $\mathbf{v}_+$ and $\mathbf{y}(s)$ have a non-transversal intersection. However according to Proposition 4.2 two stationary points $\mathbf{v}_+$ and $\mathbf{y}(s)$ of the flow Ψ^t generated by Eq. (7.12) cannot have a non-transversal intersection. Hence we have proved that $I(\mathbf{v}, \mathbf{v}_+) = 2q'$ for an arbitrary solution w not equal to w_+. $\square$

The final lemma, combined with Lemma 7.8, completes the proof of Theorem 1.2. Recall that the parameter range is $\alpha \in (\sqrt{8}, \alpha_{p,q})$, hence $\frac{q}{p} < \frac{\lambda_2}{\lambda_1}$.

7.11. Lemma. *The sequence $\{u_n\}_{n=0}^{\infty}$ does not converge to the constant solution.*

Proof. We will prove this by contradiction. Suppose that $u_n \to 1$ and let w_n be as in (7.11). Then $w^n \to w$, where w is a solution of the linear equation (7.12) and $\mathbb{E}[w] \geqslant 0$. Moreover, $\|w\|_{L^\infty} = \lim_{n\to\infty} \|w^n\|_{L^\infty} = 1$. Let T_n be the period of w^n.

First we assume that $\frac{\lambda_2}{\lambda_1}$ is irrational. Let us start with the case $\mathbb{E}[w] = 0$. It follows from Lemma 7.9 that w is not periodic and as we showed in the proof of Lemma 7.8 it must be that $T_n \to \infty$. Therefore solution w has at most $2p$ extrema on $\mathbb{R}$ and from (7.13) we conclude that $w \equiv 0$, which is in contradiction with $\|w\|_{L^\infty} = 1$.

If $\mathbb{E}[w] > 0$, then w has to be periodic, otherwise we obtain a contradiction as above. It follows from Lemma 7.9 that the only periodic solution on this energy level is w_+. Thus $w^n \to w_+$ and $\|w^n - w_+^n\|_{L^\infty} \to 0$. The fact that $\tau(\mathbf{w}_+^n) \to \tau(\mathbf{w}_+) = \frac{\lambda_1}{\lambda_2} > \frac{q}{p}$ contradicts the assumption that $\mathbf{w}^n$ is $2p$ periodic and $I(\mathbf{w}^n, \mathbf{w}_+^n) = 2q$.

In the rational case $\frac{\lambda_2}{\lambda_1}$ we argue as follows. If $\mathbb{E}[w]=0$, then $w^+\equiv 0$ and w^n cannot converge to w^+ because $\|w\|_{L^\infty}=1$. Hence, by repeating the arguments in the proof of Lemma 7.8 one gets that $\mathbf{w}$ is $2p$ periodic and intersects zero $2q$ times per period. We will obtain a contradiction by showing that $\mathbf{w}$ is $2p'$ periodic and it intersects zero $2q'$ times per period, where $p',q'\in\mathbb{N}$ such that $\frac{q'}{p'}=\frac{\lambda_2}{\lambda_1}>\frac{q}{p}$. To prove the latter claim about the extremal sequence $\mathbf{w}$, we employ solutions w_L^n of the linear equation (7.12) with the same initial conditions as solutions w^n. These functions w^n also converge to w and the energy $\mathbb{E}[w_L^n]>0$ for all $n\in\mathbb{N}$. It follows from Lemma 7.10 that $\mathbf{w}_L^n$ is $2p'$ periodic and $I(\mathbf{w}_L^n,\mathbf{w}_+^n)=2q'$. Hence as before (using the arguments in the proof of Lemma 7.8) the limit process for w_L^n implies that $\mathbf{w}$ is $2p'$ periodic and intersects zero $2q'$ times per period. This contradicts the inequality $\frac{q'}{p'}>\frac{q}{p}$.

Finally, if $\mathbb{E}[w]>0$, then solutions w^n cannot converge to w_+, since otherwise we arrive at the same contradiction as in the irrational case. Hence Lemma 7.10 implies that $\mathbf{w}$ is $2p'$ periodic and $I(\mathbf{w},\mathbf{w}^+)=2q'$. On the other hand $w^n\to w$ and by using the ideas in the proof of Lemma 7.8 (conservation of number of monotone laps and number of intersections) we get that $\mathbf{w}$ is $2p$ periodic and $I(\mathbf{w},\mathbf{w}^+)=2q$, which is a contradiction. $\square$

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