Name: Solutions

Math 165 Section 5915 *Exam 2* *October 26, 2010*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! There are 11 problems with 10 of them each worth 5 points and number 11 worth 10 points for a total of 60 points. There are also 2 bonus problems at the end both worth 3 points.

For problems 1 - 4, calculate the derivative of the function with respect to the indicated independent variable.

1. $f(x) = \cos(\tan x)$

$$f'(x) = -\sin(\tan x)(\tan x)'$$
$$= -(\sec^2 x)(\sin(\tan x))$$

2. $y(x) = \sec(2 + x^3)$

$$y'(x) = \sec(2+x^3)\tan(2+x^3)(2+x^3)'$$
$$= 3x^2 \sec(2+x^3)\tan(2+x^3)$$

3. $h(\theta) = \dfrac{\cot(3\theta)}{1 + \sin(3\theta)}$

$$h'(\theta) = \frac{(\cot(3\theta))'(1+\sin(3\theta)) - \cot(3\theta)(1+\sin(3\theta))'}{(1+\sin(3\theta))^2}$$
$$= \frac{-3\csc^2(3\theta)(1+\sin(3\theta)) - 3\cot(3\theta)\cos(3\theta)}{(1+\sin(3\theta))^2}$$

4. $g(t) = t^3 \cos(t)$

$$g'(t) = (t^3)' \cos t + t^3 (\cos t)'$$

$$= 3t^2 \cos t - t^3 \sin t$$

5. Let $f(x) = \sin x$. Calculate $f^{(38)}(\pi)$.

$$f(x) = f^{(iv)}(x) \qquad 4\overline{)38}^{\,9} \quad \begin{array}{r} 36 \\ \hline 2 \end{array} \quad \text{Rem} = 2 \quad \Rightarrow \quad f^{(38)}(x) = f''(x)$$

$$f'(x) = \cos x, \quad f''(x) = -\sin x$$

$$f^{(38)}(\pi) = f''(\pi) = -\sin(\pi) = 0$$

6. Let y be implictly defined in x by $x^2 \cos(y) + \sin(y^3) = x^2 y^6$. Compute $\dfrac{dy}{dx}$.

$$\frac{d}{dx}\left(x^2 \cos y + \sin(y^3)\right) = \frac{d}{dx}\left(x^2 y 6\right)$$

$$2x\cos y - x^2 \sin y \frac{dy}{dx} + 3y^2 \frac{dy}{dx} \cos(y^3) = 2xy^6 + 6x^2 y\ 5 \frac{dy}{dx}$$

$$\left(3y^2 \cos(y^3) - x^2 \sin y - 6x^2 y\ 5\right)\frac{dy}{dx} = 2xy^6 - 2x\cos y$$

$$\frac{dy}{dx} = \frac{2xy6 - 2x\cos y}{3y^2 \cos(y^3) - x^2 \sin y - 6x^2 y^5}$$

7. Find the equation of the tangent line of $y(x) = \sin^2(x)$ at $x = \dfrac{\pi}{4}$.

$y'(x) = 2\sin x\cos x$

$m = y'(\frac{\pi}{4}) = 2\sin(\frac{\pi}{4})\cos(\frac{\pi}{4}) = 2(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = 1$

$y(\frac{\pi}{4}) = [\sin(\frac{\pi}{4})]^2 = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$

$y - \frac{1}{2} = 1(x - \frac{\pi}{4})$

$y = x + \frac{1}{2} - \frac{\pi}{4}$

$y = x + \frac{2-\pi}{4}$

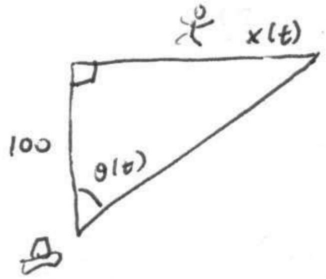
8. Evaluate the following limit $\lim_{x\to\infty} \sqrt{9x^2 + x} - 3x$. $\infty - \infty$ type

$\lim_{x\to\infty} \sqrt{9x^2+x} - 3x \cdot \frac{\sqrt{9x^2+x} + 3x}{\sqrt{9x^2+x} + 3x}$

$= \lim_{x\to\infty} \frac{9x^2+x-9x^2}{\sqrt{9x^2+x} + 3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x} + 3x} = \lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{1}{x}} + 3}$

$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

9. Larry Luckless who always has ~~back~~ bad luck is running along a straight path at a speed of 20 ft/s. A tank is located on the ground 100 ft from the path and is aiming at Larry ready to blow him up at any given moment. At what rate is the tank rotating when Larry 70 ft from the point on the path closest to the tank? Will Larry survive to see another day?



$$\tan(\theta(t)) = \frac{x(t)}{100} \Rightarrow x(t) = 100\tan(\theta(t))$$

$$\frac{dx}{dt} = 100\sec^2(\theta(t))\frac{d\theta}{dt}$$

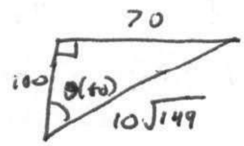
$$20 = 100\left(\frac{\sqrt{149}}{10}\right)^2\frac{d\theta}{dt}$$

$$20 = 149\frac{d\theta}{dt} \Rightarrow$$

$\frac{dx}{dt} = 20\, ft/s \quad x(t_0) = 70\, ft$

$\frac{d\theta}{dt} = ?? \quad \tan(\theta(t_0)) = \frac{7}{10}$

$\sec(\theta(t_0)) = \frac{\sqrt{149}}{10}$



$$\Rightarrow \frac{d\theta}{dt} = \frac{20}{149}\ rad/s$$

10. Estimate $\sqrt[3]{8.001}$ by using linear approximation.

let $f(x) = \sqrt[3]{x}$ and $a=8$ so $f(a) = f(8) = \sqrt[3]{8} = 2$

want to find $f(8.001)$

then $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ and $f'(8) = \frac{1}{3\sqrt[3]{8^2}} = \frac{1}{3\cdot 4} = \frac{1}{12}$

then $L(x) = f(a) + f'(a)(x-a) = 2 + \frac{1}{12}(x-8)$

then $\sqrt[3]{8.001} \approx 2 + \frac{1}{12}(8.001-8) = 2 + \frac{1}{12}\left(\frac{1}{1000}\right)$

$$= 2 + \frac{1}{12000} = \frac{24001}{12000}$$

11. Do a complete curve sketching analysis of $f(x) = x^3 - 6x^2 - 15x + 4$, in otherwords find all critical numbers, critical points, inflection points, all relative extrema, on what intervals the function is increasing/decreasing, on what intervals the function is concave up/down, find all asymptotes if there are any and sketch the curve.

$f'(x) = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5)$
$= 3(x-5)(x+1)$

$f''(x) = 6x - 12 = 6(x-2)$

C.R. $\Rightarrow$ $f'(x) = 0$ or $f'(x)$ DNE
None here

$\underline{f'(x) = 0}$

$3(x-5)(x+1) = 0$

$\Rightarrow x = 5, -1$

Possible I.P. $\Rightarrow$ $f''(x) = 0$

$\underline{f''(x) = 0}$

$\Rightarrow 6(x-2) = 0$
$x = 2$

C.R.	I.P. ??
local min $\Rightarrow (5, -96)$	$(2, -42)$ ✓
$(-1, 12) \Leftarrow$ local max	

$f \nearrow : (-\infty, -1) \cup (5, \infty)$

$f \searrow : (-1, 5)$

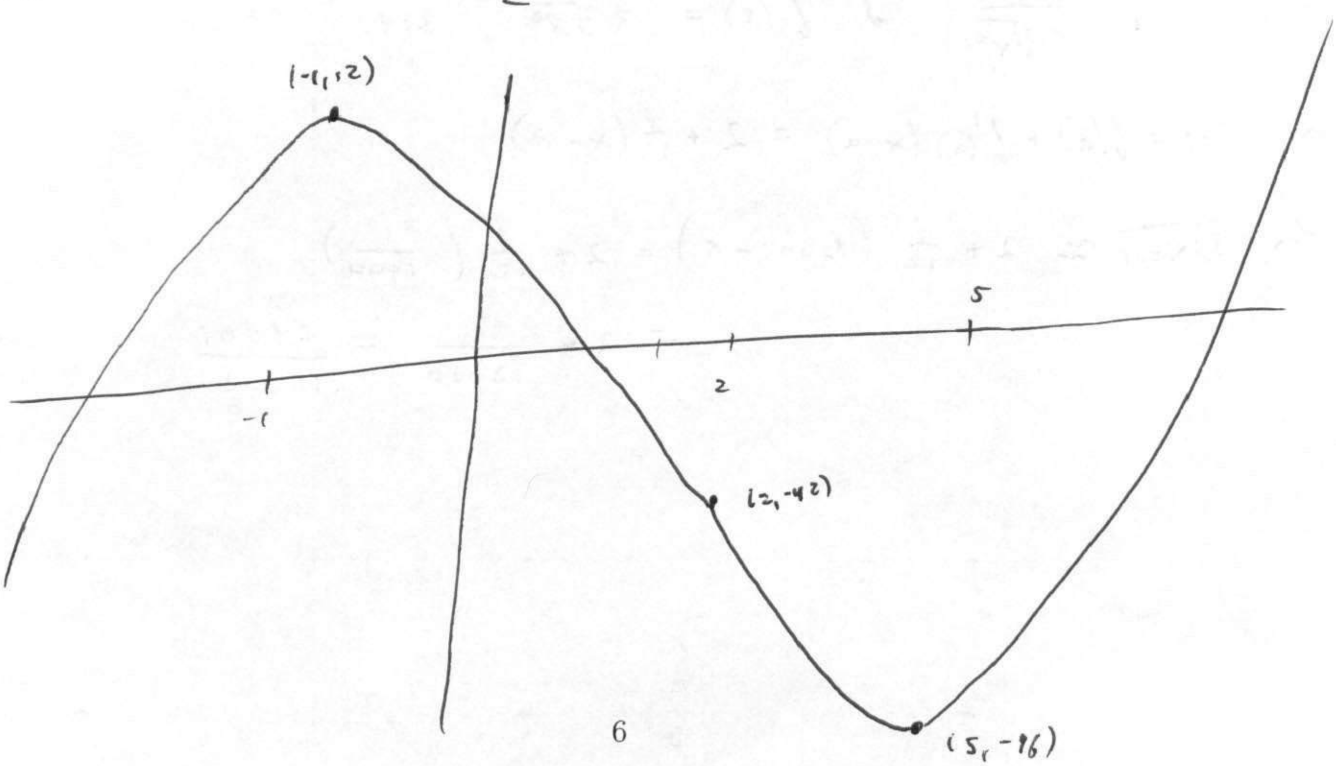
no asymptotes

$f \frown : (-\infty, 2)$

$f \smile : (2, \infty)$

$f'(x) = 3(x-5)(x+1)$: $+$ on $(-\infty,-1)$, $-$ on $(-1,5)$, $+$ on $(5,\infty)$

$f''(x) = 6(x-2)$: $-$ on $(-\infty,2)$, $+$ on $(2,\infty)$



12. (BONUS) If $f(1) = 13$ and $f'(x) \geq 4$ for $1 \leq x \leq 8$, how small can $f(8)$ possibly be?

by MVT there is a $c \in (1, 8)$ s.t. $f'(c) = \dfrac{f(8) - f(1)}{8 - 1}$

$\Rightarrow f'(c) = \dfrac{f(8) - 13}{7}$ OTH $f'(x) \geq 4 \Rightarrow 4 \leq \dfrac{f(8) - 13}{7}$

$\Rightarrow 28 \leq f(8) - 13 \Rightarrow f(8) \geq 41$

so $f(8)$ can be as small as 41

13. (BONUS) A number d is called a **fixed point** of a function f if $f(d) = d$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point. (HINT: assume f has more than one fixed point and use the Mean Value Theorem).

suppose f has two fixed pts $f(b) = b$ and $f(a) = a$

then by MVT there is a $c \in (a, b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$\Rightarrow f'(c) = \dfrac{b - a}{b - a} = 1 \Rightarrow f'(c) = 1$ can't happen since $f'(x) \neq 1$ for all x $\Rightarrow$ f has at most one fixed pt.