Name: Solutions

Math 165 Section 5915 *Exam 3* *November 30, 2010*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! There are 12 problems each of them worth 5 points for a total of 60 points. There are also 2 bonus problems at the end both worth 3 points.

For problems 1 - 4, evaluate the indefinite or definite integrals.

1. $\int (x^5 - 9x^2 + 2\sec^2 x + \cos x)\, dx$

$= \frac{x^6}{6} - 3x^3 + 2\tan x + \sin x + C$

2. $\int_0^2 t^3\sqrt{1+t^4}\, dt = \frac{1}{4}\int_1^{17} u^{\frac{1}{2}} du = \frac{1}{4}\cdot\frac{2}{3}\, u^{\frac{3}{2}}\Big|_1^{17}$

$= \frac{1}{6}\left(17\sqrt{17} - 1\right)$

$u = 1+t^4$

$du = 4t^3 dt$

$t=0 \Rightarrow u = 1$

$t=2 \Rightarrow u = 17$

3. $\int v\sqrt{v+4}\,dv = \int (u-4)\,u^{\frac{1}{2}}du = \int u^{\frac{3}{2}} - 4u^{\frac{1}{2}}du$

$u = v+4 \quad v = u-4$

$du = dv$

$= \frac{2}{5}u^{\frac{5}{2}} - 4\cdot\frac{2}{3}u^{\frac{3}{2}} + C$

$= \frac{2}{5}(v+4)^{\frac{5}{2}} - \frac{8}{3}(v+4)^{\frac{3}{2}} + C$

4. $\int_0^{\sqrt{\pi}} \theta\cos\left(\theta^2\right)\,d\theta = \frac{1}{2}\int_0^{\pi}\cos(u)du = \frac{1}{2}\sin u\Big|_0^{\pi} = 0$

$u = \theta^2$

$du = 2\theta d\theta$

$\theta = 0 \Rightarrow u = 0$

$\theta = \sqrt{\pi} \Rightarrow u = \pi$

5. Let $f(x)$ be a continuous function. If $\int_1^8 f(x)\, dx = 30$ , then evaluate $\int_1^2 x^2 f\left(x^3\right)\, dx$ .

$$\int_1^2 x^2 f(x^3)\,dx = \frac{1}{3}\int_1^8 f(u)\,du = \frac{1}{3}(30) = 10$$

$u = x^3$

$du = 3x^2\,dx$

$x = 1 \Rightarrow u = 1$

$x = 2 \Rightarrow u = 8$

6. Let $g(x) = \int_1^{\sin x} \frac{\tan(1+t^2)}{1+t^4}\, dt$ . Compute $g'(x)$.

$$g'(x) = \frac{\tan(1+\sin^2 x)}{1+\sin^4 x} \cdot (\sin x)'$$

$$= \frac{\tan(1+\sin^2 x)}{1+\sin^4 x} \cdot \cos x$$

7. Find $f(x)$ if $f''(x) = \sin x + 3\cos x$, $f(0) = 0$ and $f'(0) = 2$.

$$f'(x) = -\cos x + 3\sin x + C$$

$$2 = f'(0) = -1 + C \quad \Rightarrow \quad C = 3$$

$$f'(x) = -\cos x + 3\sin x + 3$$

$$f(x) = -\sin x - 3\cos x + 3x + D$$

$$0 = f(0) = 0 - 3 + 0 + D \quad \Rightarrow \quad D = 3$$

$$f(x) = -\sin x - 3\cos x + 3x + 3$$

8. Using the properties of integrals, prove the following estimate: $\displaystyle\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx \le \frac{\pi^3}{24}$.

notice $\sin x \le 1$ on $[0, \frac{\pi}{2}]$

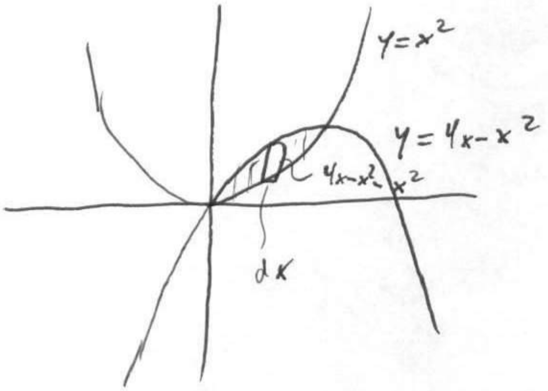
and thus $x^2 \sin x \le x^2$ Therefore

$$\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx \le \int_0^{\frac{\pi}{2}} x^2 \, dx = \frac{1}{3} x^3 \Big|_0^{\frac{\pi}{2}} = \frac{1}{3} \cdot \frac{\pi^3}{8} = \frac{\pi^3}{24}$$

hence $\displaystyle\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx \le \frac{\pi^3}{24}$

For problems 9 and 10, find the area between the indicated curves.

**9**. The curves $y = x^2$ and $y = 4x - x^2$.




$$\text{Area elmt} = (4x - x^2 - x^2)\,dx$$
$$= (4x - 2x^2)\,dx$$

$$A = \int_0^2 (4x - 2x^2)\,dx$$
$$= \left(2x^2 - \tfrac{2}{3}x^3\right)\Big|_0^2 = 8 - \tfrac{2}{3}(8)$$
$$= 8\left(1 - \tfrac{2}{3}\right) = \frac{8}{3}$$

limits

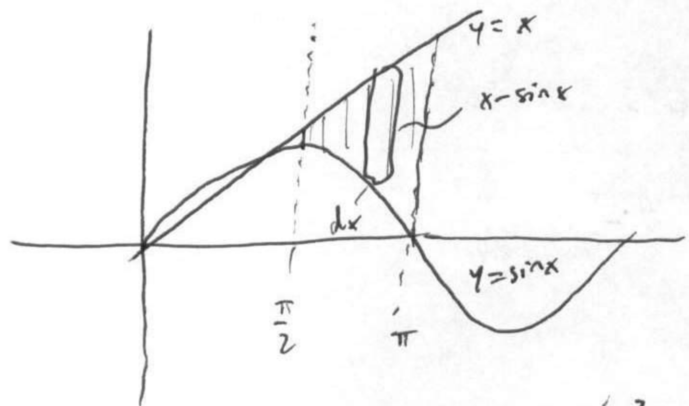
$x^2 = 4x - x^2$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

$x = 0, 2$

**10**. The curves $y = \sin x$, $y = x$, $x = \dfrac{\pi}{2}$, and $x = \pi$.




$$\text{Area elmt} = (x - \sin x)\,dx$$

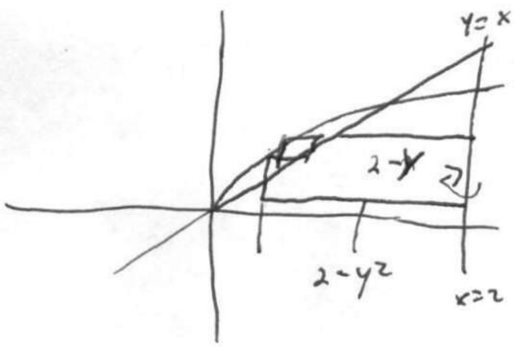
$$A = \int_{\frac{\pi}{2}}^{\pi} (x - \sin x)\,dx$$

$$= \left(\frac{x^2}{2} + \cos x\right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \left(\frac{\pi^2}{2} - 1\right) - \left(\frac{1}{2}\cdot\frac{\pi^2}{4}\right)$$

$$= \frac{\pi^2}{2} - 1 - \frac{\pi^2}{8} = \frac{3\pi^2}{8} - 1$$

**11.** Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = \sqrt{x}$ about the line $x = 2$. (Set up the integral only, DO NOT evaluate it.)




Vol element $= \pi\left((2-y^2)^2-(2-y)^2\right)dy$

$$Vol = \pi \int_0^1 \left[(2-y^2)^2 - (2-y)^2\right] dy$$

limits

$y = y^2$

$y^2 - y = 0$

$y(y-1) = 0$

$y = 0, 1$

**12.** Find two positive numbers such that the sum of the first number and four times the second number is 104 and the product of the numbers is as large as possible.

let $x, y > 0$ and $x + 4y = 104$ and $P = xy$

so $x = 104 - 4y$ and $P(y) = (104 - 4y)y = 104y - 4y^2$

then $P'(y) = 104 - 8y$ and $P'(y) = 0 \Rightarrow 8y = 104$

$y = 13$

then $x = 104 - 4(13) = 52$ $\quad P''(y) = -8$ and $P''(13) = -8 < 0$ yields max

then the #'s are $13, 52$

13. (BONUS) Evaluate the following sum $\lim_{n\to\infty}\sum_{i=0}^{n}\frac{i^5}{n^6}$. (Hint: Use the definition of the integral and left endpoint $a = 0$.)

$$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{i^5}{n^6} = \lim_{n\to\infty}\sum_{i=0}^{n}\left(\frac{i}{n}\right)^5\cdot\frac{1}{n} = \int_0^1 f(x)dx = \int_0^1 x^5 dx$$

$$= \frac{1}{6}x^6\Big|_0^1 = \frac{1}{6}$$

if $a=0$ then $\Delta x = \frac{b}{n}$ and $\Delta x = \frac{1}{n}$

gives $b=1$ and $x_i = \frac{i}{n}$

so $f(x) = x^5$

14. (BONUS) Find the following limit $\lim_{h\to 0}\frac{1}{h}\int_1^{1+h}\sqrt{1+t^4}\,dt$. (Hint: Use the Fundamental Theorem of Calculus.)

let $g(x) = \int_a^x \sqrt{1+t^4}\,dt$ then $\lim_{h\to 0}\frac{1}{h}\int_1^{1+h}\sqrt{1+t^4}\,dt = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{1+h}\sqrt{1+t^4}\,dt + \int_1^a \sqrt{1+t^4}\,dt\right)$

$$= \lim_{h\to 0}\frac{1}{h}\left(\int_a^{1+h}\sqrt{1+t^4}\,dt - \int_a^1\sqrt{1+t^4}\,dt\right) = \lim_{h\to 0}\frac{1}{h}\left(g(1+h) - g(1)\right)$$

$$= \lim_{h\to 0}\frac{g(1+h)-g(1)}{h} = g'(1)$$

but $g'(x) = \sqrt{1+x^4}$ so $g'(1) = \sqrt{1+1} = \sqrt{2}$