Name: Key

Math 166 Section 19061 | *Exam 1* | *September 20, 2011*

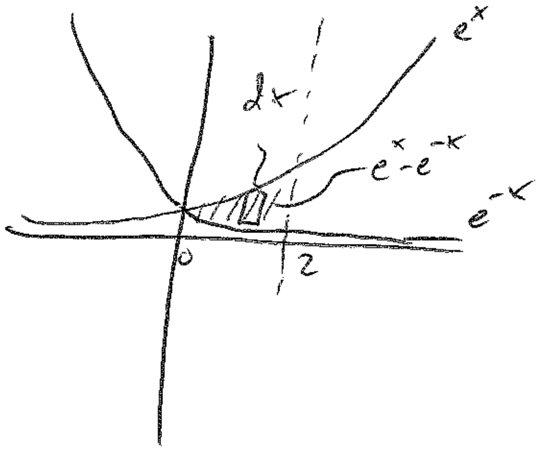
Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! There are 10 problems all of them each worth 5 points for a total of 50 points. There are is also a bonus problem at the end worth 3 points.

1. Let $f(x) = \ln x + \tan^{-1} x$. Find $(f^{-1})'\left(\frac{\pi}{4}\right)$.

$f'(x) = \frac{1}{x} + \frac{1}{x^2+1}$ and $(f^{-1})\left(\frac{\pi}{4}\right) = 1$ since $\ln 1 = 0$ and $\tan^{-1}(1) = \frac{\pi}{4}$

so $(f^{-1})'\left(\frac{\pi}{4}\right) = \frac{1}{f'(1)} = \frac{2}{3}$

2. Find the area of the region bounded by the curves $y = e^x$, $y = e^{-x}$, $x = 0$, and $x = 2$.



so Area element $= (e^x - e^{-x})\,dx$

$$A = \int_0^2 (e^x - e^{-x})\,dx = (e^x + e^{-x})\Big|_0^2$$

$$= e^2 + e^{-2} - 2$$

For problems 3 - 5 find the derivative of the given function.

3. $g(t) = \dfrac{e^t}{1+e^t}$

$$g'(t) = \frac{e^t(1+e^t) - e^t e^t}{(1+e^t)^2} = \frac{e^t}{(1+e^t)^2}$$

4. $f(x) = \arctan(\arcsin x)$

$$f'(x) = \frac{1}{1+(\arcsin x)^2} \cdot \frac{1}{\sqrt{1-x^2}}$$

5. $p(r) = \ln(\sin r) - \frac{1}{2}\sin^2 r$

$$p'(r) = \frac{1}{\sin r}\cos r - \frac{1}{2}\cdot 2\sin r\cos r$$

$$= \frac{\cos r}{\sin r} - \sin r\cos r$$

For problems 6 and 7, find the limit of the following functions if they exist.

6. $\lim\limits_{x \to 0^+} \frac{\sin(\pi x)}{\ln(1+3x)}$ $\quad \frac{0}{0}$ type

$$= \lim_{x \to 0^+} \frac{\pi\cos(\pi x)}{\left(\frac{1}{1+3x}\right)3} = \frac{\pi}{3}$$

7. $\lim_{x\to\infty}(e^x+1)^{1/x}$ ∞^0 - type

$$y=\lim_{x\to\infty}(e^x+1)^{\frac{1}{x}} \quad \text{so} \quad \ln y=\lim_{x\to\infty}\frac{1}{x}\ln(e^x+1)=\lim_{x\to\infty}\frac{\ln(e^x+1)}{x} \quad \frac{\infty}{\infty}\text{ type}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{e^x+1}\cdot e^x}{1}=\lim_{x\to\infty}\frac{e^x}{e^x+1}=1$$

$$\text{so } y=e^1=e$$

For problems 8 - 10, evaluate the following integrals.

8. $\int \sin^6 x\cos^3 x\,dx = \int \sin^6 x(1-\sin^2 x)\cos x\,dx$ $\quad u=\sin x$, $du=\cos x\,dx$

$$=\int u^6(1-u^2)\,du=\int(u^6-u^8)\,du=\frac{1}{7}u^7-\frac{1}{9}u^9+C$$

$$=\frac{1}{7}\sin^7 x-\frac{1}{9}\sin^9 x+C$$

9. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}}\, dt$ $\qquad u = \sin t \qquad du = \cos t\, dt \qquad t=0 \Rightarrow u=0 \qquad t=\frac{\pi}{2} \Rightarrow u=1$

$$= \int_0^1 \frac{du}{\sqrt{1+u^2}}$$

$u = \tan\theta$
$du = \sec^2\theta\, d\theta$
$1+u^2 = 1+\tan^2\theta = \sec^2\theta$

$u=0 \Rightarrow \theta = 0$
$u=1 \Rightarrow \theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int_0^{\frac{\pi}{4}} \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| \Big|_0^{\frac{\pi}{4}}$$

$$= \ln|\sqrt{2}+1|$$

10. $\displaystyle\int e^{4x}\cos(5x)\, dx = I$

$u = e^{4x} \qquad dv = \cos(5x)\,dx$
$du = 4e^{4x}\,dx \qquad v = \frac{1}{5}\sin(5x)$

so $I = \frac{1}{5}e^{4x}\sin(5x) - \frac{4}{5}\int e^{4x}\sin(5x)\,dx$

$u = e^{4x} \qquad dv = \sin(5x)\,dx$
$du = 4e^{4x}\,dx \qquad v = \frac{-1}{5}\cos(5x)$

$$I = \frac{1}{5}e^{4x}\sin(5x) - \frac{4}{5}\left(-\frac{1}{5}e^{4x}\cos(5x) + \frac{4}{5}\int e^{4x}\cos(5x)\,dx\right)$$

$$I = \frac{1}{5}e^{4x}\sin(5x) + \frac{4}{25}e^{4x}\cos(5x) - \frac{16}{25}I$$

so $I = \frac{5}{41}e^{4x}\sin(5x) + \frac{4}{41}e^{4x}\cos(5x) + C$

11. (BONUS) Let f be a continuous function such that

$$\int_0^x f(t)dt = e^x \sin x + \int_0^x e^{-t} f(t)dt$$

for all x. Find an explicit formula for $f(x)$. (Hint: Use the Fundamental Theorem of Calculus.)

$$\frac{d}{dx} \int_0^x f(t)dt = \frac{d}{dx}\left(e^x \sin x + \int_0^x e^{-t} f(t)dt \right)$$

$$f(x) = e^x \sin x + e^x \cos x + e^{-x} f(x)$$

$$(1 - e^{-x}) f(x) = e^x (\sin x + \cos x)$$

$$f(x) = \frac{e^x(\sin x + \cos x)}{1 - e^{-x}}$$