Name: Solutions

Math 165 Section 5915 *Exam 1* *September 23, 2010*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! There are 12 problems all of them each worth 5 points for a total of 60 points. There are also 2 bonus problems at the end both worth 3 points.

1. Let $f(x) = \sqrt{x}$ and $g(x) = \sin x$. Find the composition $g \circ f$ and its domain.

$(g \circ f)(x) = g(f(x)) = \sin(\sqrt{x})$

$\mathrm{dom}(f \circ g) = \{x \mid x \geq 0\}$

2. Let $f(x)$, $g(x)$, and $h(x)$ be functions such that: $f(3) = 5$, $f(1) = 0$, $g(6) = 2$, $g(5) = 0$, $h(4) = -2$ and $h(0) = 1$. Find $(h \circ g \circ f)(3)$.

$h(g(f(3))) = h(g(5)) = h(0) = 1$

For problems 3 and 4 find the limit of the given function if it exists.

3. $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x^2-2x-3}$ $\qquad \frac{0}{0}$ type

$$= \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x+1)} = \lim_{x\to 3} \frac{x+3}{x+1} = \frac{6}{4} = \frac{3}{2}$$

4. $\displaystyle\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x}$ $\qquad \frac{0}{0}$ type

$$= \lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x} \cdot \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \lim_{x\to 0} \frac{x^2}{x(1+\sqrt{1-x^2})} = \lim_{x\to 0} \frac{x}{1+\sqrt{1-x^2}}$$

$$= \frac{0}{1+1} = 0$$

5. Prove that $\lim_{x \to 0} x^5 \cos\left(\frac{4\pi}{x^2}\right) = 0$.

$$-1 \leq \cos\left(\frac{4\pi}{x^2}\right) \leq 1 \Rightarrow -x^5 \leq x^5 \cos\left(\frac{4\pi}{x^2}\right) \leq x^5$$

$\lim_{x \to 0} (-x^5) = \lim_{x \to 0} x^5 = 0$ so by squeeze thm

$$\Rightarrow \lim_{x \to 0} x^5 \cos\left(\frac{4\pi}{x^2}\right) = 0$$

6. Let $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x > 2 \end{cases}$. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

for continuity at $x = 2$ need $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$$\Rightarrow \lim_{x \to 2^-} cx^2 + 2x = \lim_{x \to 2^+} x^3 - cx \Rightarrow 4c + 4 = 8 - 2c$$

$$\Rightarrow 6c = 4$$

$$c = \frac{2}{3}$$

Find the derivative of the following function using the definition of the derivative. All other methods will have zero point value.

7. $g(x) = x^2 + 3x + 8$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) + 8 - x^2 - 3x - 8}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h \to 0} 2x + h + 3 = 2x + 3$$

For problems 8-10, differentiate the following functions with respect to the indicated independent variable.

8. $f(x) = 3x^7 + 5x^6 - 2x^3 + 10x^2 + 12x - 5$

$f'(x) = 21x^6 + 30x^5 - 6x^2 + 10x + 12$

9. $y(u) = (u^{-2} + u^{-3})(u^5 - 2u^2)$

$$\frac{dy}{du} = (u^{-2}+u^{-3})'(u^5-2u^2) + (u^{-2}+u^{-3})(u^5-2u^2)'$$

$$= (-2u^{-3} - 3u^{-4})(u^5 - 2u^2) + (u^{-2}+u^{-3})(5u^4 - 2u)$$

10. $h(r) = \dfrac{r^2 + 2r + 1}{1 + 2\sqrt{r}}$

$$h'(r) = \frac{(r^2+2r+1)'(1+2\sqrt{r}) - (r^2+2r+1)(1+2\sqrt{r})'}{(1+2\sqrt{r})^2}$$

$$= \frac{(2r+2)(1+2\sqrt{r}) - (r^2+2r+1)(r^{-\frac{1}{2}})}{(1+2\sqrt{r})^2}$$

For problems 11 and 12, find the equation of the tangent line of the following functions at the indicated point.

11. $y = x^5 + 2x^4 - 3x + 2$, at $(0, 2)$

$y' = 5x^4 + 8x^3 - 3$

$m = y'(0) = -3$

then $y - 2 = -3(x - 0)$

$\Rightarrow y = -3x + 2$

12. $y = \dfrac{2x^2}{x+1}$, at $(1, 1)$

$$\frac{dy}{dx} = \frac{(2x^2)'(x+1) - 2x^2(x+1)'}{(x+1)^2} = \frac{4x(x+1) - 2x^2}{(x+1)^2} = \frac{2x^2 + 4x}{(x+1)^2}$$

$$m = \left.\frac{dy}{dx}\right|_{x=1} = \frac{2+4}{(1+1)^2} = \frac{6}{4} = \frac{3}{2}$$

$\Rightarrow y - 1 = \frac{3}{2}(x - 1)$

$\Rightarrow y = \frac{3}{2}x - \frac{1}{2}$

13. (BONUS) Given that $\lim_{x \to 2} (5x - 7) = 3$, use the precise definition of a limit to find a δ that corresponds to $\varepsilon = 1$.

need $|x-2| < \delta \Rightarrow |(5x-7)-3| < 1$

$\Rightarrow |5x-10| < 1$

$\Rightarrow 5|x-2| < 1$

$\Rightarrow |x-2| < \frac{1}{5}$

pick $\delta = \frac{1}{5}$

14. (BONUS) Let $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Determine if $f'(0)$ exists and if so compute it. (Hint: Use the definition of the derivative.)

if $f'(0)$ exists then

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(\frac{1}{x}) - 0}{x}$$

$$= \lim_{x \to 0} x \sin(\tfrac{1}{x}) = 0 \qquad \text{so } f'(0) = 0$$